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MAXIMAL HYPERGRAPHS WITH RESPECT TO THE BOUNDED COST HEREDITARY PROPERTY

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Abstract

The hereditary property of hypergraphs generated by the cost colouring notion is considered in the paper. First, we characterize all maximal graphs with respect to this property. Second, we give the generating function for the sequence describing the number of such graphs with the numbered order. Finally, we construct a maximal hypergraph for each admissible number of vertices showing some density property. All results can be applied to the problem of information storage.

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1. Introduction

In the paper, we investigate the hereditary property of hypergraphs "to have a cost bounded above by a constant". An analysis of the property has been motivated by the following application. Consider an agency, which is engaged in the information storage. The company advertises itself as a secure bank whose clients can preserve data independently of each other. A unit of information corresponds to one safe. The agency takes precautions by allotment of keys, that means "to have an access to the chosen safe" is equivalent of "to use the set of keys corresponding to this safe". The management of the company distributes the keys (algorithms, passwords,...) among employees. Every worker has an associated natural cost of key preservation. Assume that different costs characterize different people.

The object of our interest is expressed in the question: how to characterize such sets of keys with their families of subsets corresponding to the safes, whose preserving cost is bounded above by a given constant for each client. Another task is to find such pairs (keys, information) which can be optimally stored with the assumed cost. That means we cannot preserve the cost adding more units of information.

Now, we present a mathematical model. Let vertices of a hypergraph H correspond to different keys. Moreover, let a set of vertices creating an edge in H correspond to one safe. Components of such a hypergraph describe different clients. With regard to the safety of information we can assume that any subset of the set of keys corresponding to one safe cannot be used to open other safes. It follows that H is a Sperner hypergraph. People preserving keys, in fact different costs of key storage, correspond to a colouring of H. Of course, the keys to one safe have to be preserved by at least two people, which guarantees suitability of the colouring. In other words, this colouring uses at least two colours for each edge. The problems stated above are equivalent to the questions: how to characterize the property of hypergraphs "to have a cost smaller than or equal to a constant" and how to characterize the maximal hypergraphs of this property. We solved the latter problem in the class of graphs (two keys are sufficient to open a safe). It is at the same time a partial solution in the class of hypergraphs (a lot of keys can be needed to open a safe). Another goal of the paper is to give the number of maximal graphs which can be constructed with respect to the above mentioned property. Finally, we constructed a maximal hypergraph for each admissible number of vertices, showing some density property.

For the terminology of the graph and hypergraph theory not presented here we refer to [1].

Let $K_{a_1,\ldots,a_k} = (V_1, \ldots, V_k; E)$ denote the complete k-partite graph, such that $a_i \geq 0$ and $|V_i| = a_i$, $i = 1, \ldots, k$.

For a given hypergraph H *a proper k-colouring of* H is a partition $f = (V_1, \ldots, V_k)$ of the set $V(H), |V_1| \geq \ldots \geq |V_k| \geq 0$ such that for every edge E of H there exist at least two indices $1 \leq i, j \leq k$, $i \neq j$ satisfying $E \cap V_i$ and $E \cap V_j$ are nonempty. The proper k-colouring of a hypergraph H can also be defined as a mapping from the set of vertices of H into the set $[k]$ of all positive integers smaller than or equal to k, called *colours*.

The chromatic sum of H, denoted by $\Sigma_N(H)$, is the smallest sum of colours over all vertices of H, among all proper k-colourings of H, $k \in N$, with N being the set of all positive integers.

A proper k-colouring f of H which achieves the chromatic sum of H with a minimum k is called *the best colouring of* H.

A Sperner hypergraph is a hypergraph which has no edge included in an other edge.

Let $\mathcal{J}$ denote a set of all unlabelled finite Sperner hypergraphs without loops. By $\mathcal{I}$ we denote a set of all unlabelled simple finite graphs.

