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SELECTIVE GENERALIZED F TESTS

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Abstract

Generalized F tests were introduced by Michalski and Zmyślony (1996) for variance components and later (1999) for linear functions of parameters in mixed linear models. We now use generalized polar coordinates to obtain, for the second case, tests that are more powerful for selected families of alternatives.

Keywords: mixed linear models, variance components, generalized polar coordinates, selective F testes.

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1. Introduction

Generalized F tests were introduced by Michalski and Zmyślony (1996), and (1999) first for variance components and later for linear combinations of parameters in mixed linear models.

We now use polar coordinates to derive generalized F tests with enhanced power for chosen alternatives. This treatment extends the one we presented for F tests, see Dias (1994) and Nunes and Mexia (2003).

Our results apply to mixed models satisfying assumptions discussed in the next section. In that section we will also consider the use of polar coordinates in selecting alternatives. In the final section, we discuss the use of polar coordinates in connection with generalized F statistics.

2. Models and hypothesis

Superscripts will indicate vector dimensions, writing $Z^n \sim N(\nu^u, W)$ when Z^n is normal with mean vector ν^u and variance-covariance matrix W. Moreover, $V \sim \sigma^2\chi^2_{g,\delta}$ indicates the product by σ^2 of a chi-square with g degrees of freedom and non-centrality parameters δ. When $\delta = 0$ we write simply $V \sim \sigma^2\chi^2_g$.

We will assume that there are mutually orthogonal matrices $Q_j = A_j^\top A_j$, $j = 1, ..., J$, and that, with Y^n the observations vector, we have the independent vectors such that

$$(1)\qquad \begin{cases} A_1Y^n \sim N\Big(\eta_1^{g_1}, \gamma_1 I_{g_1}\Big) \\ \\ A_jY^n \sim N\Big(0^{g_j}, \gamma_j I_{g_j}\Big);\ j = 2, ..., J \end{cases}$$

where

$$(2)\qquad \gamma_1 = \sum_{j=2}^{J} a_j\gamma_j$$

with $a_j \leq 0$, $j = 2, ..., r$, and $a_j \geq 0$, $j = r+1, ..., r+s = J$.

Thus we have, the independent statistics

$$(3)\qquad V_1 = \|A_1Y^n\|^2 \sim \gamma_1\chi^2_{g_1,\delta_1}$$

with

$$(4)\qquad \delta_1 = \frac{1}{\gamma_1}\|\eta_1^{g_1}\|^2$$

and

$$(5)\qquad V_j = \|A_jY^n\|^2 \sim \gamma_j\chi^2_{g_j};\ j = 2, ..., r+s.$$

Now, the hypothesis

$$H_0 : \eta_1^{g_1} = 0^{g_1} \tag{6}$$

holds if and only if $\delta_1 = 0$, and since

$$\begin{cases} E\Big(V_1\Big) = \gamma_1\Big(g_1 + \delta_1\Big) \\ \\ E\Big(V_j\Big) = \gamma_j g_j; \;\; j = 2, ..., \; r + s \end{cases} \tag{7}$$

if and only if

$$E\left(\frac{V_1}{g_1} - \sum_{j=2}^{r+s} a_j \frac{V_j}{g_j}\right) = 0. \tag{8}$$

We can thus, following Michalski and Zmyślony (1996) and (1999), consider for H_0 a generalized F test statistic

$$\Im = \frac{\frac{V_1}{g_1} + \sum_{j=2}^{r} |a_j| \frac{V_j}{g_j}}{\sum_{j=r+1}^{r+s} a_j \frac{V_j}{g_j}}, \tag{9}$$

the numerator [denominator] of this statistics being the positive [negative] part of a quadratic unbiased estimator.

As in our previous paper, Nunes and Mexia (2002), polar coordinates (r, Θ^{g_1-1}) for $\eta_1^{g_1}$, will be used to specify selected alternatives for H_0. Namely we intend to enhance the test power for alternatives such that $\Theta^{g_1-1} \in \mathcal{C}$ with $\mathcal{C}$ a conveniently chosen domain. For instance $\mathcal{C}$ may correspond to certain conditions on the ordering of the components of $\eta_1^{g_1}$.

3. Generalized F statistics and polar coordinates

In defining the selected alternatives to H_0 we will use generalized polar coordinates, see Kendall (1961, p. 15–18). If $(u, \theta_1, ..., \theta_{g_1-1})$ are these coordinates we will have, for the components $\eta_1, ..., \eta_{g_1}$ of $\eta_1^{g_1}$

$$(10)\qquad \begin{cases} \eta_1 = uc_1...c_{g_1-1} \\ \vdots \\ \eta_i = uc_1...c_{g_1-i}s_{g_1-i+1};\ i=2,...,g_1-1 \\ \vdots \\ \eta_{g_1} = us_1 \end{cases}$$

with $u = \|\eta_1^{g_1}\|$, $c_i = cos\theta_i$ and $s_i = sen\theta_i$, $i = 1,...,g_1-1$. For the central angles we have the bounds

$$(11)\qquad \begin{cases} -\frac{\pi}{2} \le \theta_i \le \frac{\pi}{2};\ i=1,...,g_1-2 \\ 0 \le \theta_{g_1-1} \le 2\pi \end{cases}$$

which define the domain $\mathcal{D}$.

