Discussiones Mathematicae
Graph Theory 25 (2005) 167–182

ARITHMETICALLY MAXIMAL INDEPENDENT SETS IN INFINITE GRAPHS

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Abstract

Families of all sets of independent vertices in graphs are investigated. The problem how to characterize those infinite graphs which have arithmetically maximal independent sets is posed. A positive answer is given to the following classes of infinite graphs: bipartite graphs, line graphs and graphs having locally infinite clique-cover of vertices. Some counter examples are presented.

Keywords: infinite graph, independent set, arithmetical maximal set, line graph.

2000 Mathematics Subject Classification: 05C69, 05C65, 05D05.

1. Introduction and Preliminaries

For a set X, the cardinality of X and the family of all subsets of X are denoted by $|X|$ and 2^X, respectively. For a family $\mathcal{F}$ of sets, let $S \subset \bigcup \mathcal{F}$ be a set. S is called *scattered* (or strong *independent*) for $\mathcal{F}$ if no two elements in S belong to the same set from $\mathcal{F}$. In the literature, see [3], "independent" for hypergraphs is considered with respect to the property "there is no $F \in \mathcal{F}$ such that $F \subset S$". We have

$$|S \cap F| \leq 1 \quad \text{for every} \quad F \in \mathcal{F}.$$

S is a *covering* of $\mathcal{F}$ if every set in $\mathcal{F}$ has an element in S, i.e., for every $F \in \mathcal{F}$ we have

$$|S \cap F| \geq 1.$$

We say that S is a *König set* of $\mathcal{F}$ if S is scattered for $\mathcal{F}$ and there exists a *choice function* f, i.e., $f : S \to \mathcal{F}$ such that $v \in f(v)$ for every $v \in S$ and

$$\bigcup_{v \in S} f(v) = \bigcup \mathcal{F}.$$

Here and subsequently, we use the following notation:

- $s\mathcal{F}$ is the family of all scattered sets for $\mathcal{F}$.
- $k\mathcal{F}$ is the family of all König sets of $\mathcal{F}$.

Let $G = (V, E)$ be a finite or infinite graph with vertices V and edges E. Let us remark that $E \subset 2^V$ is a 2-element family of vertex sets of G.

A graph is said to be *countable* if its set of vertices is countable. The *complementary* graph of G will be denoted by $\bar{G} = (V, \bar{E})$, where

$$\bar{E} = \{\{u, v\} \in 2^V \mid \{u, v\} \notin E\}.$$

A set $W \subset V$ is a *clique* of G if the induced subgraph $G[W]$ is a complete graph. A set $W \subset V$ is an *independent set* (or a *set of independent vertices*) in G if $G[W]$ has no edges. We will denote

- cG for the family of all cliques of G,
- iG for the family of all independent sets in G.

Both those families of sets are hereditary with respect to the inclusion. The family of König sets of G is defined by the requirement that it be kcG.

A set $\mathcal{F} \subset cG$ is a *clique — cover* of edges (of vertices) of G if for every $e \in E$ $(v \in V)$ there exists $W \in \mathcal{F}$ such that $e \subset W$ $(v \in W)$. Of course, both families E and cG are clique — covers of edges of G. We have

$$iG = sE = s\mathcal{F} \quad \text{for every clique — cover } \mathcal{F} \text{ of edges of } G.$$

For a family $\mathcal{F}$ of sets, we define the *star* of an element $v \in \bigcup \mathcal{F}$ as the subfamily of all sets of $\mathcal{F}$ having v as an element, with the notation:

$$St_{\mathcal{F}}(v) = \{F \in \mathcal{F} \mid v \in F\} \quad \text{and} \quad St_{\mathcal{F}}(F) = \bigcup \{St_{\mathcal{F}}(v) \mid v \in F\}.$$

The *star* of a vertex $v \in V$ in G is defined as the star v in the set of edges of G. The *neighbours* of a vertex $v \in V$ in G is the set of all vertices of G adjacent to v, with the notation:

$$St_G(v) = St_E(v) \quad \text{and} \quad Nb_G(v) = \{u \in V \mid \{u, v\} \in E\}$$

and

$$Nb_G(W) = \{u \in V \setminus W \mid \{u, v\} \in E \ \text{ for some } \ v \in W\}.$$

We assume, without lost of generality, that considered graphs are connected.