For a Sperner hypergraph H we denote by $\overline{\mathcal{E}(H)}$ a family of all subsets E of $V(H)$ none of which is an edge of H, satisfying $H + E$ is a Sperner hypergraph, excluding the empty set and all one-element subsets.

An additive hereditary hypergraph property is any set of hypergraphs from $\mathcal{J}$ which is closed under isomorphism, disjoined unions and subhypergraphs. By $\mathcal{P}^{con}$ we denote a set of all connected hypergraphs of an additive hereditary property $\mathcal{P}$. Notice that usually $\mathcal{P}^{con}$ is neither hereditary nor additive.

For any additive hereditary property $\mathcal{P} \subseteq \mathcal{J}$, the sets $\mathcal{M}(\mathcal{P})$ and $\mathcal{M}^*(\mathcal{P})$ of maximal hypergraphs of $\mathcal{P}$ are defined by $\mathcal{M}(\mathcal{P}) = \{H \in \mathcal{J} : H \in \mathcal{P}$ and $H + E \notin \mathcal{P}$ for each $E \in \overline{\mathcal{E}(H)}\}$ and $\mathcal{M}^*(\mathcal{P}) = \{H \in \mathcal{M}(\mathcal{P}) : \overline{\mathcal{E}(H)} \neq \emptyset\}$.

Let us define the family of hypergraph properties as follows $\Sigma_k = \{H \in \mathcal{J} :$ for every component H' of H holds $\Sigma_N(H') \le \binom{k+2}{2}\}$, $k \ge 1$.

It is easy to check that if $H' \subseteq H$ then $\Sigma_N(H') \le \Sigma_N(H)$.

By the above, Σ_k are additive hereditary properties of hypergraphs for $k \ge 1$.

2. Results

In the paper, we characterize the families $\mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}$ and $\mathcal{M}(\Sigma_k) \cap \mathcal{I}^{con}$ and finally $\mathcal{M}(\Sigma_k) \cap \mathcal{I}$, $k \ge 1$. We also find the number of connected graphs in $\mathcal{M}^*(\Sigma_k)$, in fact we use the generating function to count such graphs. To do this we will recall, state and prove a few lemmas. The first of them follows immediately out of the definitions.

Lemma 1. *If* $H \in \mathcal{M}^*(\Sigma_k)$, $k \ge 1$, *is a connected hypergraph, then for any best colouring* $f = (V_1, \ldots, V_p)$ *of* H *and for any* $E \in \overline{\mathcal{E}(H)}$ *there exists an index* $i \in [p]$ *satisfying* $E \subseteq V_i$.

It is easy to see that if $\Sigma_N(H) = \binom{k+2}{2}$, $\overline{\mathcal{E}(H)} \neq \emptyset$ and H is a connected hypergraph, then $H \in \mathcal{M}^*(\Sigma_k)$ if and only if the condition stated above holds.

On the other hand, it is not a sufficient condition at all. For example, consider a hypergraph $H = (V, \mathcal{E})$ such that $V = \{a, b, c, d, x, y\}$ and $\mathcal{E}$ consist of all 3 - element subsets of V, including vertex x or y. It is rather easy to see that the only best colouring of H is of the form $f = (\{a, b, c, d\}, \{x, y\})$ and the chromatic sum of H is equal to 8. Hence $H \in \Sigma_3$. Clearly, H admits the condition stated in Lemma 1. Let $E = \{a, b, c\} \in \overline{\mathcal{E}(H)}$. We have $\Sigma_N(H + E) = 10$, because $f' = (\{b, c, d\}, \{x, y\}, \{a\})$ is the best colouring of $H + E$. It follows that H is not a maximal hypergraph of Σ_3.

Moreover, we check at once that for a disconnected hypergraph H the necessary and sufficient conditions to be in $\mathcal{M}^*(\Sigma_k)$ are the following:

- for every component H' of H, $H' \in \mathcal{M}(\Sigma_k)$,
- for any two components H_1, H_2 of H, $\Sigma_N(H_1) + \Sigma_N(H_2) > \binom{k+2}{2}$.

