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LATTICE-INADMISSIBLE INCIDENCE STRUCTURES *

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Abstract

Join-independent and meet-independent sets in complete lattices were defined in [6]. According to [6], to each complete lattice $(L, \leq)$ and a cardinal number p one can assign (in a unique way) an incidence structure $\mathcal{J}_L^p$ of independent sets of $(L, \leq)$. In this paper some lattice-inadmissible incidence structures are founded, i.e. such incidence structures that are not isomorphic to any incidence structure $\mathcal{J}_L^p$.

Keywords: complete lattices, join-independent and meet-independent sets, incidence structures.

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Let $(L, \leq)$ be a complete lattice and let $\bigvee, \bigwedge$ be the supremum and the infimum of any subset of L, respectively. The least and the greatest elements in $(L, \leq)$ are denoted by 0, 1 respectively. If $x, y \in L$, then $x \| y$ means that x, y are incomparable in $(L, \leq)$. If $X \subseteq L$, then we put $X_x := X \setminus \{x\}$ for $x \in X$ and

$$J(X) = \left\{ \bigvee X_x \mid x \in X \right\}, \quad M(X) = \left\{ \bigwedge X_x \mid x \in X \right\}.$$

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Definition 1. A subset $X \subseteq L$ is said to be *join-independent (meet-independent)* if and only if $x \not\leq \bigvee X_x$ ($\bigwedge X_x \not\leq x$, resp.) for all $x \in X$.

Remark 1. The concept of independence have been studied in various types of lattices motivated by applications in algebra and geometry (refer to [1]–[3], [5], [12]). Our approach is explained in [6] in detail and it is used also in [11].

Remark 2. A set $X = \{x\}$ is *join-independent (meet-independent)* if and only if $x \neq 0$ ($x \neq 1$). If $\mathrm{card}(X) = |X| \geq 2$, then X is join-independent (meet-independent) if and only if $x \| \bigvee X_x$ ($x \| \bigwedge X_x$, resp.) for all $x \in X$.

To avoid trivial cases we will suppose that $|X| > 2$ in what follows. The notions of join- and meet-independent sets are dual in complete lattices. Each assertion about join-independent sets admits its corresponding dual one which will not be stated explicitly.

The set of all join-independent (meet-independent) sets of cardinality $p > 2$ will be denoted by G^p (M^p, respectively).

The following proposition is obvious:

Proposition 1. *Let x, y be distinct elements of a set $X \in G^p$. Then $x \| y$ and $\bigvee X_x \| \bigvee X_y$.* ■

To every subset $X \subseteq L$ we assign a system U_X of subsets of L by setting $Y \in U_X$ iff there exists a bijective mapping $\alpha : X \to Y$ such that $\bigvee X_x \leq \alpha(x)$ and $\alpha(x) \| x$ for all $x \in X$. This mapping is called a *U-mapping.*

Dually, to a subset $X \subseteq L$ we assign a system V_X of subsets of L by setting $Z \in V_X$ iff there exists a bijective mapping $\beta : X \to Z$ such that $\beta(x) \leq \bigwedge X_x$ and $\beta(x) \| x$ for all $x \in X$. This mapping is called a *V-mapping.* It is easy to show: If α is a U-mapping, then α^{-1} is a V-mapping.
The proof of the following proposition is straightforward.

Proposition 2. *Let $X \subseteq L$. Then the following statements are equivalent:*

(1) $X \in G^p$,

(2) $J(X) \in U_X$,

(3) $U_X \neq \emptyset$. ■

Proposition 3. *Let $X \subseteq L$ where $|X| = p$. If $Y \in U_X$, then $Y \in M^p$ and $X \in V_Y$.*

Proof. Let $Y \in U_X$. Then a U-mapping $\alpha : X \to Y$ exists. Let us put $Y_{\alpha(x)} = Y \setminus \{\alpha(x)\}$ for all $x \in X$. If $\alpha(y) \in Y_{\alpha(x)}$, then $y \in X_x$ and $x \in X_y$ which yields $x \leq \bigvee X_y \leq \alpha(y)$. Hence, $x \leq \bigwedge Y_{\alpha(x)}$. If $\bigwedge Y_{\alpha(x)} \leq \alpha(x)$, then $x \leq \alpha(x)$ which is a contradiction. Thus, $Y \in M^p$. Since $\alpha^{-1} : Y \to X$ is a V-mapping we get $X \in V_Y$. ■

Proposition 4. *Let $X \subseteq L$. Then the following statements are equivalent:*

(1) *$X \in G^p$,*

(2) *$J(X) \in M^p$.*

Proof. (1) ⇒ (2) : It follows from Proposition 2 and 3.

