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RECURRENCE RELATIONS FOR CONDITIONAL MOMENT GENERATING FUNCTIONS OF ORDER STATISTICS AND RECORD VALUES

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Abstract

In this paper, recurrence relations for conditional moment generating functions and conditional moments of order statistics and record values based on random samples drawn from members of a class of doubly truncated distributions $\Im_d$ are obtained.

Keywords: order statistics, record values, conditional moment generating functions, conditional moments, recurrence relations.

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1. Introduction

Order statistics and records are used in a variety of disciplines and have extensively appeared in statistical literature. Many authors have investigated either of the topies or both, among others: Sarhan and Greenberg (1962), Reiss (1989), Arnold, Balakrishnan and Nagaraja (1992, 1998), Ahsanullah (1995) and Ahsanullah and Nevzorov (2001).

In this paper, recurrence relations of conditional moment generating functions and conditional moments of powers of order statistics and records based on random samples drawn from a population whose distribution is a member of a general class of distributions, denoted by $\Im_d$, are obtained.

Suppose that a random variable X having an absolutely continuous distribution function (*df*), considered by AL-Hussaini and Osman (1997), AL-Hussaini (1999) and AL-Hussaini and Ahmad (2003a, 2003b), is given by

$$F(x) \equiv F_X(x;\theta) = 1 - e^{-\lambda(x;\theta)} \equiv 1 - e^{-\lambda(x)}, \qquad x > 0,$$

and the probability density function (*pdf*), given by

$$f(x) = \lambda'(x)\, e^{-\lambda(x)}, \quad x > 0,$$

where $\lambda(x) \equiv \lambda(x;\theta)$ is a nonnegative, monotone increasing and differentiable function of x such that $\lambda(x) \to 0$ as $x \to 0^+$ and $\lambda(x) \to \infty$ as $x \to \infty$, $\lambda'(x)$ is the derivative of $\lambda(x)$ with respect to x and the parameter θ (may be a vector) belongs to some parameter space.

We shall write the class $\Im$ of distributions as

$$\Im = \left\{ F : F(x) = 1 - e^{-\lambda(x)}, \quad x > 0 \right\}. \tag{1.1}$$

A doubly truncated *pdf* on $[P_1, Q_1]$, denoted by $f_d(x)$, is given by

$$f_d(x) = A_d\, \lambda'(x)\, e^{-\lambda(x)}, \quad P_1 \le x \le Q_1, \quad (P_1 \ge 0, Q_1 \le \infty), \tag{1.2}$$

where

$$A_d = 1/\left[e^{-\lambda(P_1)} - e^{-\lambda(Q_1)} \right]. \tag{1.3}$$

The corresponding doubly truncated *df* and the survival function (sf) are given, respectively, for $0 \le P_1 \le x \le Q_1 \le \infty$, by

$$F_d(x) = A_d \left[e^{-\lambda(P_1)} - e^{-\lambda(x)} \right] \tag{1.4}$$

and

$$\bar{F}_d(x) = Q_2 + \frac{f_d(x)}{\lambda'(x)}, \tag{1.5}$$

where $\bar{F}_d(.) = 1 - F_d(.)$, $F_d(.)$ is given by (1.4) and

$$Q_2 = -A_d\, e^{-\lambda(Q_1)} = e^{-\lambda(Q_1)}/[e^{-\lambda(Q_1)} - e^{-\lambda(P_1)}]. \tag{1.6}$$

Notice that $\bar{F}_d(P_1) = 1$ and $\bar{F}_d(Q_1) = 0$.

We shall write $\Im_d$ to denote the doubly truncated class. So that, for $P_1 \leq x \leq Q_1$, $P_1 \geq 0, Q_1 \leq \infty$,

$$\Im_d = \left\{F_d : F_d(x) = \left[e^{-\lambda(P_1)} - e^{-\lambda(x)}\right] \Big/ \left[e^{-\lambda(P_1)} - e^{-\lambda(Q_1)}\right]\right\}. \tag{1.7}$$

Special cases of the doubly truncated class $\Im_d$ are the non-truncated, left and right truncated classes, denoted by $\Im$, $\Im_L$ and $\Im_R$, where the non-truncated class $\Im$ is given by (1.1), the left truncated class is given by

$$\Im_L = \left\{F_L : F_L(x) = 1 - e^{-[\lambda(x)-\lambda(P_1)]}, \quad x \geq P_1, \quad P_1 > 0\right\}, \tag{1.8}$$

in which case, it is only required for $\lambda(x)$ to satisfy the condition $\lambda(x) \to \infty$ as $x \to \infty$. The right truncated class $\Im_R$ is given by

$$\Im_R = \left\{F_R : F_R(x) = \left[1 - e^{-\lambda(x)}\right]\Big/\left[1 - e^{\lambda(Q_1)}\right],\ 0 \leq x \leq Q_1,\ Q_1 < \infty\right\}, \tag{1.9}$$

in which case, it is only required for $\lambda(x)$ to satisfy the condition $\lambda(x) \to 0$ as $x \to 0^+$.