An easy computation shows that the largest complete graph which is in Σ_k has $k + 1$ vertices. It follows that

$$(\mathcal{M}(\Sigma_k) \cap \mathcal{I}) - (\mathcal{M}^*(\Sigma_k) \cap \mathcal{I}) = \{K_i,\ i = 1, \ldots, k + 1\}.$$

Next we are going to formulate some necessary and sufficient conditions for a connected graph, which is not complete, to be a maximal hypergraph of Σ_k.

It is clear that $\mathcal{M}^*(\Sigma_1) \cap \mathcal{I}^{con} = \emptyset$.

Lemma 2 ([3]). *If $K_{a_1,a_2,\ldots,a_t} = (V_1, \ldots, V_t; \mathcal{E})$ and $a_1 \geq a_2 \geq \ldots \geq a_t \geq 1$, then $f = (V_1, V_2, \ldots, V_t)$ is the unique best colouring of $K_{a_1,a_2,\ldots,a_t}$ up to order of parts of the same size.*

Lemma 3. *Let $G = K_{a_1,a_2,\ldots,a_t} = (V_1, \ldots, V_t; \mathcal{E})$ be a given hypergraph and $a_1 \geq a_2 \geq \ldots \geq a_t \geq 1$. Moreover let $\overline{\mathcal{E}(G)} \ni E \subseteq V_j$ for some j, $1 \leq j \leq t$. Then every best colouring of the hypergraph $G + E$ is of the form*

$$f = (V_1, \ldots, V_{j-1}, V_k, V_{j+1}, \ldots, V_{k-1}, V_j - \{v\}, V_{k+1} \ldots, V_t, \{v\}),$$

where $v \in V_j$ and $k = \max\{i : |V_i| = |V_j|\}$.

Proof. Suppose that $f' = (V_1', V_2', \ldots, V_p')$ is the best colouring of the hypergraph $G + E$. Applying the definition of $\overline{\mathcal{E}(G)}$ it is clear that for any $m, 1 \leq m \leq p$, there exists $w, 1 \leq w \leq t$, such that $V_m' \subseteq V_w$. It is also easy to see that there exist at least two sets V_n', V_l' such that $V_n' \subseteq V_j$ and $V_l' \subseteq V_j$ and $V_l' \cap E \neq \emptyset$, $V_n' \cap E \neq \emptyset$. Suppose that there exists x, $x \in [t] - \{j\}$ satisfying $V_w' \subseteq V_x$ and $V_m' \subseteq V_x$ for some w, m, $1 \leq w < m \leq p$. Then the sum of the colouring $f^* = (V_1', V_2', \ldots, V_w' \cup V_m', \ldots, V_p')$ is less than the sum of f', which contradicts the fact that f' is the best colouring of $G + E$.

In the same manner we can see that the partition $(V_j - \{v\}, \{v\})$, where $v \in E$, is the only partition of the set V_j which realizes the least possible sum of $G + E$. The statement finishes the proof. ∎

Theorem 4. *Let $k \geq 2$. $G \in \mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}$ if and only if $G \cong K_{a_1,a_2,\ldots,a_k}$, where $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$ are integers $t + 1 - p > \binom{k+2}{2} - s$, $\binom{k+2}{2} - k + 1 \leq s \leq \binom{k+2}{2}$ and $a_2 \geq 1$, with $s = \sum_{j=1}^{k} j a_j$, $t = \max\{i : \ a_i \neq 0\}$, $p = \max\{i : \ a_i \geq 2\}$.*