2. Arithmetically Maximal Sets

The paper deals with a special kind of maximality which we call arithmetical maximality. For a family of sets $\mathcal{F} \subset 2^X$ which consists of finite sets only, a set $A \in \mathcal{F}$ of maximal cardinality is called an arithmetically maximal set in the family. This notion is generalized on arbitrary families.

Definition 2.1. Let $\mathcal{F}$ be a family of sets. A set $A \in \mathcal{F}$ is an *arithmetically maximal set* (a.m.s. for short) in $\mathcal{F}$ if the following implication holds:

$$\text{if } F \in \mathcal{F} \text{ and } A \setminus F \text{ is finite, then } |A \setminus F| \geq |F \setminus A|.$$

In other words, see Komar and Łoś [5], $A \in \mathcal{F}$ is a.m.s. in $\mathcal{F}$ iff for every finite set B included in A and every set C satisfying $C \cap A = \emptyset$, the following implication holds:

$$\text{if } (A \setminus B) \cup C \in \mathcal{F}, \text{ then } |B| \geq |C|. \tag{1}$$

Of course, such A is maximal in $\mathcal{F}$ (with respect to the inclusion). We denote:

- $m\mathcal{F}$ is the family of all maximal sets in $\mathcal{F}$,
- $am\mathcal{F}$ is the family of all a.m.s. in $\mathcal{F}$.

Hence we have

$$am\mathcal{F} \subset m\mathcal{F}$$

and

$$k\mathcal{F} \subset ams\mathcal{F}. \tag{2}$$

We will consider the behavior of the family of all independent sets in a graph. An a.m.s. in the family iG is said to be *arithmetically maximal independent set* (a.m.i.s.) in G. The structures of a.m.i. sets in finite graphs where studied in [9] and [4]. It is worth to mention, that the family of all finite graphs having a König set (defined as $\{G \mid kcG \neq \emptyset\}$) is not hereditary with respect to induced subgraphs.

Example 2.1. Let us denote by

$$I_n = \left\{ \frac{n(n-1)}{2} + 1, \ldots, \frac{n(n-1)}{2} + n \right\}, \quad \text{for } n = 1, 2, \ldots$$

and

$$E_n = \{\{i,j\} \mid i,j \in I_n,\ i \neq j\} \cup \{\{\max I_n, \max I_n + 1\}\}, \quad \text{for } n = 1, 2, \ldots.$$

Define $G = (V, E)$, where V is the set of all positive integers and

$$E = E_1 \cup E_2 \cup \ldots.$$

Every set $S = \{i_1, i_2, \ldots\}$ such that $i_n \in I_n$ and $i_{n+1} \neq i_n + 1$ for every $n = 1, 2, \ldots$ is both König and a.m.i.s. in G. Observe that $S \in kcG$ but for the family E we have $kE = \emptyset$.

It is easy to check that for the complement of G there is no arithmetically maximal independent set, i.e., $ami\tilde{G} = \emptyset$.

3. Independent Sets of n-partite and Matrix Graphs

We say a graph $G = (V, E)$ is *n-partite* if G admits a partition $V = V_1 \cup \ldots \cup V_n$ of its vertex set, such that $V_k \in iG$ for every $k = 1, \ldots, n$.

A *matching* in $G = (V, E)$ is a set $M \subset E$ satisfying:

$$e_1 \cap e_2 = \emptyset \ \text{ for all } e_1, e_2 \in M, \text{ such that } e_1 \neq e_2.$$

The line graph $L(G)$ of a graph G has vertices corresponding to the edges of G such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. G is *a line graph* if it is isomorphic to $L(H)$ of a graph H.

It is easy to see that for line graphs we have

$$cL(G) = \{St_G(v) \mid v \in V\}$$

and

$$M \text{ is a matching in } G \text{ if and only if } M \in iL(G).$$

A graph is a *matrix graph* if it is isomorphic to the line graph of a bipartite graph.

Theorem 3.1 (König duality theorem, 1936)**.** *For any finite bipartite graph $G = (V, E)$ there exists a pair (C, M) (called König covering of G) such that C is a covering of E, M is a matching in G, and C consists of exactly one vertex from every edge of M.*

For every graph G, if C is a covering of E and M is a matching in G, then

$$|C| \geq |M|.$$

Clearly if (C, M) is a König covering of G, then $|C| = |M|$ and $M \in iL(G)$. Additionally,

$$f(e) = St_G(e \cap C) \quad \text{for} \quad e \in M$$

is the suitable choice function $f : M \to cL(G)$. Therefore, M is a König set of $L(G)$. Therefore by (2), we obtain the following:

Corollary 3.2. *For any finite bipartite graph $G = (V, E)$, if a pair (C, M) is a König covering of G, then $V \setminus C$ is an a.m.i.s. in G (in other words, a.m.s. in iG) and M is an a.m.s. in $iL(G)$.*

For infinite graphs we can find in [5], the following statement:

(3) $\quad kcG = amiG$ for every countable matrix graph G.