AL-Hussaini, Ahmad and El-Boghdady (2004a, 2004b) have obtained recurrence relations of multivariate moment generating functions of powers of order statistics and records, respectively, based on random samples drawn from a population whose distribution is a member of the doubly truncated class of distributions $\Im_d$.

Members of $\Im_d$ include important distributions, used in areas as life testing and other areas of statistics as well, such as the doubly truncated distributions of each of the Weibull, Compound Weibull, Pareto, power function, Gompertz and compound Gompertz distributions. Recurrence relations obtained in this paper are applied to such members as illustrative examples.

2. RECURRENCE RELATIONS FOR CONDITIONAL MOMENT GENERATING FUNCTION OF ORDER STATISTICS

Suppose that $X_1, \ldots, X_n$ are independently identically distributed random variables as a random variable X having a *df* $F_d(x)$, $x \in [P_1, Q_1]$. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of $X_1, \ldots, X_n$. For integers r, s such that $1 \leq r < s \leq n$, the conditional density function of $X_{s:n}$ given $X_{r:n}$ is known to be given by

$$f_{X_{s:n}|X_{r:n}}(y \mid x) = A_1 \Big[\bar{F}_d(x) - \bar{F}_d(y)\Big]^{s-r-1} \Big[\bar{F}_d(y)\Big]^{n-s} f_d(y), \quad P_1 \leq x < y \leq Q_1, \tag{2.1}$$

where

$$A_1 = (n-r)! \,/\, \Big[(s-r-1)!\,(n-s)!\,\{\bar{F}_d(x)\}^{n-r}\Big]. \tag{2.2}$$

(See, for example David 1981).

The following theorem gives recurrence relations for the conditional moment generating function or conditional moments of order statistics.

Theorem 1. *The necessary and sufficient condition for a random variable X to be distributed as* (1.4)*, is that, for integers r, s and a such that $1 \leq r < s \leq n$ and $a \geq 1$,*

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$(2.3) \qquad = \frac{a\,t}{n-s+1}\, E\left[\frac{X^{a-1}_{s:n} e^{tX^a_{s:n}}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right] + \frac{(n-r)\,Q_2}{(n-s+1)\,\bar{F}_d(x)}$$

$$\Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big],$$

which implies that

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$(2.4) \qquad = \frac{a}{n-s+1}\, E\left[\frac{X^{a-1}_{s:n}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right] + \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)}$$

$$\Big\{E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\}.$$

It is assumed that all of the moment generating functions and conditional moments involved exist.

Proof.

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x)$$

$$= E\Big[e^{tX^a_{s:n}} \mid X_{r:n} = x\Big]$$

$$(2.5) \qquad = \int_x^{Q_1} e^{ty^a} f_{X_{s:n}|X_{r:n}}(y \mid x)\, dy$$

$$= A_1 \int_x^{Q_1} e^{ty^a}\Big[\bar{F}_d(x) - \bar{F}_d(y)\Big]^{s-r-1}\Big[\bar{F}_d(y)\Big]^{n-s} f_d(y)dy$$

$$= -A_2 \int_x^{Q_1} e^{ty^a}\Big[\bar{F}_d(x) - \bar{F}_d(y)\Big]^{s-r-1} d\Big[\bar{F}_d(y)\Big]^{n-s+1},$$

where

$$A_2 = \frac{A_1}{n-s+1} = \frac{(n-r)!}{(n-s+1)!\,(s-r-1)!\,\left[\bar{F}_d(x)\right]^{n-r}}. \tag{2.6}$$

Integrating by parts, we then have

$$\begin{aligned}
&M_{X^a_{s:n}|X_{r:n}}(t \mid x)\\
&= A_2 \int_x^{Q_1} \left[\bar{F}_d(y)\right]^{n-s+1} \Big\{a\,t\,y^{a-1}\,e^{ty^a}\left[\bar{F}_d(x)-\bar{F}_d(y)\right]^{s-r-1}\\
&+e^{ty^a}\,(s-r-1)\left[\bar{F}_d(x)-\bar{F}_d(y)\right]^{s-r-2} f_d(y)\Big\}\,dy \qquad (2.7)\\
&= a\,t\,A_2 \int_x^{Q_1} y^{a-1}e^{ty^a}\left[\bar{F}_d(x)-\bar{F}_d(y)\right]^{s-r-1}\left[\bar{F}_d(y)\right]^{n-s+1} dy\\
&+A_3 \int_x^{Q_1} e^{ty^a}\left[\bar{F}_d(x)-\bar{F}_d(y)\right]^{s-r-2}\left[\bar{F}_d(y)\right]^{n-s+1} f_d(y)\,dy,
\end{aligned}$$

where

$$A_3 = (s-r-1)\,A_2 = \frac{(n-r)!}{(n-s+1)!\,(s-r-2)!\,\left[\bar{F}_d(x)\right]^{n-r}}.$$