Proof. ($\Leftarrow$) Let $G = K_{a_1,a_2,\ldots,a_k} = (V_1, \ldots, V_k; \mathcal{E})$ and the above conditions hold. Since $a_2 \geq 1$, G is a connected graph. Lemma 2 now yields $\Sigma_N(G) = s \leq \binom{k+2}{2}$, hence $G \in \Sigma_k$. If G was a complete graph, we would have $1 \geq a_1 \geq \ldots a_k \geq 0$ and therefore $s \leq \binom{k+1}{2} < \binom{k+2}{2} - k + 1$, a contradiction. Hence $\overline{\mathcal{E}(G)}$ is nonempty and p, t are well defined. Let $E \in \overline{\mathcal{E}(G)}$. It is clear that there exists an index j, $1 \leq j \leq p$, such that $E \subseteq V_j$. Let $m = \max\{i : a_i = a_j\}$. According to Lemma 3, we have

$$\Sigma_N(G + E) = \sum_{x=1}^{m-1} a_x x + (a_m - 1)m + \sum_{x=m+1}^{t} a_x x + (t + 1)$$

$$= s - m + t + 1 \geq s + t + 1 - p > \binom{k+2}{2},$$

the last inequality holds from the assumption. Hence $G + E \notin \Sigma_k$ and $G \in \mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}$.

($\Rightarrow$) Let now $G \in \mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}$ and consider the best colouring $f = (V_1, \ldots, V_l)$, $|V_l| \geq 1$, of G. Moreover, let $a_j = |V_j|$ for $j = 1, \ldots, l$. Since $G \in \Sigma_k$ it is clear that $\Sigma_N(G) = \sum_{j=1}^{l} a_j j = s \leq \binom{k+2}{2}$. Since $G \in \mathcal{M}^*(\Sigma_k)$, we can see that G is not a complete graph. We conclude from the form of the colouring f and Lemma 1 that G is a complete multipartite graph $K_{a_1,a_2,\ldots,a_l}$. We must have $l \leq k$, because the only connected graph $K_{a_1,a_2,\ldots,a_l}$, where

$l \geq k+1$, satisfying $\sum_{j=1}^{l} a_j j \leq \binom{k+2}{2}$ is $K_{k+1} \notin \mathcal{M}^*(\Sigma_k)$. Moreover $s = \sum_{j=1}^{l} a_j j \geq \binom{k+2}{2} - k + 1$. Otherwise, since for any $E \in \overline{\mathcal{E}(G)}$, $\Sigma_N(G+E) - \Sigma_N(G) \leq k$ by Lemmas 2 and 3, we have

$$\Sigma_N(G+E) \leq k + s < k + \binom{k+2}{2} - k + 1 = \binom{k+2}{2} + 1.$$

Hence $\Sigma_N(G+E) \leq \binom{k+2}{2}$, which contradicts the maximality of G.

It remains to prove that $t+1-p > \binom{k+2}{2} - s$, with t, p as it was described. Since $t = l$, Lemma 3 shows that $\Sigma_N(G+E) = s + t + 1 - m$, where $E \in V_w$ and $m = \max\{i : \ |V_i| = |V_w|\}$. As $m \leq p$ we have

$$1 + \binom{k+2}{2} \leq \min\{\Sigma_N(G+E) : E \in \overline{\mathcal{E}(G)}\} = s + t + 1 - p,$$

which completes the proof. ■

As a consequence of the last proof we have the identity between the set of connected graphs in $\mathcal{M}^*(\Sigma_k)$ and the set of connected graphs in $\mathcal{M}^*(\Sigma_k \cap \mathcal{I})$. This completes the characterization of the families $\mathcal{M}(\Sigma_k) \cap \mathcal{I}$ and $\mathcal{M}(\Sigma_k \cap \mathcal{I})$.

Moreover, since every connected hypergraph in Σ_k has at most $\binom{k+2}{2} - 1$ vertices and K_{k+1} is a maximal hypergraph of Σ_k, it is clear that the order of the connected hypergraph in $\mathcal{M}^*(\Sigma_k)$ is limited. In the next theorem, we will show that there exists such a hypergraph for each admissible number of vertices.

