Problems Column

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ESTIMATION OF CUT-VERTICES IN EDGE-COLOURED COMPLETE GRAPHS

Adam Idzik

Akademia Świętokrzyska
15 Świętokrzyska street, 25–406 Kielce, Poland
and
Institute of Computer Science
Polish Academy of Sciences
21 Ordona street, 01–237 Warsaw, Poland

e-mail: adidzik@ipipan.waw.pl

All graphs considered here are finite simple graphs, i.e., graphs without loops, multiple edges or directed edges. For a graph $G = (V, E)$, where V is a vertex set and E is an edge set, we write sometimes $V(G)$ for V and $E(G)$ for E to avoid ambiguity. We shall write $G \setminus v$ instead of $G_{V\setminus\{v\}} = (V \setminus \{v\}, E \cap 2^{V\setminus\{v\}})$, the subgraph induced by $V \setminus \{v\}$. A vertex $v \in V(G)$ is called a *cut-vertex* of G if G is connected and $G \setminus v$ is not. By a *k-edge-colouring* of a graph we mean any finite partition of the set of its edges into k subsets. A graph (V, E) with a given k-edge-colouring $(E^1, \cdots, E^k)$ $(E^i \cap E^j = \emptyset$ for $i \neq j$; $i, j \in \{1, \cdots, k\}$ and $\bigcup_{i\in\{1,\cdots,k\}} E^i = E)$ is denoted by $(V, E^1, \cdots, E^k)$. The graphs (V, E^i) are called monochromatic subgraphs of $(V, E^1, \cdots, E^k)$, $i \in \{1, \cdots, k\}$. As usual, by K_m we denote the complete graph with m vertices.

Let $\mathrm{c}(G^i)$ denote the number of cut-vertices of G^i in a monochromatic subgraph $G^i = (V, E^i)$ of a k-edge-coloured complete graph $K_m = (V, E^1, \cdots, E^k)$ $(i \in \{1, \cdots, k\})$.

Given a k-edge-coloured graph $G = (V, E^1, \cdots, E^k)$, we define $F^i = E \setminus E^i$, $G^i = (V, E^i)$, $\bar{G}^i = (V, F^i)$, where $E = \bigcup_{i\in\{1,\cdots,k\}} E^i$ and $i \in \{1, \cdots, k\}$. Here G^i is a monochromatic subgraph of G and $\bar{G}^i$ its complement in G.

Theorem (Idzik, Tuza, Zhu)**.** *Let* $(E^1, \cdots, E^k)$ *be a* k*-edge-colouring of* K_m $(k \geq 2,\ m \geq 4)$*, such that all the graphs* $\bar{G}^1, \cdots, \bar{G}^k$ *are connected.*

(i) *If one of the subgraphs* $G^1, \cdots, G^k$ *is* 2*-connected, say* G^i*, then* $c(\bar{G}^i) \leq m-2$ *and* $c(\bar{G}^j) = 0$ *for* $j \neq i$ $(i, j \in \{1, \ldots, k\})$*.*

(ii) *If none of the graphs* $G^1, \cdots, G^k$ *is* 2*-connected, and one of them is connected, say* G^i*, then* $c(\bar{G}^i) \leq 2$ $(i \in \{1, \cdots, k\})$*.*

(iii) *If none of the graphs* $G^1, \cdots, G^k$ *is* 2*-connected, and one of them is disconnected, say* G^i*, then* $c(\bar{G}^i) \leq 1$ $(i \in \{1, \cdots, k\})$*.*

Problem. Let $(E^1, \cdots, E^k)$ be a k-edge-colouring of K_m $(k \geq 2,\ m \geq 4)$. What is the cardinality of the set of the sum of cut-vertices of $\bar{G}^i$ in the case none of G^i is 2-connected and (a) two of G^i are connected or (b) two of G^i are disconnected and $c(\bar{G}^i) = 1$ $(i \in \{1, \cdots, k\})$?

Observe that in both cases (a) and (b) all the graphs $\bar{G}^1, \cdots, \bar{G}^k$ are connected.

This problem is related to some theorems presented in [1] and [2].

References

[1] J. Bosák, A. Rosa and Š. Znám, *On decompositions of complete graphs into factors with given diameters*, in: P. Erdős and G. Katona, eds., Theory of Graphs, Proceedings of the Colloquium Held at Tihany, Hungary (Academic Press, New York, 1968) 37–56.

[2] A. Idzik and Z. Tuza, *Heredity properties of connectedness in edge-coloured complete graphs*, Discrete Math. **235** (2001) 301–306.

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