(2) ⇒ (1) : Let $J(X) \in M^p$. If we put $P_x = J(X) \setminus \bigvee X_x$ for $x \in X$, then $\bigwedge P_x \not\leq \bigvee X_x$ and $\bigwedge P_x \leq \bigvee X_y$ for each $y \in X_x$. Let us assume that $x \leq \bigvee X_x$. Then $\bigvee X_x = \bigvee X$ and $\bigvee X_y \leq \bigvee X_x$ for all $y \in X_x$. Thus, $\bigwedge P_x \leq \bigvee X_x$ which is a contradiction. Hence, $x \not\leq \bigvee X_x$ and $X \in G^p$. ■

Proposition 5. *Let $X \in G^p$ and $Y \subseteq L$. Then*

(1) *$Y \in U_X$*

if and only if

(2) *there exists a bijective mapping $\gamma : J(X) \to Y$ such that $m \leq \gamma(m)$ for each $m \in J(X)$ and $\gamma(m) \| n$ for all $n \in J(X)$ distinct from m.*

Proof. Since X is a join-independent set the mapping $\beta : x \mapsto \bigvee X_x$, $x \in X$, is a bijection of X onto $J(X)$.

(1) ⇒ (2) : It follows from $Y \in U_X$ that there exists a U-mapping $\alpha : X \to Y$. Let us put $\gamma = \alpha\beta^{-1}$. If $m \in J(X)$, then $m = \bigvee X_x$ for a certain $x \in X$ and $\gamma(\bigvee X_x) = \alpha(x)$. Thus, $\bigvee X_x \leq \gamma(\bigvee X_x)$. Consider $n \in J(X)$ where $n \neq m$. Then $n = \bigvee Y_y$ where $y \neq x$. If $\alpha(x) \leq \bigvee X_y$, then $\bigvee X_x \leq \alpha(x) \leq \bigvee X_y$ which contradicts Proposition 1. If $\bigvee X_y \leq \alpha(x)$, then $x \leq \bigvee X_y \leq \alpha(x)$, a contradiction again. Hence, $\alpha(x) \| \bigvee X_y$ and $\gamma(m) \| n$.

(2) ⇒ (1) : The mapping $\alpha = \gamma\beta$ is a bijection of X onto Y with $\alpha(x) = \gamma(\bigvee X_x)$ for $x \in X$. It suffices to show that α is a U-mapping. ■

Proposition 6. *If $X \subseteq L$ and $Y \in V_X$, then $U_X \cap U_Y = \emptyset$.*

Proof. If $|X| = p$, then $Y \in V_X$ yields $Y \in G^p$ and $J(Y) \in M^p$. By Proposition 3, $X \in U_Y$ and there exists a mapping $\gamma : J(Y) \to X$ given in Proposition 5. Assume that $A \in U_X$. According to Proposition 5, for each $a \in A$ there is a unique element $\bigvee X_x \in J(X)$ such that $\bigvee X_x \leq a$. Then $z \leq a$ for all $z \in X_x$. It follows from $p > 2$ that X_x contains at least two distinct elements z_1, z_2. If we put $\gamma^{-1}(z_1) = m_1$, $\gamma^{-1}(z_2) = m_2$, then we obtain $m_1 \leq z_1 \leq a$, $m_2 \leq z_2 \leq a$. Thus, by Proposition 5, $A \notin U_Y$. ■

Proposition 7. *Let $X, Y \in G^p$. Then $J(X) = J(Y)$ if and only if $U_X = U_Y$.*

Proof.

