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# BAYESIAN ESTIMATION OF AR(1) MODELS WITH UNIFORM INNOVATIONS

Hocine Fellag and Karima Nouali

*Department of Mathematics,*
*Faculty of Sciences*
*University of Tizi Ouzou*
*Tizi-Ouzou 15000, Algeria*

**Abstract**

The first-order autoregressive model with uniform innovations is considered. In this paper, we propose a family of BAYES estimators based on a class of prior distributions. We obtain estimators of the parameter which perform better than the maximum likelihood estimator.

**Keywords:** autoregressive model, Bayesian estimator, prior distribution, uniform distribution.

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## 1. Introduction

Let us consider the following autoregressive model

$$X_t = \rho X_{t-1} + \varepsilon_t, \qquad t = \ldots, -1, 0, 1, \ldots, \tag{1.1}$$

where $0 < \rho < 1$ and the $\varepsilon_t$'s are $i.i.d$ and distributed according to uniform distribution $U(0,1)$. $X_1$ is assumed to be distributed according to $U(0, 1/(1-\rho))$ such that the process is mean stationary. The likelihood function based on the observations $x = (x_1, x_2, \ldots, x_n)$ is then

$$p(x|\rho) = (1-\rho)\ I_A(x),$$

where $A = \{x : 0 \leq x_1 \leq 1/(1-\rho), 0 \leq x_t - \rho x_{t-1} \leq 1\ ,\ t = 2, \ldots, n\}$.

Let $\rho_1$ the maximum likelihood estimator of $\rho$ introduced by Bell and Smith (1986):

$$\rho_1 = min\,(1, x_2/x_1, \ldots, x_n/x_{n-1})\,.$$

The problem of bayesian analysis of AR(1) models is studied by Turkmann (1990) and Ibazizen and Fellag (2003) when the errors are exponential.

In this paper, we propose a Bayesian estimator of the parameter of the model (1.1) using a family of prior distributions for the parameter $\rho$ proposed by Ibazizen and Fellag (2003). The estimators obtained with this method under quadratic loss structure appear to be closer to the parameter than the usual maximum likelihood estimator $\rho_1$.

## 2. Bayesian estimation of the ar(1) parameter

Consider the following family of prior distributions for the parameter $\rho$

$$p(\rho;\beta) \propto \frac{\rho^{\beta-1}}{1-\rho}\ I_{(0,1)}(\rho), \quad \beta > 0\ . \tag{2.1}$$

Suppose that our data consists of the segment of the observations $x = (x_1, x_2, \ldots, x_n)$. Then, the prior assessment on $\rho$ is transformed via BAYES theorem into

$$p(\rho|x) \propto p(x|\rho)\ \ .\ p(\rho;\beta)$$

and then,

$$p(\rho|x) = C\ .\ \rho^{\beta-1} I_{(\rho_0,\rho_1)}(\rho)$$

with $\rho_1$ given above,

$$\rho_0 = \max\left(0, \frac{x_1-1}{x_1}, \frac{x_2-1}{x_1}, \ldots, \frac{x_n-1}{x_{n-1}}\right) \quad \text{and} \quad C = \frac{\beta}{\rho_1^\beta - \rho_0^\beta}\ .$$

Under quadratic loss structure, the BAYES estimator of $\rho$ is the posterior mean and is given by the formula

$$\hat{\rho}_B(\beta) = \int_{\rho_0}^{\rho_1} \rho \, p(\rho|x) d\rho \, .$$

This leads to

$$\hat{\rho}_B(\beta) = \frac{\beta}{\beta+1} \; \frac{\rho_1^{\beta+1} - \rho_0^{\beta+1}}{\rho_1^{\beta} - \rho_0^{\beta}} \quad , \quad \beta > 0 \, . \tag{2.2}$$

The posterior variance is

$$\sigma_B^2(\beta) = E\left[(\rho - \hat{\rho}_B(\beta))^2 | \, x\right] = \frac{\beta}{\beta+2} \; \frac{\rho_1^{\beta+2} - \rho_0^{\beta+2}}{\rho_1^{\beta} - \rho_0^{\beta}} - \hat{\rho}_B(\beta)^2 \, .$$

In order to illustrate our formulas, consider one segment of 20 observations from the model (1.1) simulated with the true value $\rho = 0.4$ . We found

$$\rho_0 = 0.3685 \quad \text{and} \quad \rho_1 = 0.4193$$

Simulated values of the Bayesian estimator and the posterior variance (given in parentheses) of $\rho$ for different values of $\beta$ are given in the following table