Therefore, for countable matrix graphs, the existence of an a.m.i.s. is equivalent to the existence of a König covering.

Podewski and Steffens [7, 8] showed that every countable infinite bipartite graph has a König covering. Aharoni [1] showed that every uncountable bipartite graph has a König covering.

Theorem 3.3. *Let G be a graph.*

(i) *If G is a matrix graph, then G has an arithmetically maximal independent set;*

(ii) *If G is a bipartite graph, then G has an a.m.i.s. (i.e., $amiG \neq \emptyset$).*

Proof. We refer to the Podewski-Steffens theorem (respectively Aharoni's theorem) as the König duality theorem for countable (respectively uncountable) bipartite graphs.

By the same arguments as for Corollary 3.2, from (3) follows (i).

Let (C, M) be a König covering of $G = (V, E)$ and we set $S = V \setminus C$. Then $S \in iG$ and every edge of G has a vertex in C. From (2) follows that S is a.m.s. in iG. ■

Problem. Two questions with respect to possible generalizations of Theorem 3.3 are natural. Is there an a.m.i.s. in any n-partite graph as well as in any line graph?

The first question has a negative answer for 3-partite countable graphs, because of the following example:

Example 3.4. Let $G = (V, E)$, where V is the sum of three disjoint sets, $V = A \cup B \cup C$, with

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}, \quad C = \{c_1, c_2, \ldots\},$$

and $E = E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{\{a_i, b_j\} \mid j \geq 2i\},$$
$$E_2 = \{\{b_i, c_j\} \mid j \geq 2i\},$$
$$E_3 = \{\{c_i, a_j\} \mid j \geq 2i\}.$$

Observation 1. Assume $S \in iG$ (i.e., S is an independent set of vertices in G).

1. If $|S \cap A| = \infty$ then $S \cap B$ is a finite set and $S \cap C = \emptyset$.
2. If $|S \cap B| = \infty$ then $S \cap C$ is a finite set and $S \cap A = \emptyset$.
3. If $|S \cap C| = \infty$ then $S \cap A$ is a finite set and $S \cap B = \emptyset$.

Observation 2. All sets A, B, C as well as the sets

$$B_n = \begin{cases} \{b_1, \ldots, b_n, a_{\frac{n+1}{2}}, a_{\frac{n+1}{2}+1}, \ldots\} & \text{for odd } n, \\ \{b_1, \ldots, b_n, a_{\frac{n}{2}+1}, a_{\frac{n}{2}+2}, \ldots\} & \text{for even } n, \end{cases}$$

$$C_n = \begin{cases} \{c_1, \ldots, c_n, b_{\frac{n+1}{2}}, b_{\frac{n+1}{2}+1}, \ldots\} & \text{for odd } n, \\ \{c_1, \ldots, c_n, b_{\frac{n}{2}+1}, b_{\frac{n}{2}+2}, \ldots\} & \text{for even } n, \end{cases}$$

$$A_n = \begin{cases} \{a_1, \ldots, a_n, c_{\frac{n+1}{2}}, c_{\frac{n+1}{2}+1}, \ldots\} & \text{for odd } n, \\ \{a_1, \ldots, a_n, c_{\frac{n}{2}+1}, c_{\frac{n}{2}+2}, \ldots\} & \text{for even } n \end{cases}$$

are independent sets of vertices in G for $n = 1, 2, \ldots$. Additionally, A_n, B_n, C_n with odd n are maximal in iG.

From Observations 1 and 2 we conclude:

Observation 3. Assume $S \in iG$ be infinite. There exists an odd n such that $S \subset A_n$ or $S \subset B_n$ or $S \subset C_n$. In each case, S is not arithmetically maximal because (1) and

$$B_{2k+1} = B_{2k-1} \setminus \{a_k\} \cup \{b_{2k}, b_{2k+1}\},$$

$$C_{2k+1} = C_{2k-1} \setminus \{b_k\} \cup \{c_{2k}, c_{2k+1}\}$$

and

$$A_{2k+1} = A_{2k-1} \setminus \{c_k\} \cup \{a_{2k}, a_{2k+1}\}$$

for every $k = 1, 2, \ldots$.