1. Let $J(X) = J(Y)$ and consider $C \in U_X$. Then, by Proposition 5, there exists a mapping $\gamma : J(X) \to C$. Since $J(X) = J(Y)$, we obtain $C \in U_Y$ and thus, $U_X \subseteq U_Y$. It is also obvious that $U_Y \subseteq U_X$.
2. Let $U_X = U_Y$. Since $J(X) \in U_X$ and $J(Y) \in U_Y$, we get $J(X) \in U_Y$ and $J(Y) \in U_X$. It follows from $J(X) \in U_Y$ that there exists a bijection $\gamma : J(X) \to J(Y)$ established in Proposition 5 and for each $\bigvee X_x \in J(X)$ there exists a unique element $\bigvee Y_y$ such that $\bigvee X_x \leq \bigvee Y_y$. If we put $\xi_1(x) = y$, we get a bijective mapping of X onto Y. Similarly, with the help of $J(X) \in U_Y$ we define a bijective mapping $\xi_2 : Y \to X$ such that $\xi_2(m) = n$ if and only if $\bigvee Y_m \leq \bigvee X_n$. For $x \in X$ we get $\bigvee X_x \leq \bigvee Y_{\xi_1(x)} \leq \bigvee X_{\xi_2\xi_1(x)}$ and, by Proposition 1, $x = \xi_2\xi_1(x)$. Consider $\bigvee X_x \in J(X)$. Then $\bigvee X_x \leq \bigvee Y_{\xi_1(x)}$ and, with respect to $\xi_1(x) \in Y$, we obtain $\bigvee Y_{\xi_1(x)} \leq \bigvee X_{\xi_2\xi_1(x)} = \bigvee X_x$. Thus, $\bigvee X_x = \bigvee Y_{\xi_1(x)}$ and $\bigvee X_x \in J(Y)$. Therefore, $J(X) \subseteq J(Y)$ and $J(Y) \subseteq J(X)$ can be obtained similarly. ■

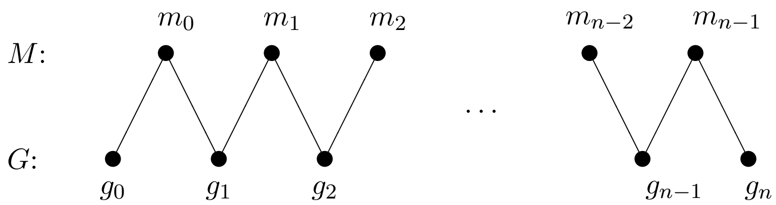
As in [6], to $(L, \leq)$ and p an incidence structure can be assigned. We recall the definition and some basic facts (more thoroughly, see [4]) about incidence structures needed in what follows.

Definition 2. An *incidence structure (context)* is a triple of sets $\mathcal{J} = (G, M, I)$, where $I \subset G \times M$. An incidence structure $\mathcal{J}_1 = (G_1, M_1, I_1)$ is a *substructure* of $\mathcal{J}$ if $G_1 \subseteq G$, $M_1 \subseteq M$ and $I_1 = I \cap (G_1 \times M_1)$.

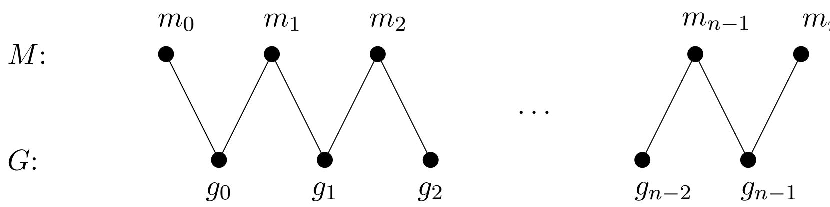
Remark 3. Incidence structures are often given by their graphs: The elements of sets G, M are represented by points and those corresponding to elements $g \in G$, $m \in M$ are joined by a line-segment iff gIm.