Table 1. Simulated values of the Bayesian estimator and posterior variance of $\rho$ for n=20 and $\rho = 0.4$

| $\beta$ | 0.2 | 0.6 | 1.0 | 2.0 | 5.0 |
|---|---|---|---|---|---|
| $\hat{\rho}_B(\beta)$ | 0.3934 | 0.3936 | 0.3939 | 0.3944 | 0.3960 |
| $\|\hat{\rho}_B - \rho\|$ | 0.00653725 | 0.0063186 | 0.0061 | 0.00555404 | 0.00392878 |
| $\sigma_B^2(\beta)$ | 0.000215034 | 0.000215072 | 0.000215053 | 0.000214755 | 0.000211759 |

Since $|\rho_1 - \rho| = 0.0193$, one can note that, for every $\beta$, the estimator $\hat{\rho}_B(\beta)$ is closer than $\rho_1$ to the parameter $\rho$. Also, the value of $\hat{\rho}_B$ tends to the true value when $\beta$ grows . The posterior variance is near zero for every $\beta$ and changes slightly when $\beta$ increases.

## 3. Simulation study

Consider the following exhaustive simulation study. We simulate samples from the model (1.1) for $\beta = 0.5, 1.0, 1.5, 2.0$ and for $n = 10, 20$. The value of $\hat{\rho}_B(\beta)$ and its posterior variance are calculated. The computations are based on 100000 replications of the process. The results are given in Table 2.

Table 2. Simulated values $\hat{\rho}_B(\beta)$ and its variance for $\rho$=0.1,0.4,0.9, $n$=10,20 and $\beta$=0.5,1.0,1.5,2.0 . Variances are given in parentheses

| $n$ | $\rho$ | $\rho_1$ | $\beta = 0.5$ | $\beta = 0.8$ | $\beta = 1.0$ | $\beta = 1.5$ | $\beta = 2.0$ |
|---|---|---|---|---|---|---|---|
| | 0.1 | 0.1052 | 0.0988 | 0.1030 | 0.1134 | 0.1237 | 0.1315 |
| | | (0.0060) | (0.0018) | (0.0185) | (0.0019) | (0.0022) | (0.0026) |
| 10 | 0.4 | 0.4211 | 0.3954 | 0.3960 | 0.3978 | 0.4000 | 0.4022 |
| | | (0.0024) | (0.0017) | (0.0017) | (0.0016) | (0.0014) | (0.0014) |
| | 0.9 | 0.9101 | 0.8999 | 0.8999 | 0.8999 | 0.9000 | 0.9000 |
| | | $(1.08\ 10^{-4})$ | $(0.5525\ 10^{-4})$ | $(0.5522\ 10^{-4})$ | $(0.5521\ 10^{-4})$ | $(0.5517\ 10^{-4})$ | $(0.5514\ 10^{-4})$ |
| | 0.1 | 0.1833 | 0.0981 | 0.1087 | 0.1127 | 0.1228 | 0.1305 |
| | | (0.0051) | (0.0017) | (0.0017) | (0.0016) | (0.0019) | (0.0022) |
| 20 | 0.4 | 0.4540 | 0.3924 | 0.3938 | 0.3949 | 0.3972 | 0.3995 |
| | | (0.0023) | (0.0018) | (0.0017) | (0.0016) | (0.0015) | (0.0014) |
| | 0.9 | 0.9100 | 0.8992 | 0.8992 | 0.8992 | 0.8993 | 0.8993 |
| | | $(1.0434\ 10^{-4})$ | $(0.7808\ 10^{-4})$ | $(0.778\ 10^{-4})$ | $(0.7689.10^{-4})$ | $(0.7574\ 10^{-4})$ | $(0.7464\ 10^{-4})$ |

We can remark that the Bayesian estimator $\hat{\rho}_B(\beta)$ has smaller standard deviation than the maximum likelihood estimator $\rho_1$. In our exhaustive simulation, we remark that, when $n$ is not very small, the Bayesian estimator $\hat{\rho}_B(\beta)$ is better than the maximum likelihood estimator of $\rho$ for every $\beta$. Also, this is true if $n$ is too small (e.g; $n = 10$) and $\rho$ not near zero. However, when $\rho$ is near zero, $\hat{\rho}_B(\beta)$ is the best only if $0 < \beta < 1$. So our conclusion can be as follows : if we choose a prior distribution given by the formula (2.1) with $0 < \beta < 1$, then the Bayesian estimator obtained under quadratic loss structure performs better than the maximum likelihood estimator.

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