Finally observe that $amiG = \emptyset$.

4. Independent Sets in Line Graphs

A family $\mathcal{F}$ is called a *reverse n-regular* family if for any v we have $|St_{\mathcal{F}}(v)| = n$. Let $\mathcal{K} \subset \mathcal{F}$ be families of sets. We say that F is a *representation* of $\mathcal{K}$ in $\mathcal{F}$ if $F \in s\mathcal{K}$ and $St_{\mathcal{F}}(F) = \mathcal{K}$. We call a subfamily *representable* if it has a representation. A family $\mathcal{K}$ is a *maximal representable* subfamily of $\mathcal{F}$ if it has a representation and for any $\mathcal{K}' \neq \mathcal{K}$ such that $\mathcal{K} \subset \mathcal{K}' \subset \mathcal{F}$ there is no representation.

Theorem 4.1. *Let $\mathcal{F}$ be a countable reverse 2-regular family. If $S \in s\mathcal{F}$ and $St_{\mathcal{F}}(S)$ is a maximal representable subfamily of $\mathcal{F}$, then S is a.m.s. in the family of scattered sets for $\mathcal{F}$, i.e., $S \in ams\mathcal{F}$.*

Proof. Let S satisfies the assumption and $\mathcal{K} = St_{\mathcal{F}}(S)$. The family $s\mathcal{F}$ is hereditary and $S \in ms\mathcal{F}$. Suppose to the contrary that $S \notin ams\mathcal{F}$. From (1), there exist two finite sets $A \subset S$ and $B \in s\mathcal{F}$ such that

$$B \cap S = \emptyset, \ |\ B\ | > |\ A\ | \quad \text{and} \quad (S \setminus A) \cup B \in s\mathcal{F}.$$

The bipartite graph $G = (A \cup B, E)$ with

$$E = \{\{a, b\} \mid a \in A, \ b \in B \ \text{ and } \ St_{\mathcal{F}}(a) \cap St_{\mathcal{F}}(b) \neq \emptyset\} \tag{4}$$

satisfies

$$|St_E(v)| \leq 2 \text{ for every } v \in A \cup B$$

and

$$|St_E(v)| \geq 1 \text{ for every } v \in B.$$

Because $|B| > |A|$, there exists a connected component of G which is a simple path

$$P = (b_1, a_1, \dots, b_{n-1}, a_{n-1}, b_n) \text{ with } |St_E(b_1)| = |St_E(b_n)| = 1$$

and

$$a_i \in A, \quad b_i \in B \text{ for each } i.$$

Let $\tilde{A} = \{a_1, \dots, a_{n-1}\}$ and $\tilde{B} = \{b_1, \dots, b_n\}$. Denote $St_{\mathcal{F}}(a_i) = \{X_i, Y_i\}$. From the construction (4) and revers 2-regularity of $\mathcal{F}$, we have

$$St_{\mathcal{F}}(b_i) = \{Y_{i-1}, X_i\} \text{ for } i = 2, \dots, n.$$

Additionally,

$$St_{\mathcal{F}}(b_1) = \{Y_0, X_1\} \text{ with } Y_0 \notin \mathcal{K} \text{ and } X_n \notin \mathcal{K}.$$

Therefore, we have

$$St_{\mathcal{F}}(\tilde{B}) = St_{\mathcal{F}}(\tilde{A}) \cup \{Y_0, X_n\} \text{ with } \mathcal{K} \cap \{Y_0, X_n\} = \emptyset. \tag{5}$$

The set

$$F = (S \setminus \tilde{A}) \cup \tilde{B}$$

is scattered for $\mathcal{F}$ and

$$St_{\mathcal{F}}(F) = (St_{\mathcal{F}}(S) \setminus St_{\mathcal{F}}(\tilde{A})) \cup St_{\mathcal{F}}(\tilde{B}).$$

From (5), we have

$$St_{\mathcal{F}}(F) = \mathcal{K} \cup \{Y_0, X_n\}$$

which is not possible because $\mathcal{K}$ is a maximal representable subfamily of $\mathcal{F}$.