Definition 3. An incidence structure $\mathcal{J} = (G, M, I)$ having the following incidence graph is called a *simple connection*

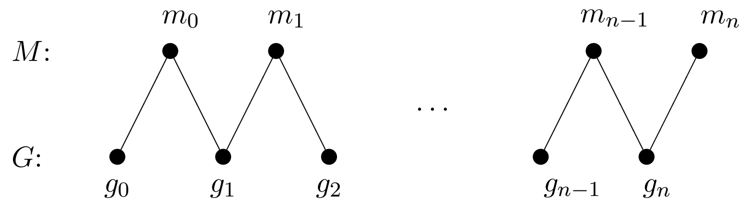
(a) of type 1:



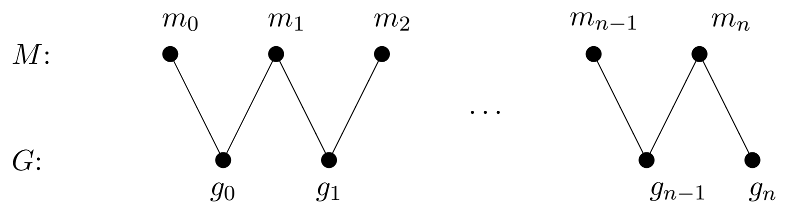
(b) of type 1':



(c) of type 2:



(d) of type 2':



The positive integer n is said to be a *length* of this connection.

Let $\mathcal{J} = (G, M, I)$ be an incidence structure. Then for every subset $A \subseteq G$, respectively $B \subseteq M$, we put $A^{\uparrow} = \{m \in M \mid (\forall g \in A)[gIm]\}$, $B^{\downarrow} = \{g \in G \mid (\forall m \in B)[gIm]\}$. In [7], *independent sets* in G and M are defined and to each cardinal number p the incidence structure $\mathcal{J}^p$ of independent sets of cardinality p is assigned.

If $(L, \leq)$ is a complete lattice, then $\mathcal{J}_L = (L, L, I)$ is an incidence structure in which aIb iff $a \leq b$ for $a, b \in L$. Join- and meet-independent sets in $(L, \leq)$ are independent in $\mathcal{J}_L$ in the sense of [7]. To $(L, \leq)$ and a cardinal p the incidence structure $\mathcal{J}_L^p = (G^p, M^p, I^p)$ is assigned, where AI^pB iff $B \in U_A$ for any $A \in G^p$, $B \in M^p$ (see [6]). It is obvious that $A^{\uparrow} = U_A$, $B^{\downarrow} = V_B$ for $A \in G^p$, $B \in M^p$.

Definition 4. An incidence structure $\mathcal{J}$ is said to be *lattice-inadmissible* if there do not exist a complete lattice L and a cardinal number $p > 2$ such that the associated incidence structure $\mathcal{J}_L^p$ is isomorphic to $\mathcal{J}$. Otherwise, $\mathcal{J}$ is called *lattice-admissible.*

Remark 4. Each incidence structure $\mathcal{J} = (G, M, I)$ with $\{g\}^\uparrow = \emptyset$ ($\{m\}^\downarrow = \emptyset$, respectively) for some $g \in G$ ($m \in M$) is lattice-inadmissible, since $U_A \neq \emptyset$ ($V_B \neq \emptyset$) for every $A \in G^p$ ($B \in M^p$, resp.).

Some other examples of lattice-inadmissible incidence stuctures are given below.

Proposition 8. *Let $X \in G^p \cap M^p$. Then*

(1) $X \not{I}^p X$,

and

(2) *if XI^pC and BI^pX, then $B \not{I}^p C$.*

Proof. From BI^pX, we get $B \in V_X$ and, by Proposition 6, $U_X \cap U_B = \emptyset$. If XI^pC and BI^pC, then $C \in U_X \cap U_B$ which is a contradiction. Obviously, $XI^pJ(X)$ and $M(X)I^pX$. Since $M(X) \in V_X$, we obtain $U_X \cap U_{M(X)} = \emptyset$ again. If XI^pX, then $X \in U_X \cap U_{M(X)}$which is a contradiction. ∎

Corrolary 1.