The second term in (2.7) is the same as (2.5) when s is replaced by $s-1$. Therefore, (2.7) can be written as

$$\begin{aligned}
&M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)\\
&= a\,t\,A_2 \int_x^{Q_1} y^{a-1}e^{ty^a}\left[\bar{F}_d(x)-\bar{F}_d(y)\right]^{s-r-1}\left[\bar{F}_d(y)\right]^{n-s+1} dy.
\end{aligned} \tag{2.8}$$

By using (1.5), we can write

$$\left[\bar{F}_d(y)\right]^{n-s+1} = \left[\bar{F}_d(y)\right]^{n-s}\left[\bar{F}_d(y)\right] = \left[\bar{F}_d(y)\right]^{n-s}\left[Q_2 + \frac{f_d(y)}{\lambda'(y)}\right].$$

By substituting in (2.8) with A_2 being written in terms of A_1 as in (2.6), we have

$$
\begin{aligned}
&M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)\\
&= \frac{a\ t\ A_1}{n-s+1}\int_x^{Q_1} \frac{y^{a-1}e^{ty^a}}{\lambda'(y)}\Big[\bar F_d(x)-\bar F_d(y)\Big]^{s-r-1}\Big[\bar F_d(y)\Big]^{n-s} f_d(y)\,dy\\
&+\frac{a\ t\ Q_2\ A_1}{n-s+1}\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar F_d(x)-\bar F_d(y)\Big]^{s-r-1}\Big[\bar F_d(y)\Big]^{n-s} dy\\
&= \frac{a\ t}{n-s+1}E\left[\frac{X^{a-1}_{s:n}e^{tX^a_{s:n}}}{\lambda'(X_{s:n})} \mid X_{r:n}=x\right]\\
&+\frac{a\ t\ Q_2\ A_1}{n-s+1}\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar F_d(x)-\bar F_d(y)\Big]^{s-r-1}\Big[\bar F_d(y)\Big]^{n-s} dy.
\end{aligned}
\tag{2.9}
$$

By replacing n by $n-1$, in (2.8), we obtain

$$
\begin{aligned}
&\int_x^{Q_1} y^{a-1}\ e^{ty^a}\Big[\bar F_d(x)-\bar F_d(y)\Big]^{s-r-1}\Big[\bar F_d(y)\Big]^{n-s} dy\\
&= \frac{(n-r)}{a\ t\ A_1\ \bar F_d(x)}\Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big].
\end{aligned}
\tag{2.10}
$$

Notice, from (2.6) and (2.2), that if n is replaced by $n-1$ in (2.6), then $A_2 = A_1\bar F_d(x)\ /\ (n-r)$ where A_1 is given by (2.2).

Substituting in (2.9), we obtain (2.3).

On the other hand, if (2.3) is satisfied, its left hand side is then given, from (2.8), by

$$
a\ t\ A_2\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar F_d(x)-\bar F_d(y)\Big]^{s-r-1}\Big[\bar F_d(y)\Big]^{n-s+1} dy.
\tag{2.11}
$$

The right hand side of (2.3) is given, from definition and the use of (2.10) and (2.9), by

$$\frac{a\ t\ A_1}{n-s+1}\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar{F}_d(x)-\bar{F}_d(y)\Big]^{s-r-1}\Big[\bar{F}_d(y)\Big]^{n-s}\left[\frac{f_d(y)}{\lambda'(y)}\right]dy$$

$$+\frac{a\ t\ Q_2\ A_1}{n-s+1}\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar{F}_d(x)-\bar{F}_d(y)\Big]^{s-r-1}\Big[\bar{F}_d(y)\Big]^{n-s}dy \tag{2.12}$$

$$=a\ t\ A_2\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar{F}_d(x)-\bar{F}_d(y)\Big]^{s-r-1}\Big[\bar{F}_d(y)\Big]^{n-s}\left[Q_2+\frac{f_d(y)}{\lambda'(y)}\right]dy.$$

By equating (2.11) and (2.12), we obtain

$$0=\int_x^{Q_1} y^{a-1}e^{ty^a}\Big[\bar{F}_d(x)-\bar{F}_d(y)\Big]^{s-r-1}\Big[\bar{F}_d(y)\Big]^{n-s}\left[\bar{F}_d(y)-Q_2-\frac{f_d(y)}{\lambda'(y)}\right]dy.$$

By applying the extension of Müntz-Szazás theorem [see, Hwang and Lin (1984)], it follows that

$$\bar{F}_d(y)=Q_2+\frac{f_d(y)}{\lambda'(y)}.$$

By differentiating both sides of (2.3) and then setting $t=0$, the recurrence relation (2.4) of conditional moments of order statistics is obtained.

2.1. Left, right and nontruncated cases

Special conditional doubly truncated cases are the conditional left, right and nontruncated distributions. Recurrence relations of moment generating functions and product moments of order statistics corresponding to each one of such cases characterize its members.