■

Remark 4.2. Theorem 4.1 fails to be true without the assumption of reverse 2-regularity. We can not replace it neither by the assumption $|St_{\mathcal{F}}(v)| \leq 2$ nor by the assumption that $\mathcal{F}$ is a reverse n-regular family for any $n > 2$.

Below we indicate how the considered notions may be used to graphs with possible multiple edges. By a multigraph we mean a triple $H = (V, E, \tau)$ — two arbitrary sets (of vertices V and of edges E) and a function τ from E to the family of all 2-element subsets of V. We have $\tau(e) = \{u, v\}$ iff u and v are the ends of e. Let us notice, that every line graph of a multigraph without loops has a revers 2-regular clique-cover of edges. The existence of such clique-cover is sufficient for the graph to be the line-graph of a multigraph (see Bermond and Meyer [2] for finite graphs).

Theorem 4.3. *Every countable line-graph* (*of a multigraph*) *has an arithmetically maximal independent set.*

Proof. Let $H = (V, E, \tau)$ be a countable multigraph and $G = L(H) = (E, \mathcal{E})$, where $\mathcal{E} = \{\{e_1, e_2\} | \tau(e_1) \cap \tau(e_2) \neq \emptyset\}$. We can assume that H is connected multigraph (otherwise we can deal with every component of H separately) with $|V| > 2$. If $|V| = 2$ then G is a complete graph and $amiG \neq \emptyset$. In natural way, we extend the definition of the operator St_G on multigraphs:

$$St_H(v) = \{e \in E \mid v \in \tau(e)\}.$$

The family

$$\mathcal{F} = \{St_H(v) \mid v \in V\}$$

is a clique-cover of edges of G. It is reverse 2-regular and $s\mathcal{F} = iG$.

From Steffens existence theorem [8] (which is evidently true also for multigraphs), there exists a matching $S \subset E$ such that S is a complete matching of $H[V^*]$ and V^* is a maximal (with respect to the inclusion) matchable subset of V. Therefore,

$$\mathcal{K} = \{St_H(v) \mid v \in V^*\}$$

is a maximal representable subfamily of $\mathcal{F}$. It follows that $S \in ams\mathcal{F}$. ■

Remark 4.4. We have proved Theorem 4.3 for all line-graphs of countable multigraphs. The assumption on countability is used only in the proof of existence of a maximal matchable subset of vertices (Steffens [8]). Therefore, Theorem 4.3 may be generalized to all line graphs of multigraphs which possess maximal matchable subsets of vertices – for example, the line graphs of multigraphs without infinite paths. On the other hand, the property of having a maximal matchable subset of vertices is not necessary in general as the next example shows. The graph $G = L(K_{\aleph_0,\aleph_1})$ (the line graph of the complete bipartite graph with bipartition: a countable set and a set of size $\aleph_1$) as a matrix graph has an a.m.i.s. though $K_{\aleph_0,\aleph_1}$ has no maximal matchable subset of vertices.

5. Arithmetically Maximal Independent Sets of Cc-locally Finite Graphs

We shall need the following properties of arithmetically maximal independent sets.

Lemma 5.1. *If a graph G has no infinite independent set, then either $amiG \neq \emptyset$ or there exists an infinite sequence $\{S_n\}_{n=1}^{\infty}$ of pair-wise disjoint independent sets such that $|S_n| < |S_{n+1}|$ for every $n = 1, 2, \ldots$.*

Proof. Since iG is a family of finite sets, then the existence of the sequence $\{S_n\}_{n=1}^{\infty}$ in iG implies $amiG = \emptyset$. If $amiG = \emptyset$, then there exists an infinite sequence $\{A_n\}_{n=1}^{\infty}$ such that $A_n \in iG$ and $|A_n| < |A_{n+1}|$ for every $n = 1, 2, \ldots$. As its subsequence $\{S_n\}_{n=1}^{\infty}$ can be constructed. ∎

Lemma 5.2. *If $G = (V, E)$ is a graph and $S \in amiG$, then for every $W \subset V$ the set $W \cap S$ is an a.m.i.s. in the graph $G[W \setminus Nb_G(S \setminus W)]$. Additionally, for every $X \in amiG[W \setminus Nb_G(S \setminus W)]$ the set $X \cup (S \setminus W)$ is an a.m.i.s. in G.*