1. *If an incidence structure $\mathcal{J} = (G, M, I)$ contains an element $x \in G \cap M$ such that xIx, then $\mathcal{J}$ is lattice-inadmissible. In particular, for any (non-empty) complete lattice $(L, \leq)$, the incidence structure $\mathcal{J}_L$ is lattice-inadmissible, since aIa for all $a \in L$.*

2. *If $\mathcal{J} = (G, M, I)$ contains elements $x \in G \cap M$, $b \in G$, $c \in M$ such that xIc, bIx and bIc, then $\mathcal{J}$ is lattice-inadmissible.* ∎

Theorem 1. *Let $(L, \leq)$ be a complete lattice and $p > 2$. Then, in L, there do not exist pairwise distinct subsets $A, B, C \in G^p$, $X, Y, Z \in M^p$ such that $U_A = \{X\}$, $U_B = \{X, Y\}$, $U_C = \{Y, Z\}$, $V_X = \{A, B\}$, $V_Y = \{B, C\}$.*

Proof. Let us suppose that such subsets exist. Then obviously $X = J(A)$. If furthermore $X = J(B)$, then $U_A = U_B$, by Proposition 7, which is a contradiction. Hence, $Y = J(B)$ and similarly $Z = J(C)$. Since $X = J(A) = \{\bigvee A_x \mid x \in A\} \in M^p$, we get $M(X) = \{\bigwedge P_x \mid x \in A\}$, where

$P_x = X \setminus \{\bigvee A_x\}$. Moreover, $a \leq \bigwedge P_a$ for all $a \in A$ and $a \| \bigwedge P_x$ for all $x \in A_a$. It follows from $V_X = \{A, B\}$ that either $A = M(X)$ or $B = M(X)$. Let $B = M(X)$. Then there is a unique $a \in A$ such that $B = \{\bigwedge P_a\} \cup A_a$, where $a < \bigwedge P_a$ and $x = \bigwedge P_x$ for all $x \in A_a$. Obviously, $B \setminus \{\bigwedge P_a\} = A_a$ and $\bigvee B_{\bigwedge P_a} = \bigvee A_a$. For $y \in A_a$, we get $x \leq \bigvee A_y$ for all $x \in A_y \setminus \{a\}$ and also $a \leq \bigwedge P_a \leq \bigvee A_y$. This yields $\bigvee A_y = \bigvee B_y$ and $X = J(B)$, which is a contradiction. Thus, $A = M(X)$. In a similar way, from $V_Y = \{B, C\}$, we show that $B = M(Y)$.

Since $V_X = \{A, B\}$ and $A = M(X)$, there exists precisely one element $a \in A$ such that $B = \{b\} \cup A_a$, where $b < a$ and $b \parallel x$ for all $x \in A_a$. Then $B_b = A_a$ and $\bigvee B_b = \bigvee A_a$. It follows from $U_B = \{X, Y\}$ and $Y = J(B)$ that there exists a unique $y \in A_a$ such that $\bigvee B_y < \bigvee A_y$ and $\bigvee B_x = \bigvee A_x$ for each $x \in A_a \setminus \{y\}$. Hence, $Y = \{\bigvee B_y\} \cup \{\bigvee A_a\} \cup \{\bigvee A_x \mid x \in A_a \setminus \{y\}\}$. Since $Y \in M^p$, we get $\bigvee B_y \parallel \bigvee A_x$ for all $x \in A_y$.

It follows from $V_Y = \{B, C\}$ and $B = M(Y)$ that $C = \{c\} \cup B_z$ for some $z \in B$, where $c < z$ and $c \parallel x$ for all $x \in B_z$.

Since $Z = J(C)$, it is obvious that $Z = \{\bigvee C_q \mid q \in C\}$. It follows from $U_C = \{Y, Z\}$ that $|Y \cap Z| = p - 1$. Let us prove that $X \in U_C$ by assigning a mapping γ of the set $J(C) = Z$ onto the set X (from Proposition 5). We examine all particular cases.