Selected alternatives will be those for which $\Theta^{g_1-1} \in \mathcal{C} \subset \mathcal{D}$, where $\mathcal{C}$ may be chosen from conditions on the ordering of the $\eta_1,...,\eta_{g_1}$.

For the components $Z_1,...,Z_{g_1}$ of Z^{g_1} we have, generalized polar coordinates $(V_1^{1/2},\Theta_1,...,\Theta_{g_1-1})$, with

$$(12)\qquad \begin{cases} Z_1 = V_1^{1/2}\ell_1\Big(\Theta^{g_1-1}\Big) = UC_1,...,C_{g_1-1} \\ \vdots \\ Z_i = V_1^{1/2}\ell_i\Big(\Theta^{g_1-1}\Big) = UC_1...C_{g_1-i}S_{g_1-i+1};\ i=2,...,g_1-1 \\ \vdots \\ Z_{g_1} = V_1^{1/2}\ell_{g_1}\Big(\Theta^{g_1-1}\Big) = US_1 \end{cases}$$

where $C_i = cos\Theta_i$ and $S_i = sen\Theta_i$, $i = 1,...,g_1$.

In carrying out selective tests our statistics will be the pair $(\Im, \Theta^{g_1-1})$.

To obtain distribution results we start by pointing that, see Kendall (1961, p. 16), the Jacobian of transformations (10) is $\frac{1}{2}v^{\frac{g_1}{2}-1}h(\theta^{g_1-1})$ with $h(\theta^{g_1-1}) = c_1^{g_1-2}...c_{g_1-2} \ge 0$.

Since the density of $Z^{g_1} = A_1 Y^n$ is

$$n\Big(z^{g_1}\Big) = \frac{e^{-\frac{1}{2\gamma_1}\|z^{g_1}-\eta_1^{g_1}\|^2}}{(2\pi)^{g_1/2}\gamma_1^{g_1/2}} = \frac{e^{-\frac{1}{2\gamma_1}(\|z^{g_1}\|^2-2\eta_1^\top z+\|\eta_1^{g_1}\|^2)}}{(2\pi)^{g_1/2}\gamma_1^{g_1/2}}, \tag{13}$$

when we carry out the transformation for generalized polar coordinates we easily obtain for (V_1, Θ^{g_1-1}) the joint density

$$\begin{aligned}
&(14)\quad f^0\Big(v_1, \theta^{g_1-1}|\delta_1\Big)\\
&= \frac{e^{-\frac{\delta_1}{2}}}{(2\pi)^{g_1/2}\gamma_1^{g_1/2}} h\Big(\theta^{g_1-1}\Big)\frac{v_1^{-1/2}}{2}\sum_{i=0}^{+\infty}\frac{e^{-\frac{v_1}{2\gamma_1}}v_1^{\frac{g_1-1+i}{2}}}{i!}\left(\frac{\eta_1^\top \ell(\theta)}{\gamma_1}\right)^i\\
&= \frac{e^{-\frac{\delta_1}{2}}}{(\pi)^{g_1/2}}\sum_{i=0}^{+\infty}\frac{e^{-\frac{v_1}{2\gamma_1}}}{2\gamma_1\Gamma(\frac{g_1+i}{2})}\left(\frac{v_1}{2\gamma_1}\right)^{\frac{g_1+i}{2}-1}\frac{\Gamma(\frac{g_1+i}{2})2^{i/2-1}}{i!\gamma_1^{i/2}}\Big(\eta_1^\top \ell(\theta)\Big)^i h\Big(\theta^{g_1-1}\Big)\\
&= \frac{e^{-\frac{\delta_1}{2}}}{\pi^{g_1/2}}\sum_{i=0}^{+\infty}\frac{1}{\gamma_1}g\left(\frac{v_1}{\gamma_1}|g_1+i\right)m_i\Big(\theta^{g_1-1}\Big)
\end{aligned}$$

where $m_i(\theta^{g_1-1}) = \frac{\Gamma(\frac{g_1+i}{2})2^{i/2-1}}{i!\gamma_1^{i/2}} (\eta_1^\top \ell(\theta^{g_1-1}))^i \ h(\theta^{g_1-1})$, $i = 0, \ldots$ and $g(x|g_1+i)$ is the density of a central chi-square with g_1+i degrees of freedom. Now, when H_0 holds $\delta_1 = 0$ as well as $m_i(\theta^{g_1-1}) = 0$, $i = 1, \ldots$, and so

$$f^0\Big(v_1, \theta^{g_1-1}|0\Big) = \frac{1}{\pi^{g_1/2}}\frac{1}{\gamma_1}g\left(\frac{v_1}{\gamma_1}|g_1\right)\frac{\Gamma(\frac{g_1}{2})}{2}h\Big(\theta^{g_1-1}\Big) \tag{15}$$

thus, when H_0 holds, V_1 and Θ^{g_1-1} are independent.