Theorem 5. *For each integers k, p, $k \geq 2$, $k+2 \leq p \leq \binom{k+2}{2} - 1$ there exists a connected graph $G \in \mathcal{M}^*(\Sigma_k)$ satisfying $|V(G)| = p$.*

Proof. Let p satisfy $k+2 \leq p \leq \binom{k+2}{2} - 2$. First, we show that $G = K_{l,\underbrace{1,\ldots,1}_{w-1}}$, where w, l are given by

$$w^2 - w \leq k^2 + 3k + 2(1-p) < w^2 + w,$$
$$l = p - w + 1,$$

has the desired property. We use Theorem 4 to show it. It is clear that for fixed k, p there is exactly one number w satisfying the first two inequalities, because $(w-1)^2+(w-1)=w^2-w$. Consider the function $f(p)=k^2+3k+2-2p$ of an integer p, $k+2\le p\le \binom{k+2}{2}-1$. It is obvious that f has a maximum at $p=k+2$, namely $f(k+2)=k^2+k-2$, and has a minimum equal to 4 at $p=\binom{k+2}{2}-2$. Hence $k\ge w\ge 2$. The task is now to show that such a graph G satisfies the conditions stated in Theorem 4. It is clear that $\max\{i:\ a_i\neq 0\}=w$. Since $l=p-w+1\ge k+2-k+1\ge 3$, $a_1\ge a_2\ge \ldots \ge a_k\ge 0$ and $\max\{i:\ a_i\ge 2\}=1$. By $w\ge 2$ we have $a_2\ge 1$. As $w^2-w\le k^2+3k+2(1-p)$ we obtain $s\le \binom{k+2}{2}$. As $w^2+w>k^2+3k+2(1-p)$ and $k\ge w$ the conditions $s\ge \binom{k+2}{2}-k+1$ and $w>\binom{k+2}{2}-s$ hold.

Similar considerations can be applied to $G=K_{\binom{k+2}{2}-2,1}$. ■

In the next theorem, we give the generating function for the sequence describing $|M^*(\Sigma_k)\cap I^{con}|, k\ge 1$. To do this we will recall and prove two lemmas. Both of these observations result easily from the known formula for the number of ways of distinct decompositions of the positive integer into positive summands with some restrictions [4].

Lemma 6. *Let k be an arbitrary positive integer. The number of solutions $(x_1,x_2,\ldots)$ of the equation $\sum_{i=1}^{\infty} x_i\binom{i+1}{2}=\binom{k+2}{2}$ in sequences of nonnegative integers is equal to $c_{\binom{k+2}{2}}$ where*

$$f(x)=\sum_{n=1}^{\infty} c_n x^n=\prod_{n=2}^{\infty}\frac{1}{1-x^{\binom{n}{2}}}.$$

Proof. It is obvious that

$$\prod_{n=2}^{\infty}\frac{1}{1-x^{\binom{n}{2}}}=(1+x+x^2+\ldots)(1+x^3+x^6+\ldots)\cdot$$
$$\cdot\ldots\cdot\left(1+x^{\binom{n}{2}}+x^{2\binom{n}{2}}\right)\cdot\ldots.$$

According to the formula for the series product we have the right side of the above equation being the sum of components of the form

$$x^{\lambda_1}x^{\lambda_2\binom{3}{2}}x^{\lambda_3\binom{4}{2}}\cdot\ldots=x^{\lambda_1+\lambda_2\binom{3}{2}+\lambda_3\binom{4}{2}+\cdots},$$

where λ_i is the number of expression taken from the ith series, the numeration starting from zero. The coefficient of $x^{\binom{k+2}{2}}$ is thus equal to the number of sequences $(\lambda_1, \lambda_2, \ldots,)$ such that $\lambda_1 + \lambda_2\binom{3}{2} + \lambda_3\binom{4}{2} + \cdots = \binom{k+2}{2}$, which completes the proof. ∎