Corollary 1. *The necessary and sufficient condition for a random variable X to be distributed as a member of the left truncated class* (1.8) *is that, for integers r, s and a such that $1 \le r < s \le n$ and $a \ge 1$,*

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

(2.13)
$$= \frac{a\, t}{n-s+1} E\left[\frac{X^{a-1}_{s:n} e^{tX^a_{s:n}}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right],$$

which implies that

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

(2.14)
$$= \frac{a}{n-s+1}\, E\left[\frac{X^{a-1}_{s:n}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right].$$

Corollary 2. *The necessary and sufficient condition for a random variable X to be distributed as a member of the right truncated class* (1.9) *is that, for integers r, s and a such that $1 \leq r < s \leq n$ and $a \geq 1$,*

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x) = \frac{at}{n-s+1}$$

(2.15)
$$E\left[\frac{X^{a-1}_{s:n} e^{tX^a_{s:n}}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right] + \frac{(n-r)e^{-\lambda(Q_1)}}{[e^{-\lambda(Q_1)} - 1](n-s+1)\bar{F}_d(x)}$$

$$\Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big],$$

which implies that

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big] = \frac{a}{n-s+1}$$

(2.16)
$$E\left[\frac{X^{a-1}_{s:n}}{\lambda'(X_{s:n})} \mid X_{r:n} = x\right] + \frac{(n-r)e^{-\lambda(Q_1)}}{\big[e^{-\lambda(Q_1)} - 1\big](n-s+1)\bar{F}_d(x)}$$

$$\Big\{E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\}.$$

Remarks.

(1) In the non-truncated case $\Im$, the characterization condition is the same as (2.13).

(2) A referee has pointed out that relation (2.3) can also be shown to be a consequence of Eq. (2.9) obtained by Ahmad and Fawzy (2003).

2.2 Examples

(1) Doubly truncated Weibull distribution:

$\lambda(x) = \beta x^{\gamma}$ and $\lambda'(x) = \beta\gamma x^{\gamma-1}$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \frac{a\, t}{\beta\gamma\,(n-s+1)} E\Big[X^{a-\gamma}_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x\Big]$$

$$+\frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)} \Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$= \frac{a}{\beta\gamma\,(n-s+1)}\, E\Big[X^{a-\gamma}_{s:n} \mid X_{r:n} = x\Big]$$

$$+\frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)}\Big\{E\, \Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\},$$

where $\bar{F}_d(x) = \{exp[-(x^{\gamma} - Q_1^{\gamma})] - 1\}/\{exp[-\beta(P_1^{\gamma} - Q_1^{\gamma})] - 1\}$ and $a > \gamma$.

[Recurrence relations for product moments of order statistics under the doubly truncated exponential and doubly truncated Rayleigh distributions can be obtained from the Weibull distribution by setting $\gamma = 1$ and 2, respectively].

(2) Doubly truncated compound Weibull distribution (three - parameter Burr type XII distribution):

$\lambda(x) = \gamma\ \ln(1 + x^{\theta}/\beta)$ and $\lambda'(x) = \gamma\ \theta\ x^{\theta-1}/(\beta + x^{\theta})$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \frac{a\ t}{\gamma\ \theta\ (n-s+1)} \Big\{ \beta E\Big[X^{a-\theta}_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x\Big] + E\Big[X^a_{s:n} \exp^{tX^a_{s:n}} \mid X_{r:n} = x\Big]\Big\}$$

$$+ \frac{(n-r)\ Q_2}{(n-s+1)\ \bar{F}_d(x)} \Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$= \frac{a}{\gamma\ \theta\ (n-s+1)} \Big\{ \beta E\Big[X^{a-\theta}_{s:n} \mid X_{r:n} = x\Big] + E\Big[X^a_{s:n} \mid X_{r:n} = x\Big]\Big\}$$

$$+ \frac{(n-r)\ Q_2}{(n-s+1)\bar{F}_d(x)} \Big\{ E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\},$$

where $\bar{F}_d(x) = \{[(\beta + Q_1^{\theta})/(\beta + x^{\theta})]^{\alpha} - 1\}/\{[(\beta + Q_1^{\theta})/(\beta + P_1^{\theta})]^{\alpha} - 1\}$.

[Recurrence relations for product moments of order statistics under the doubly truncated compound exponential, doubly truncated compound Rayleigh and doubly truncated two-parameter Burr type XII distributions can be obtained from the compound Weibull distribution by setting $\alpha = 1, \alpha = 2$ and $\beta = 1$, respectively].

(3) Doubly truncated Pareto I distribution:

$\lambda(x) = -\gamma \ln(\alpha/x)$ and $\lambda'(x) = \gamma/x$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \frac{a\, t\, \beta}{\gamma(n-s+1)}\, E\Big[X^a_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x\Big]$$

$$+ \frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)} \Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$= \frac{a}{\gamma(n-s+1)}\, E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] + \frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)}$$

$$\Big\{E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\},$$

where $\bar{F}_d(x) = [(Q_1/x)^\gamma - 1]/[(Q_1/P_1)^\gamma - 1]$.