Nextly, since the $|a_j|V_j$, $j = 2, ..., r+s$, have densities $\frac{1}{|a_j|\gamma_j} g(\frac{v_j}{|a_j|\gamma_j}|g_j)$, $j = 2, ..., r+s$, the joint density of $\Theta^{g_1-1}, V_1, ..., V_{r+s}$ will be

(16)
$$f^0\Big(v^{r+s}, \theta^{g_1-1}|\delta_1\Big)$$

$$= f^0\Big(v_1, \theta^{g_1-1}|\delta_1\Big) \prod_{i=2}^{r+s} \frac{1}{|a_i|\gamma_i} g\left(\frac{v_i}{|a_i|\gamma_i}|g_i\right)$$

$$= \frac{e^{-\frac{\delta_1}{2}}}{\pi^{g_1/2}} \sum_{i=0}^{+\infty} m_i\Big(\theta^{g_1-1}\Big) \frac{1}{\gamma_1} g\left(\frac{v}{\gamma_1}|g_1 + i\right) \prod_{i=2}^{r+s} \frac{1}{|a_i|\gamma_i} g\left(\frac{v_i}{|a_i|\gamma_i}|g_i\right).$$

When H_0 holds we have, with $a_1 = 1$,

(17)
$$f^0\Big(v^{r+s}, \theta^{g_1-1}|0\Big)$$

$$= \frac{\Gamma(\frac{g_1}{2})}{2\pi^{g_1/2}} \prod_{j=1}^{r+s} \frac{1}{|a_j|\gamma_j} g\left(\frac{v_j}{|a_j|\gamma_j}|g_j\right) h\Big(\theta^{g_1-1}\Big)$$

$$= \left[\frac{1}{\pi^{g_1/2}} \prod_{j=1}^{r+s} \frac{1}{|a_j|\gamma_j} g\left(\frac{v_j}{|a_j|\gamma_j}|g_j\right)\right] m_0\Big(\theta^{g_1-1}\Big).$$

We thus established

Proposition 1. *When H_0 holds $\Im$ is independent from Θ^{g_1-1} which has density $m_0(\theta^{g_1-1})$.*

Proof. The thesis follows from (17) observing that only $\Im$ depends on the $|a_j|V_j$, $j = 1, ..., r+s$. ■

Thus, to test H_0 having selected the alternatives for which $\Theta^{g_1-1} \in \mathcal{C}$ we can reject H_0 whenever $\Theta^{g_1-1} \in \mathcal{C}$ and $\Im > k$. The test level will be the product of $p(\Theta^{g_1-1} \in \mathcal{C}|H_0)$ by $p(\Im > k|H_0)$ since, when H_0 holds, both statistics are independent.

When $\mathcal{C}$ is given by conditions on the ordering of the $\eta_1, ..., \eta_{g_1}$ it is usually simple to use these in order to compute $p(\Theta^{g_1-1} \in \mathcal{C}|H_0)$, for instance if we require $\eta_1 = min\{\eta_1, ..., \eta_{g_1}\}$, we have

$$p\Big(\Theta^{g_1-1} \in \mathcal{C}|H_0\Big) = \frac{1}{g_1}. \tag{18}$$

As for the second factor we point out that, when H_0 holds, $\Im$ is the quotient of two linear combinations $\sum_{j=1}^{r} b_j\chi^2_{g_j}$ and $\sum_{j=r+1}^{r+s} b_j\chi^2_{g_j}$ with $b_j = \frac{|a_j|}{g_j}$, $j = 1, ..., r+s$. Now, Fonseca *et al.* (2002), gave explicit expressions for the distribution of such quotients when all degrees of freedom in the numerator or in the denominator are even. They also showed how to use Monte-Carlo methods in the general case.

For a q level test we will have

$$q = p\Big(\Theta^{g_1-1} \in \mathcal{C}|H_0\Big)p\Big(\Im > k|H_0\Big) \tag{19}$$

so that, when $p(\Theta^{g_1-1} \in \mathcal{C}|H_0)$ is small, we are led to choose k small. Thus a considerable increase of power for selected alternatives is to be expected.

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