1. Suppose that $z = b$. Then $c < b < a \leq \bigvee A_x$ for all $x \in A_a$ and $C = \{c\} \cup A_a$. Obviously $c \parallel \bigvee A_a$ and $\bigvee C_c = \bigvee A_a$. Moreover, $\bigvee C_y \leq \bigvee B_y < \bigvee A_y$ and $\bigvee C_x \leq \bigvee A_x$ for all $x \in A_a \setminus \{y\}$, where, since $|Y \cap Z| = p - 1$, precisely one inequality $\leq$ is replaced by the strict one. Thus, $Z = \{\bigvee A_a\} \cup \{\bigvee C_x \mid x \in A_a\}$. Consider a mapping $\gamma : Z \to X$ defined by setting $\gamma(\bigvee A_a) = \bigvee A_a$, $\gamma(\bigvee C_x) = \bigvee A_x$ for all $x \in A_a$. It is easy to see that $m \leq \gamma(m)$ for all $m \in Z$. We prove that $\gamma(m) \| n$ for all $n \in Z \setminus \{m\}$.

   a) Let $\bigvee C_y < \bigvee B_y$. Then $Z = \{\bigvee C_y\} \cup \{\bigvee A_x \mid x \in A_y\}$. It suffices to show that $\bigvee C_y \| \bigvee A_q$ for $q \in A_y$. Let $\bigvee C_y \leq \bigvee A_a$. Then, from $c \leq \bigvee C_y$, we get $c \leq \bigvee A_a$, which is a contradiction. Let $\bigvee C_y \leq \bigvee A_x$ for $x \in A_a \setminus \{y\}$. Then $x \leq \bigvee C_y$, which is a contradiction again.

   b) Let $\bigvee C_q < \bigvee A_q$ for a certain $q \in A_a \setminus \{y\}$. Then $Z = \{\bigvee B_y\} \cup \{\bigvee C_q\} \cup \{\bigvee A_x \mid x \in A \setminus \{q, y\}\}$. It suffices to show that $\bigvee C_q \| \bigvee A_x$ for $x \in A_q$. Suppose that $\bigvee C_q \leq \bigvee A_a$. Then, from $c \leq \bigvee C_q$, we get $c \leq \bigvee A_a$, which is a contradiction. If $\bigvee C_q \leq \bigvee A_x$ for $x \in A_a \setminus \{q\}$, then we obtain a contradiction again, because of $x \leq \bigvee C_q$.

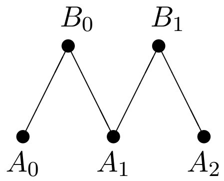
2. Let $z = y$. Then $c \| \bigvee A_y$ and $\bigvee C_c = \bigvee B_y < \bigvee A_y$, $\bigvee C_b \leq \bigvee A_a$, $\bigvee C_x \leq \bigvee A_x$ for all $x \in A_a \smallsetminus \{y\}$. It is easy to see that $Z = \{\bigvee B_y\} \cup \{\bigvee C_q \mid q \in B_y\}$. The mapping γ is defined by setting $\gamma(\bigvee B_y) = \bigvee A_y$, $\gamma(\bigvee C_b) = \bigvee A_a$, $\gamma(\bigvee C_x) = \bigvee A_x$ for $x \in A_a \smallsetminus \{y\}$. Further, we proceed similarly to the case 1.
   a) Let $\bigvee C_b < \bigvee A_a$. Then $Z = \{\bigvee B_y\} \cup \{\bigvee C_b\} \cup \{\bigvee A_x \mid x \in A_a \smallsetminus \{y\}\}$. If $\bigvee C_b \leq \bigvee A_y$, then $c \leq \bigvee C_b$ yields $c \leq \bigvee A_y$, which is a contradiction. If $\bigvee C_b \leq \bigvee A_x$ for $x \in A \smallsetminus \{y\}$, then $x \in \bigvee A_x$.
   b) Let $\bigvee C_q < \bigvee A_q$ for a certain $q \in A_a \smallsetminus \{y\}$. Then $Z = \{\bigvee B_y\} \cup \{\bigvee C_q\} \cup \{\bigvee A_x \mid x \in B \smallsetminus \{q, y\}\}$. Similarly to the preceding case, we show that $\bigvee C_x \| \bigvee A_x$ for $x \in A_q$.
3. Let $z \in A_a \smallsetminus \{y\}$. Then $c \| \bigvee A_z$ and $\bigvee C_c = \bigvee B_z = \bigvee A_z$, $\bigvee C_b \leq \bigvee A_a$, $\bigvee C_y \leq \bigvee B_y < \bigvee A_y$ and $\bigvee C_x \leq \bigvee A_x$ for remaining $x \in A$. Let us put $\gamma(\bigvee C_c) = \bigvee A_z$, $\gamma(\bigvee C_b) = \bigvee A_a$, $\gamma(\bigvee C_y) = \bigvee A_y$ and $\gamma(\bigvee C_x) = \bigvee A_x$ for remaining $x \in A$.
   a) Let $\bigvee C_b < \bigvee A_a$. If $\bigvee C_b \leq \bigvee A_z$, then $c \leq \bigvee A_z$, which is a contradiction. For $x \in A_a \smallsetminus \{z\}$, it follows from $\bigvee C_b \leq \bigvee A_x$ that $x \leq \bigvee A_x$.
   b) Let $\bigvee C_y < \bigvee B_y$. Then $\bigvee C_y \leq \bigvee A_a$ implies $b \leq \bigvee A_a$, $\bigvee C_y \leq \bigvee A_z$ implies $c \leq \bigvee A_z$, and for remaining $x \in A$, we get $x \leq \bigvee A_x$, which is a contradiction in all cases.
   c) Let $\bigvee C_q < \bigvee A_q$ for $q \in A_a \smallsetminus \{y, z\}$. Similarly to the preceding cases, we show that $\bigvee C_x \| \bigvee A_x$ for $x \in A_q$.