(4) Doubly truncated beta distribution:

$\lambda(x) = \beta \ln[1/(1-x)]$ and $\lambda'(x) = \beta/(1-x)$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \frac{a\,t}{\beta(n-s+1)} \Big\{ E\Big[X^{a-1}_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x\Big] - E\Big[X^a_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x\Big]\Big\}$$

$$+ \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)} \Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$= \frac{a}{\beta(n-s+1)} \Big\{ E\Big[X^{a-1}_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s:n} \mid X_{r:n} = x\Big]\Big\}$$

$$+ \frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)} \Big\{ E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\},$$

where $\bar{F}_d(x) = \{[(1-x)/(1-Q_1)]^\beta - 1\}/\{[(1-P_1)/(1-Q_1)]^\beta - 1\}$ and $a > 1$.

(5) Doubly truncated Gompertz distribution:

$\lambda(x) = (1/\sigma\gamma)[e^{\gamma x} - 1]$ and $\lambda'(x) = (1/\sigma)e^{\gamma x}$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \left(\frac{a\ t\ \sigma}{n-s+1}\right) E\Big[X^{a-1}_{s:n} e^{tX^a_{s:n} - \gamma X_{s:n}} \mid X_{r:n} = x\Big]$$

$$+\frac{(n-r)Q_2}{(n-s+1)\bar{F}_d(x)}\Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x)\Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x\Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x\Big]$$

$$= \frac{a\ \sigma}{n-s+1}\Big\{E\Big[X^{a-1}_{s:n} e^{-\gamma X_{s:n}} \mid X_{r:n} = x\Big] + E\Big[X^a_{s:n} \mid X_{r:n} = x\Big]\Big\}$$

$$+\frac{(n-r)\ Q_2}{(n-s+1)\ \bar{F}_d(x)}\Big\{E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x\Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x\Big]\Big\},$$

where $\bar{F}_d(x) = \{exp[-\frac{1}{\sigma\gamma}(e^{\gamma x} - e^{\gamma Q_1})] - 1\}/\{exp[-\frac{1}{\sigma\gamma}(e^{\gamma P_1} - e^{\gamma Q_1})] - 1\}$ and $a > 1$.

(6) Doubly truncated compound Gompertz distribution:

$\lambda(x) = \delta \ln[1 + (e^{\gamma x} - 1)/\beta\gamma]$ and $\lambda'(x) = \delta\gamma/[1 + (\beta\gamma - 1)e^{-\gamma x}]$.

Recurrence relations (2.3) and (2.4) reduce, respectively, to

$$M_{X^a_{s:n}|X_{r:n}}(t \mid x) - M_{X^a_{s-1:n}|X_{r:n}}(t \mid x)$$

$$= \frac{at}{\gamma\delta(n-s+1)} \Big\{ E\Big[X^{a-1}_{s:n} e^{tX^a_{s:n}} \mid X_{r:n} = x \Big]$$

$$+ (\beta\gamma - 1) E\Big[X^{a-1}_{s:n} e^{tX^a_{s:n} - \gamma X_{s:n}} \mid X_{r:n} = x \Big] \Big\}$$

$$+ \frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)} \Big[M_{X^a_{s:n-1}|X_{r:n-1}}(t \mid x) - M_{X^a_{s-1:n-1}|X_{r:n-1}}(t \mid x) \Big]$$

and

$$E\Big[X^a_{s:n} \mid X_{r:n} = x \Big] - E\Big[X^a_{s-1:n} \mid X_{r:n} = x \Big]$$

$$= \frac{a}{\gamma\delta(n-s+1)} \Big\{ E\Big[X^{a-1}_{s:n} \mid X_{r:n} = x \Big] + (\beta\gamma - 1) E\Big[X^{a-1}_{s:n} e^{-\gamma X_{s:n}} \mid X_{r:n} = x \Big] \Big\}$$

$$+ \frac{(n-r)\, Q_2}{(n-s+1)\, \bar{F}_d(x)} \Big\{ E\Big[X^a_{s:n-1} \mid X_{r:n-1} = x \Big] - E\Big[X^a_{s-1:n-1} \mid X_{r:n-1} = x \Big] \Big\},$$

where $\bar{F}_d(x) = \{[(\beta\gamma - 1 + e^{\gamma x})/(\beta\gamma - 1 + e^{\gamma Q_1})]^{-\delta} - 1\}/\{[(\beta\gamma - 1 + e^{\gamma P_1})/(\beta\gamma - 1 + e^{\gamma Q_1})]^{-\delta} - 1\}$ and $a > 1$.

3. RECURRENCE RELATION FOR CONDITIONAL MOMENT GENERATING FUNCTION OF RECORD VALUES

A different type of ordering is that of records. Suppose that $X_1, X_2, \dots$ is a sequence of i.i.d. random variables as a random variable X having a *df* $F_d(x)$. Let, for $n \geq 1, X_{U(n)} = max\{X_1, \dots, X_n\}$.