Proof. On the contrary, suppose that

$$W \cap S \notin amiG[W \setminus Nb_G(S \setminus W)].$$

From (1), there exist two finite sets

$$A \subset W \cap S \quad \text{and} \quad B \in iG[W \setminus Nb_G(S \setminus W)]$$

such that

$$B \cap (W \cap S) = \emptyset, \ | \ B \ | > | \ A \ | \quad \text{and} \quad ((W \cap S) \setminus A) \cup B \in iG[W \setminus Nb_G(S \setminus W)].$$

It is evident that

$$B \cap S = \emptyset, \quad \text{and} \quad (S \setminus A) \cup B \in iG$$

in spite of the assumption. The last statement follows immediately from the definition of a.m.s. ∎

Lemma 5.3. *Let $G = (V, E)$ be a graph and $V = V_1 \cup V_2 \cup \dots$ be a partition of V. The following conditions are equivalent:*

(i) $S \in amiG$.

(ii) $S \in miG$ *and for every finite set* $X \subset S$ *we have*

$$X \in amiG[X \cup (Nb_G(X) \setminus Nb_G(S \setminus X))].$$

(iii) $S \in miG$ *and for every* n *the set*

$$S_n = S \cap \bigcup_{i=1}^{n} V_i \in amiG[\bigcup_{i=1}^{n} V_i \setminus Nb_G(S \setminus S_n)].$$

Proof. (i) $\Rightarrow$ (iii). It follows easily from Lemma 5.2.

(iii) $\Rightarrow$ (ii). Assume (ii) to be false. Then there exists a finite set $X \subset S$ such that

$$X \notin amiG[X \cup (Nb_G(X) \setminus Nb_G(S \setminus X))].$$

It follows that there exist two finite sets

$$A \subset X \quad \text{and} \quad B \subset Nb_G(X) \setminus Nb_G(S \setminus X)$$

such that

$$(X \setminus A) \cup B \in iG \quad \text{and} \quad |A| < |B|.$$

There exists n such that

$$X \cup A \cup B \subset \bigcup_{i=1}^{n} V_i.$$

In addition, we have

$$A \subset S_n, \;\; B \cap S_n = \emptyset \;\; \text{and} \;\; B \subset \bigcup_{i=1}^{n} V_i \setminus Nb_G(S \setminus S_n).$$

Therefore, $(S_n \setminus A) \cup B \in iG$ which contradicts (iii).

(ii) $\Rightarrow$ (i). If $S \notin amiG$, then there exist two finite sets $X \subset S$ and $Y \subset V \setminus Nb_G(S \setminus X)$ such that

$$(S \setminus X) \cup Y \in iG \;\; \text{and by (ii)} \;\; |X| < |Y|.$$

Since $S \in miG$, we have

$$Y \subset Nb_G(X) \ \text{ and } \ X \in amiG[X \cup (Nb_G(X) \setminus Nb_G(S \setminus X))],$$

which contradicts (ii) with respect to X. ∎

Definition 5.1. A graph G is called a *cc-locally finite* graph if for every clique K of G the induced subgraph $G[Nb_G(K)]$ has a finite clique-cover of vertices.

Theorem 5.4. *Let G be a cc-locally finite graph such that there is no infinite sequence $\{K_n\}_{n=1}^{\infty}$ of infinite cliques of G with $Nb_G(K_i) \cap Nb_G(K_j) = \emptyset$ for all $i \neq j$. Then there exists an a.m.i.s. in G.*

Proof. We can assume that $G = (V, E)$ is a connected graph (otherwise we can deal with every component of G separately). Note that if the graph has a finite clique-cover of vertices, then it has finite a.m.i.s.

Assume G has no finite clique-cover of vertices. Let K be a clique of G. We define the sequence of the orbits of K as follows:

$$V_0 = K \ \text{ and } \ V_n = Nb_G\left(\bigcup_{i=0}^{n-1} V_i\right) \neq \emptyset \ \text{ for every } \ n \geq 1.$$

It is easy to see that

$$V = \bigcup_{i=0}^{\infty} V_i \ \text{ and } \ V_n \cap V_m = \emptyset \ \text{ for every } n \neq m.$$

We shall denote

$$\tilde{V}_n = \bigcup_{i=0}^{n} V_i \ \text{ for } n = 0, 1, \ldots.$$

Claim 1. For every $n \geq 0$ the graph $G[\tilde{V}_n]$ has a finite a.m.i.s. and there exists N_0 such that V_n is a finite set for every $n \geq N_0$.