Thus, we have obtained $X \in U_C$, which contradicts our assumption $U_C = \{Y, Z\}$. ∎

Remark 5. The dual statement also holds, where $V_X = \{A\}$, $V_Y = \{A, B\}$, $V_Z = \{B, C\}$ and $U_A = \{X, Y\}$, $U_B = \{Y, Z\}$.

Corrolary 2. *Every simple connection (of type 1, 1', 2, 2') of the length greater than 1 is a lattice-inadmissible incidence structure.*

Proof. Consider a complete lattice $(L, \leq)$. Let $\mathcal{J}_L^p = (G^p, M^p, I^p)$ be a simple connection of type 1 and of the length 2. Thus, its graph can be sketched as follows:



Obviously, $B_0 = J(A_0)$. If $B_0 = J(A_1)$, then $U_{A_0} = U_{A_1}$, which is a contradiction. Hence, $B_1 = J(A_1)$. However, it means that $B_1 = J(A_2)$, which is a contradiction again. Dually, we can proceed for any simple connection of type 1' and of the length 2.

Consider a simple connection $\mathcal{J}_L^p$ of type 1 and of the length greater than 2 or a simple connection of type 2 and of the length at least 2. Then $\mathcal{J}_L^p$ contains sets $A_0, A_1, A_2 \in G^p$ and $B_0, B_1, B_2 \in M^p$ such that $U_{A_0} = \{B_0\}$, $U_{A_1} = \{B_0, B_1\}$, $U_{A_2} = \{B_1, B_2\}$, $V_{B_0} = \{A_0, A_1\}$, $V_{B_1} = \{A_1, A_2\}$. According to Theorem, such sets cannot exist. Similar assertion for simple connections of types 1', 2' holds dually. ■

Remark 6. Simple connections of the length 1 are lattice-admissible incidence structures (refer to [6] for an example of a simple connection of type 2).

Remark 7. There exists a complete lattices $(L, \leq)$ and a cardinal p such that the incidence structure $\mathcal{J}_L^p$ contains a simple connection of the length greater than 1 as its substructure.

There exist (general) incidence structures $\mathcal{J}$ such that their corresponding incidence structures $\mathcal{J}^p$ of independent sets are simple connections. In [8]–[10], there are such incidence structures $\mathcal{J}$ investigated that $\mathcal{J}^p$ are simple connections of type 1.

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