We say that $X_{U(n)}$ is an upper record value of $\{X_n, n \geq 1\}$, if $X_{U(j)} > X_{U(j-1)}$, for $j > 1$. The sequence $\{U(n), n \geq 1\}$ is called upper record times, where $U(1) = 1$ and $U(n) = min\{j : j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$. Lower record times and values are similarly defined. For details, see Arnold, Balakrishnan and Nagaraja (1998). In this book, it was shown that the conditional density function $f_{U(n)|U(m)}(y \mid x)$ is given by

$$f_{U(n)|U(m)}(y \mid x) = \frac{[R(y) - R(x)]^{n-m-1}}{(n-m-1)!} \frac{f(y)}{\bar{F}(x)}, \qquad y > x.$$

The conditional density function based on the doubly truncated distribution $\bar{F}_d(.)$ (and density $f_d(.)$) is then given by

$$f_{U(n)|U(m)}(y \mid x) = \frac{[R_d(y) - R_d(x)]^{n-m-1}}{(n-m-1)!} \frac{f_d(y)}{\bar{F}_d(x)}, \qquad P_1 \leq x < y \leq Q_1, \tag{3.1}$$

where

$$R_d(.) = -\ln[\bar{F}_d(.)]. \tag{3.2}$$

For a given record value, we may be interested in knowing what is expected in the next record. The following theorem gives recurrence relations for the conditional moment generating function or conditional moments of record values.

Theorem 2. *The necessary and sufficient condition for a random variable X to be distributed as* (1.4), *is that, for integers $1 \leq m < n$ and $a \geq 1$,*

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x)$$

$$= atE\left[\frac{X^{a-1}_{U(n+1)} e^{tX^a_{U(n+1)}}}{\lambda'(X_{U(n+1)})}\left\{1 - e^{[\lambda(X_{U(n+1)})-\lambda(Q_1)]}\right\} \mid X_{U(m)} = x\right], \tag{3.3}$$

which implies that

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= aE\left[\frac{X^{2a-1}_{U(n+1)}}{\lambda'(X_{U(n+1)})}\left\{1 - e^{[\lambda(X_{U(n+1)})-\lambda(Q_1)]}\right\} \mid X_{U(m)} = x\right]. \tag{3.4}$$

It is assumed that all of the conditional moment generating functions and conditional moments involved exist.

Proof. By definition,

$$M_{X^a_{U(n)}|X_{U(m)}}(t \mid x) = E\Big[e^{tX^a_{U(n)}} \mid X_{U(m)} = x\Big]$$

$$= \int_x^{Q_1} e^{ty^a} f_{X_{U(n)}|X_{U(m)}}(y \mid x)\, dy \tag{3.5}$$

$$= B \int_x^{Q_1} e^{ty^a} \Big[R_d(y) - R_d(x)\Big]^{n-m-1} f_d(y)\, dy,$$

where

$$B = 1 \;/\; \Big[(n-m-1)!\; \bar{F}_d(x)\Big], \qquad R_d(.) = -\ln\Big[\bar{F}_d(.)\Big], \tag{3.6}$$

and $\bar{F}_d(.)$, $f_d(.)$ are given by (1.4) and (1.1). Therefore

$$M_{X^a_{U(n)}|X_{U(m)}}(t \mid x) = B^\star \int_x^{Q_1} e^{ty^a}\ \bar{F}_d(y) d\Big[R_d(y) - R_d(x)\Big]^{n-m}$$

(3.7)
$$= -B^\star \int_x^{Q_1} \Big[R_d(y) - R_d(x)\Big]^{n-m} \Big\{e^{ty^a}\left[- f_d(y)\right] + a\, t\, y^{a-1} e^{ty^a} \bar{F}_d(y)\Big\} dy$$

$$= B^\star \int_x^{Q_1} e^{ty^a} \Big[R_d(y) - R_d(x)\Big]^{n-m} f_d(y) dy$$

$$-a\ t\ B^\star \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{\bar{F}_d(y)}{f_d(y)}\right] \Big[R_d(y) - R_d(x)\Big]^{n-m} f_d(y)\ dy,$$

where $B^\star = B\ /\ (n-m)! = 1/[(n-m)!\ \bar{F}_d(x)]$.

It may be observed that the first term in (3.7) is the same as (3.5) if n is replaced by $n-1$. In the second term of (3.7),

$$\frac{\bar{F}_d(y)}{f_d(y)} = \frac{1}{\lambda'(y)} \Big[1 - e^{[-\lambda(Q_1) - \lambda(y)]}\Big].$$

Therefore, (3.7) can be rewritten in the form

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x)$$

$$= a\ t\ E\left[\frac{X^{a-1}_{U(n+1)} e^{tX^a_{U(n+1)}}}{\lambda'(X_{U(n+1)})} \Big\{1 - e^{[\lambda(X_{U(n+1)}) - \lambda(Q_1)]}\Big\} \mid X_{U(m)}\right].$$

On the other hand, if condition (3.3) is satisfied, then its left hand side is given from (3.7) by

(3.8)
$$a\ t\ B^\star \int_x^{Q_1} y^{a-1} e^{ty^a} \left[\frac{\bar{F}_d(y)}{f_d(y)}\right] \Big[R_d(y) - R_d(x)\Big]^{n-m} f_d(y)\ dy.$$