Since G is a cc-locally finite graph, we can deduce by induction that $G[\tilde{V}_n]$ has a finite clique-cover of vertices for every $n \geq 0$. Therefore, $G[\tilde{V}_n]$ has a finite a.m.s. of its independent vertices. For any two cliques $K_1 \subset V_{n_1}$ and $K_2 \subset V_{n_2}$ such that $|n_1 - n_2| > 2$ we have

$$Nb_G(K_1) \cap Nb_G(K_2) = \emptyset.$$

By the assumption on cliques of G there exists a number N_0 such that in $G[V_n]$ there is no infinite clique for every $n \geq N_0$. Since for every $n \geq N_0$ the graph $G[V_n]$ has a finite clique-cover of the vertices and its cliques are finite sets, V_n ought to be finite.

Claim 2. Let N_0 be as in Claim 1. Then there exists

$$S_k \in amiG[\tilde{V}_{N_0+k}] \text{ for every } k \geq 1,$$

such that the sequence $\{S_k\}_1^\infty$ is hereditary, i.e.:

If $S_k \cap V_{N_0+n} = S_{k'} \cap V_{N_0+n}$ for some $n < k' < k$, then

$$S_k \cap \tilde{V}_{N_0} = S_{k'} \cap \tilde{V}_{N_0} \text{ and } S_k \cap V_{N_0+i} = S_{k'} \cap V_{N_0+i} \text{ for each } 1 \leq i < n.$$

Let $\{T_k\}_1^\infty$ be a sequence such that $T_k \in amiG[\tilde{V}_{N_0+k}]$. For every k consider the partition of $T_k = T_{0,k} \cup T_{1,k} \cup \ldots \cup T_{k,k}$

$$\text{where } T_{0,k} = \tilde{V}_{N_0} \cap T_k \text{ and } T_{n,k} = V_{N_0+n} \cap T_k \text{ for } n > 0.$$

By Lemma 5.2,

$$T_{n,k} \in amiG[\tilde{V}_{N_0+n} \setminus Nb_G(T_{n+1,k})] \text{ for every } 0 \leq n < k. \tag{6}$$

Let us denote for $n = 1, 2, \ldots$

$$\mathcal{W}_n = \{T_{n,k} \mid k = n, n+1, \ldots\} \text{ and } \mathcal{W} = \bigcup_{n=1}^{\infty} \mathcal{W}_n.$$

Define two functions l and α from $\mathcal{W}$ to the set of positive integers and to the family of independent sets of G, respectively. We set for $X \in \mathcal{W}_n$

$$l(X) = \min\{k \geq n \mid X = T_{n,k}\} \text{ and } \alpha(X) = T_{n-1,l(X)}. \tag{7}$$

It is obvious that every family $\alpha(\mathcal{W}_n) = \{\alpha(X) | X \in \mathcal{W}_n\}$ is finite and

$$\alpha(\mathcal{W}_1) \subset iG[\tilde{V}_{N_0}] \text{ and } \alpha(\mathcal{W}_n) \subset \mathcal{W}_{n-1} \text{ for every } n > 1.$$

Additionally, by (6), for each $X \in \mathcal{W}_1$

$$\alpha(X) \in amiG[\tilde{V}_{N_0} \setminus Nb_G(X)] \text{ and } \alpha(T_{1,1}) \cup T_{1,1} \in amiG[\tilde{V}_{N_0+1}]. \tag{8}$$

Let us denote

$$\alpha^n(X) = \alpha(\alpha^{n-1}(X)), \alpha^1(X) = \alpha(X) \text{ and } \Lambda(X) = \bigcup_{j=1}^{n} \alpha^j(X) \text{ for } X \in \mathcal{W}_n.$$

Let us prove, by induction on n, the following generalization of the formula (8): For every $n \geq 1$

$$\Lambda(T_{n,n}) \cup T_{n,n} \in amiG[\tilde{V}_{N_0+n}] \text{ and} \tag{9}$$

$$\Lambda(X) \in amiG[\tilde{V}_{N_0+n-1} \setminus Nb_G(X)] \text{ for each } X \in \mathcal{W}_n.$$

We first observe that for $n = 1$ it is exactly Formula (8).