Lemma 7. *Let i, k, $1 \le i < k$ be any positive integers. The number of solutions $(x_1, x_2, \ldots)$ of the equation $\sum_{j=1}^{\infty} x_j\binom{j+1}{2} = \binom{k+2}{2} - i$, in sequences of nonnegative integers so that there exist indices t, p, $t \ge p + i$ satisfying $x_t = 1$, $x_p \ge 1$, $x_j = 0$ for all $j \ge t+1$, $x_j = 0$ for all $p < j < t$ is equal to $b_{\binom{k+2}{2}-i}$, where*

$$f(x) = \sum_{j=1}^{\infty} b_j x^j = \sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} \prod_{n=1}^{p} \frac{x^{\binom{p+1}{2}}}{1 - x^{\binom{n+1}{2}}}.$$

Proof. We first observe that the right side of the above equation can be written as

$$\sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} x^{\binom{p+1}{2}} (1 + x + x^2 + \ldots)(1 + x^3 + x^6 + \ldots)\cdot$$

$$\cdot \ldots \cdot \left(1 + x^{\binom{p+1}{2}} + x^{2\binom{p+1}{2}} + \ldots\right).$$

According to the formulas for the series product, the sums product and sums sum we have the right side of the above equation being the sum of the components of the form

$$x^{1\cdot\binom{t+1}{2}} x^{1\cdot\binom{p+1}{2}} x^{\lambda_1} x^{\lambda_2\binom{3}{2}} x^{\lambda_3\binom{4}{2}} \cdot \ldots x^{\lambda_p\binom{p+1}{2}}$$
$$= x^{\binom{t+1}{2} + (\lambda_p+1)\binom{p+1}{2} + \lambda_{p-1}\binom{p}{2} + \ldots + \lambda_1\binom{2}{2}},$$

where $p + i \le t$ and λ_j is the number of expression taken from the jth series of the last sum, the numeration starting from zero. The coefficient of $x^{\binom{k+2}{2}-i}$ is thus equal to the number of sequences $(\lambda_1, \lambda_2, \ldots,)$ such that $\lambda_t = 1$, $p + i \le t$, $\lambda_j = 0$, $p < j < t$ and

$$\lambda_1\binom{2}{2} + \lambda_2\binom{3}{2} + \lambda_3\binom{4}{2} + \cdots + \lambda_{p-1}\binom{p}{2}$$
$$+(\lambda_p + 1)\binom{p+1}{2} + \binom{t+1}{2} = \binom{k+2}{2} - i,$$

which completes the proof. ∎

Theorem 8. *The generating function for the sequence* $(a_n)_{n=1}^{\infty}$, *where for* $k = 1, 2, \ldots$

$$a_{\binom{k+2}{2}} = |\mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}| - 2,$$

is of the form

$$f(x) = \sum_{j=1}^{\infty} a_j x^j = \sum_{i=1}^{\infty} x^i \sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} \prod_{n=1}^{p} \frac{x^{\binom{p+1}{2}}}{1 - x^{\binom{n+1}{2}}} + \prod_{n=2}^{\infty} \frac{1}{1 - x^{\binom{n}{2}}}.$$

Proof. Theorem 4 shows that $|\mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}|$ is equal to the number of connected graphs $G \simeq K_{a_1,\ldots,a_k}$, where $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$ are integers and $\binom{k+2}{2} - k + 1 \leq s \leq \binom{k+2}{2}$, $t + 1 - p > \binom{k+2}{2} - s$, $a_2 \geq 1$ with $s = \sum_{j=1}^{k} j a_j$, $t = \max\{i : \ a_i \neq 0\}$, $p = \max\{i : \ a_i \geq 2\}$. It follows that $|\mathcal{M}^*(\Sigma_k) \cap \mathcal{I}^{con}|$ is equal to the number of solutions $(a_1, a_2, \ldots, a_k)$ in the finite sequences of nonnegative integers of the identity