The right hand side of condition (3.3) is given by

$$(3.9) \qquad a\,t\,B^{\star}\int_x^{Q_1} y^{a-1}e^{ty^a}\left[\frac{1-e^{-[\lambda(Q_1)-\lambda(y)]}}{\lambda'(y)}\right]\Big[R_d(y)\text{–}R_d(x)\Big]^{n-m} f_d(y)\,dy.$$

Equating (3.8) and (3.9) we then have

$$0=\int_x^{Q_1} y^{a-1}e^{ty^a}\left[\frac{\bar F_d(y)}{f_d(y)}-\frac{1-e^{-[\lambda(Q_1)-\lambda(y)]}}{\lambda'(y)}\right]\Big[R_d(y)-R_d(x)\Big]^{n-m} f_d(y)\,dy.$$

It then follows from the extension of Müntz-Sazás theorem [see, Hwang and Lin (1984)] that

$$\frac{\bar F_d(y)}{f_d(y)}=\frac{1-e^{-[\lambda(Q_1)-\lambda(y)]}}{\lambda'(y)}=\frac{e^{-\lambda(y)}-e^{-\lambda(Q_1)}}{\lambda'(y)\;e^{-\lambda(y)}},$$

which has a solution given by

$$\bar F_d(y)=A[e^{-\lambda(Q_1)}-e^{-\lambda(y)}],$$

so that

$$f_d(y)=A\;[\lambda'(y)\;e^{-\lambda(y)}],\qquad P_1\le y\le Q_1,$$

where A is such that $\bar F_d(y)$ is a survival function, or $f_d(y)$ is a *pdf*.

Differentiating both sides of (3.3) with respect to t and then setting $t=0$, recurrence relation (3.4), for conditional moments, is obtained.

Remark. A referee has pointed out that relation (3.3) can also be shown to be a consequence of Eq. (2.7) obtained by Ahmad and Fawzy (2003).

3.1. Left, right and nontruncated cases

In the left truncated or nontruncated cases, conditions (3.3) and (3.4) become, for integers $n<m$ and $a\ge 1$,

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x)$$

$$= a\ t\ E\left[\frac{X^{a-1}_{U(n+1)} e^{tX^a_{U(n+1)}}}{\lambda'(X_{U(n+1)})} \mid X_{U(m)}\right],$$

and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= a\ E\left[\frac{X^{2a-1}_{U(n+1)}}{\lambda'(X_{U(n+1)})} \mid X_{U(m)}\right],$$

In the left truncated case, $x \geq P_1$, $(P_1 > 0)$ and $\lambda(x) \to \infty$ as $x \to \infty$.

In the non-truncated case, $x > 0$, $(P_1 > 0)$ and $\lambda(x) \to 0$ as $x \to 0^+$ and $\lambda(x) \to \infty$ as $x \to \infty$.

In the right truncated case. conditions (3.3) and (3.4) remain the same, provided that $0 \leq x \leq Q_1$, $Q_1 < \infty$ and $\lambda(x) \to 0$ as $x \to 0^+$.

3.2 Examples

(1) Doubly truncated Weibull distribution:

$\lambda(x) = \beta x^\gamma$ and $\lambda'(x) = \beta\gamma x^{\gamma-1}$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{r:n}}(t \mid x)$$

$$= \frac{a\ t}{\beta\gamma} E\Big[X^{a-\gamma}_{U(n+1)} \exp^{tX^a_{U(n+1)}} \Big(1 - e^{\beta(x^\gamma - Q_1^\gamma)}\Big) \mid X_{U(m)} = x\Big],$$

where $a > \gamma$ and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(n)} = x\Big]$$

$$= \frac{a}{\beta\gamma} E\Big[X^{2a-\gamma}_{U(n+1)}\Big(1 - e^{\beta(x^\gamma - Q_1^\gamma)}\Big) \mid X_{U(m)} = x\Big],$$

where $2a > \gamma$.

[Recurrence relations for product moments of record values under the doubly truncated exponential and doubly truncated Rayleigh distributions can be obtained from the Weibull distribution by setting $\alpha = 1$ and 2, respectively].

(2) Doubly truncated compound Weibull distribution (three - parameter Burr type XII distribution):

$\lambda(x) = \gamma\ \ln(1 + x^\theta/\beta)$ and $\lambda'(x) = \gamma\ \theta\ x^{\theta-1}/(\beta + x^\theta)$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x)$$

$$= \frac{a\ t}{\gamma\theta} E\Big[\Big(\beta X^{a-\theta}_{U(n+1)} + X^a_{U(n+1)}\Big) e^{tX^a_{U(n+1)}} \Big[1 - \Big(\frac{\beta + X^\theta_{U(n+1)}}{\beta + Q_1^\theta}\Big)^\gamma\Big] \mid X_{r:n} = x\Big],$$

where $a \geq \theta$ and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= \frac{a}{\gamma\theta} E\Big[\Big(\beta X^{2a-\theta}_{U(n+1)} + X^{2a}_{U(n+1)}\Big)\Big[1 - \Big(\frac{\beta + X^\theta_{U(n+1)}}{\beta + Q_1^\theta}\Big)^\gamma\Big] \mid X_{r:n} = x\Big],$$

where $2a \geq \theta$.