Let $X \in \mathcal{W}_{n+1}$ $n \geq 1$. By (7), $\alpha(X) = T_{n,l(X)}$ and $X = T_{n+1,l(X)}$. We have

$$\bigcup_{j=0}^{n-1} T_{j,l(X)} \in amiG\left[\tilde{V}_{N_0+n-1} \setminus Nb_G\left(\bigcup_{j=n}^{l(X)} T_{j,l(X)}\right)\right]$$

$$= amiG[\tilde{V}_{N_0+n-1} \setminus Nb_G(\alpha(X))],$$

because

$$T_{l(X)} = \bigcup_{j=0}^{l(X)} T_{j,l(X)} \in amiG[\tilde{V}_{N_0+l(X)}]$$

and Lemma 5.2.

On the other hand, from induction hypothesis we obtain

$$\Lambda(\alpha(X)) \in amiG[\tilde{V}_{N_0+n-1} \setminus Nb_G(\alpha(X))].$$

Therefore, by Lemma 5.2

$$\Lambda(\alpha(X)) \cup \alpha(X) \cup \bigcup_{j=n+1}^{l(X)} T_{j,l(X)} \in amiG[\tilde{V}_{N_0+l(X)}] \tag{10}$$

and, additionally,

$$\Lambda(\alpha(X)) \cup \alpha(X) \in amiG[\tilde{V}_{N_0+n} \setminus Nb_G(X)].$$

This clearly forces the second part of (9). If $X = T_{n+1,n+1}$ then $l(X) = n+1$ and (10) becomes the first part of (9).

Define

$$S_k = \Lambda(T_{k,k}) \cup T_{k,k} \in amiG[\tilde{V}_{N_0+k}] \quad \text{for every } k \geq 1.$$

The sequence $\{S_k\}_1^\infty$ is hereditary, because

$$S_k \cap V_{N_0+n} = (\Lambda(T_{n,n}) \cup T_{n,n}) \cap V_{N_0+n} = T_{n,n}$$

independently on k, which completes the proof of Claim 2.

We take a hereditary sequence $\{S_n\}_1^\infty$ as in Claim 2 to define a special graph $\Gamma = (\mathcal{V}, \mathcal{E})$, such that $\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \ldots$, where

$$\mathcal{V}_0 = \{\tilde{V}_{N_0} \cap S_k \mid k \geq n\} \quad \text{and} \quad \mathcal{V}_n = \{V_{N_0+n} \cap S_k \mid k \geq n\} \quad \text{for } n \geq 1.$$

and

$$\begin{aligned} \mathcal{E} = \{\{X, Y\} \;\mid\; & X \in \tilde{V}_{N_0} \cap S_k,\ Y \in V_{N_0+1} \cap S_k \ \text{ and } \ k \geq 0\} \\ \cup\, \{\{X, Y\} \;\mid\; & X \in V_{N_0+n} \cap S_k,\ Y \in V_{N_0+n+1} \cap S_k \ \text{ and } \ k > n \geq 1\}. \end{aligned}$$

It is worth to notice that for every $n \geq 0$ the set $\mathcal{V}_n$ is non-empty and finite. It contains the set $V_{N_0+n} \cap S_n$ with a possibility, that $\emptyset \in \mathcal{V}_n$.

The graph Γ is an infinite forest. It has only a finite number of connected components (trees). Additionally, it is a locally finite graph (i.e., every vertex of Γ has a finite number of neighbours). Königs Lemma states that locally finite infinite tree has an infinite path, see [6]. Then it follows the existence of an infinite path $P = (P_0, P_1, P_2, \ldots)$ in Γ. To prove the theorem, it remains to notice that, by Lemma 5.3 (iii) the set $S = \bigcup_{n=0}^\infty P_n$ is an a.m.i.s. in the graph G. ■

The next example shows that the assumption on infinite cliques in Theorem 5.4 is essential.

Example 5.5. Let $V = \bigcup_{n=1}^\infty V_n$, where $V_n = \{v_{n,1}, v_{n,2}, \ldots\}$ for $n = 1, 2, \ldots$ are infinite mutually disjoint sets of vertices and

$$E = \{\{x, y\} \subset V_n \mid n = 1, 2, \ldots\} \cup \left(\bigcup_{n=1}^\infty \bigcup_{i=1}^\infty \{\{v_{n,i}, v_{n+1,j}\} \mid j \geq i\} \right).$$

The graph $G = (V, E)$ is a cc-locally finite graph but has no a.m.i.s..

Acknowledgement

The author is grateful to Jan Komar for valuable discussions and comments.

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Received 28 November 2003
Revised 8 March 2005