$$\sum_{i=1}^{k} x_i i = s \tag{1}$$

satisfying $a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$ and

$$\begin{cases} \binom{k+2}{2} - k + 1 \leq s \leq \binom{k+2}{2}, \\ t + 1 - p > \binom{k+2}{2} - s, \\ t = \max\{i : \ a_i \neq 0\}, \\ p = \max\{i : \ a_i \geq 2\}, \end{cases} \tag{2}$$

decreased by one, because the solution $a_1 = \binom{k+2}{2}$, $a_i = 0$ for all $2 \leq i \leq k$ should be excluded with respect to the condition $a_2 \geq 1$. It is easily seen that the problem stated above can be written as follows: find the number of solutions $(a_1, a_2, \ldots)$ in sequences of nonnegative integers of the identity

$$\sum_{i=1}^{\infty} x_i i = s \tag{3}$$

satisfying $a_1 \geq a_2 \geq \ldots$ and (2). Then decrease it by two, because the solutions

- $a_1 = \binom{k+2}{2}$, $a_i = 0$ for all $i \geq 2$,
- $a_1 = a_2 = \ldots = a_{k+1} = 1$, $a_i = 0$ for all $i \geq k + 2$

should be excluded. The second solution refers to a graph K_{k+1}, which is in $\mathcal{M}(\Sigma_k) - \mathcal{M}^*(\Sigma_k)$.

We can now rewrite (3) as

$$\sum_{i=1}^{\infty}(x_i - x_{i+1})(1 + \ldots + i) = s = \sum_{i=1}^{\infty}(x_i - x_{i+1})\binom{i+1}{2}. \tag{4}$$

Taking $y_n = (x_n - x_{n+1})$ for $n \geq 1$ we obtain

$$\sum_{i=1}^{\infty} y_i \binom{i+1}{2} = s \tag{5}$$

where the condition $x_1 \geq x_2 \geq \ldots$ is substituted by $y_i \geq 0$ for $i \geq 1$. It is clear that (2) is always true if $s = \binom{k+2}{2}$ or equivalently $t = p$. We can now rewrite (2) as

$$\begin{cases} \exists_{1 \leq i \leq k-1}\ s = \binom{k+2}{2} - i \\ \exists_{t,p \in N,\, t \geq p+i}\ y_t = 1, y_p \geq 1 \text{ and } \forall_{p<j<t \vee j \geq t+1}\ y_j = 0 \end{cases} \tag{6}$$

or

$$s = \binom{k+2}{2}. \tag{7}$$

Lemma 6 shows that the number of solutions of (5) satisfying (7) is equal to $c_{\binom{k+2}{2}}$ where

$$\sum_{n=0}^{\infty} c_n x^n = \prod_{n=2}^{\infty} \frac{1}{1 - x^{\binom{n}{2}}}.$$

Lemma 7 yields that the number of such solutions satisfying (6) is equal to $\sum_{i=1}^{k-1} d_{\binom{k+2}{2}-i}$ where

$$\sum_{j=1}^{\infty} d_j x^j = \sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} \prod_{n=1}^{p} \frac{x^{\binom{p+1}{2}}}{1 - x^{\binom{n+1}{2}}}$$

which means it is equal to $g_{\binom{k+2}{2}}$ where

$$\sum_{j=1}^{\infty} g_j x^j = \sum_{i=1}^{k-1} x^i \sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} \prod_{n=1}^{p} \frac{x^{\binom{p+1}{2}}}{1-x^{\binom{n+1}{2}}}$$

$$= \sum_{i=1}^{\infty} x^i \sum_{t=i+1}^{\infty} x^{\binom{t+1}{2}} \sum_{p=1}^{t-i} \prod_{n=1}^{p} \frac{x^{\binom{p+1}{2}}}{1-x^{\binom{n+1}{2}}}.$$

and the proof is complete. ∎

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