[Recurrence relations for product moments of record values under the doubly truncated compound exponential, doubly truncated compound Rayleigh and doubly truncated Burr type XII dstributions can be obtained from the compound Weibull distribution by setting $\gamma = 1, \gamma = 2$ and $\beta = 1$, respectively].

(3) Doubly truncated Pareto distribution:

$\lambda(x) = -\gamma \ln(\alpha/x)$ and $\lambda'(x) = \gamma/x$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x)$$

$$= \frac{a\,t}{\gamma} \left\{ E\Big[X^a_{U(n+1)} e^{tX^a_{U(n+1)}} \left(1 - \left(\frac{X_{U(n+1)}}{Q_1}\right)^{\gamma}\right) \mid X_{U(m)} = x \Big] \right\}$$

and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= \frac{a}{\gamma} \left\{ E\Big[X^a_{U(n+1)} \left(1 - \left(\frac{X_{U(n+1)}}{Q_1}\right)^{\gamma}\right) \mid X_{U(m)} = x \Big] \right\}.$$

(4) Doubly truncated beta distribution:

$\lambda(x) = \beta \ln[1/(1-x)]$ and $\lambda'(x) = \beta/(1-x)$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x)$$

$$= \frac{at}{\beta} E\Big[\Big(X^{a-1}_{U(n+1)} + X^a_{U(n+1)}\Big) e^{tX^a_{U(n+1)}} \Big(1 - \Big(\frac{1-Q_1}{1-X_{U(n+1)}}\Big)^{\beta}\Big) \mid X_{U(m)} = x\Big]$$

and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= \frac{a}{\beta} E\left[\left(X^{a-1}_{U(n+1)} + X^a_{U(n+1)}\right)\left(1 - \left(\frac{1-Q_1}{1-X_{U(n+1)}}\right)^{\beta}\right) \mid X_{U(m)} = x\right],$$

where $a > 1$.

(5) Doubly truncated Gompertz distribution:

$\lambda(x) = (1/\sigma\gamma)[e^{\gamma x} - 1]$ and $\lambda'(x) = (1/\sigma)e^{\gamma x}$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n)}|X_{U(m)}}(t \mid x) = a\ t\ \sigma$$

$$\times E\Big[X^{a-1}_{U(n+1)} e^{tX^a_{U(n+1)} - \gamma X_{U(n+1)}}$$

$$\times \left\{1 - exp\left(\frac{1}{\sigma\ \gamma}\Big(e^{\gamma X_{U(n+1)}} - e^{\gamma Q_1}\Big)\right)\right\} \mid X_{U(m)} = x\Big]$$

and

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= a\ \sigma\ E\Big[X^{a-1}_{U(n+1)} e^{-\gamma X_{U(n+1)}}$$

$$\times \left\{1 - exp\left(\frac{1}{\sigma\ \gamma}(e^{\gamma X_{U(n+1)}} - e^{\gamma Q_1})\right)\right\} \mid X_{U(m)} = x\Big],$$

where $a > 1$.

(6) Doubly truncated compound Gompertz distribution:

$\lambda(x) = \delta \ln[1 + (e^{\gamma x} - 1)/\beta\gamma]$ and $\lambda'(x) = \delta\gamma/[1 + (\beta\gamma - 1)e^{-\gamma x}]$.

Recurrence relations (3.3) and (3.4) reduce, respectively, to

$$M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x) - M_{X^a_{U(n+1)}|X_{U(m)}}(t \mid x)$$

$$= \frac{at}{\gamma\delta} E\Big[X^{a-1}_{U(n+1)}\, e^{tX^a_{U(n+1)}} \Big(1 + \Big(\beta\gamma - 1\Big) e^{-\gamma X_{U(n+1)}}\Big)$$

$$\times \left[1 - \left(\frac{\beta\gamma + e^{\gamma X_{U(n+1)}} - 1}{\beta\gamma + e^{\gamma Q_1} - 1}\right)^{\delta}\right] \mid X_{U(m)} = x\Big]$$

and, for $a > 1$,

$$E\Big[X^a_{U(n+1)} \mid X_{U(m)} = x\Big] - E\Big[X^a_{U(n)} \mid X_{U(m)} = x\Big]$$

$$= \frac{a}{\gamma\delta} E\Big[X^{a-1}_{U(n+1)} \Big(1 + \Big(\beta\gamma - 1\Big) e^{-\gamma X_{U(n+1)}}\Big)$$

$$\times \left[1 - \left(\frac{\beta\gamma + e^{\gamma X_{U(n+1)}} - 1}{\beta\gamma + e^{\gamma Q_1} - 1}\right)^{\delta}\right] \mid X_{U(m)} = x\Big].$$

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