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ON NEUMANN BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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Abstract

We provide two existence results for the nonlinear Neumann problem

$$\begin{cases} -div(a(x)\nabla u(x)) = f(x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, a is a weight function and f a nonlinear perturbation. Our approach is variational in character.

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1. Introduction and results

In this paper, we deal with problems of the form

$$(*) \qquad \begin{cases} -div(a(x)\nabla u(x)) = f(x,u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a C^1 boundary $\partial\Omega$, $f(.,.)$ is a Carathéodory function and $a(.)$ is a positive weight on Ω. Our work is motivated by the results in [1] and [2] concerning the Dirichlet problem. We provide two existence results for $(*)$, the first for a Carathéodory function f with sublinear growth at infinity and the other for a continuous function f which is independent of the space variable. We refer to [6] for a similar result but with a different behavior of f at infinity and to [4] for an unbounded domain Ω.

For the first existence result we make the following assumptions:

$H(a)$ the weight function $a : \Omega \to \mathbb{R}$ is positive a.e. in $x \in \Omega$ and $a,\, a^{-s} \in L^1(\Omega)$ where $s > \frac{N}{2}$.

$H(f)$ $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (that is, $f(x,u)$ is measurable in x for every u in $\mathbb{R}$ and continuous in u for almost every $x \in \Omega$) such that

(i) $|f(x,u)| \le A$, $A \in \mathbb{R}$, for every $u \in \mathbb{R}$ and almost every $x \in \Omega$.

(ii) $\lim\limits_{u\to\pm\infty} f(x,u) sign u = f^+(x)$, where $f^+ \in L^\infty(\Omega)$, $f^+ \ge 0$, with a strict inequality holding in a set of positive measure.

(iii) $\limsup\limits_{u\to 0} \frac{F(x,u)}{|u|^2} \le \theta(x)$ uniformly in x for almost every $x \in \Omega$, where $\theta \in L^\infty(\Omega)$, $\theta(x) \le 0$ with a strict inequality holding in a set of positive measure.

Remark. Hypothesis $H(a)$ implies that the space $H^1(\Omega,a) = \{u \in L^2(\Omega) : \int_\Omega a(x)|Du|^2 dx < +\infty\}$ supplied with the norm

$$||u|| = \left(\int_\Omega a(x)\,|Du|^2\,dx + \int_\Omega |u|^2 dx \right)^{\frac{1}{2}}$$

is reflexive. For more details we refer to [5].

Consider the Euler-Lagrange functional associated with $(*)$,

$$\Phi(u) := \frac{1}{2} \int_\Omega a(x)\,|\nabla u|^2 - \int_\Omega F(x,u)dx,$$

where

$$F(x,u) := \int_0^u f(x,t)dt.$$

It is well known that if the growth of $f(.,.)$ is up to critical, then $\Phi(.)$ is a well defined C^1 functional on $H^1(\Omega,a)$.

We need two auxiliary lemmas.

Lemma 1. $\Phi(.)$ *satisfies the Palais-Smale condition.*

Proof. Suppose not. Then, there exists a sequence $\{u_n\}_{n\in N}$ in $H^1(\Omega, a)$ such that $|\Phi(u_n)| \leq c$, $c \in \mathbb{R}$, and $\Phi'(u_n) \to 0$ and $\|u_n\| \to +\infty$. Let $y_n = \frac{u_n}{\|u_n\|}$. By passing to a subsequence if necessary, we may assume that $y_n \to y$ weakly in $H^1(\Omega, a)$, $y_n \to y$ strongly in $L^2(\Omega)$ and $y_n(x) \to y(x)$ a.e. Since $|\Phi(u_n)| \leq c$ we have that

$$\left| \frac{1}{2} \int_\Omega a(x) \left| Dy_n \right|^2 dx - \frac{1}{\left\| u_n \right\|^2} \int_\Omega \int_0^{u_n} f(x,s) ds dx \right| \leq \frac{c}{\left\| u_n \right\|^2}.$$

By the Sobolev embedding

$$\left| \int_\Omega \int_0^{u_n} f(x,s) ds dx \right| \leq A \int_\Omega |u_n| \, dx \leq c_1 \left\| u_n \right\|_2 \leq c_2 \left\| u_n \right\|,$$

c_1, $c_2 \in \mathbb{R}$. Therefore

$$\int_\Omega a(x) \left| Dy_n \right|^2 \to 0.$$

Exploiting the lower semicontinuity of the norm of $H^1(\Omega, a)$ we deduce that

$$\int_\Omega a(x) \left| Dy \right|^2 dx = 0,$$

so $y = \xi, \;\; \xi \neq 0$. Consequently, $|u_n(x)| \to +\infty$ a.e. in Ω. Since $\Phi'(u_n) \to 0$, there exists a decreasing sequence $\{\varepsilon_n\}_{n\in N}$ of positive real numbers such that $\varepsilon_n \to 0$ and

$$\left\langle \Phi'(u_n), v \right\rangle \leq \varepsilon_n \left\| v \right\| \tag{1}$$

for every $n \in N$ and every $v \in H^1(\Omega, a)$. By taking $v = u_n$ and dividing (1) by $\|u_n\|$ we get

$$\left| \int_\Omega a(x) \left| Dy_n \right|^2 dx - \int_\Omega \frac{f(x, u_n) u_n}{\left\| u_n \right\|} dx \right| \leq \varepsilon_n.$$

So

$$
\begin{aligned}
0 &= \liminf \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|}dx = \liminf \int_\Omega f(x,u_n) sign u_n \frac{|u_n|}{\|u_n\|}dx \\
&\geq \int_\Omega f^+ |\xi| \, dx, \qquad (2)
\end{aligned}
$$

a contradiction. Therefore the sequence $\{u_n\}_{n\in N}$ is bounded. So there exists $u \in H^1(\Omega, a)$ such that, up to a subsequence, $u_n \to u$ weakly in $H^1(\Omega, a)$, $u_n \to u$ strongly in $L^2(\Omega)$ and $u_n(x) \to u(x)$ a.e. By taking $v = u_n - u$ in (1) we get

$$
\left| \int_\Omega a(x) Du_n (Du_n - Du) dx - \int_\Omega f(x,u_n)(u_n - u) dx \right| \leq \varepsilon_n \|u_n - u\| .
$$

Since $f(\cdot,\cdot)$ is bounded $\int_\Omega f(x,u_n)(u_n - u)dx \to 0$, and consequently $\int_\Omega a(x)Du_n(Du_n - Du)dx \to 0$ as $n \to +\infty$. Thus

$$
\begin{aligned}
&\int_\Omega a(x)|Du_n - Du|^2 dx \\
&= \int_\Omega a(x) Du_n (Du_n - Du) dx - \int_\Omega a(x) Du (Du_n - Du) dx \to 0,
\end{aligned}
$$

so $u_n \to u$ strongly in $H^1(\Omega, a)$. ■

Lemma 2. *There exist $\rho, \eta > 0$ such that $\Phi(u) > \eta$ for every $u \in H^1(\Omega, a)$ with $\|u\| = \rho$.*

Proof. We will show that if $\|u_n\| = \rho_n \downarrow 0$, then $\Phi(u) > 0$. For if this is not true, then there exists a sequence $\{u_n\}_{n\in N}$ such that $\|u_n\| = \rho_n \downarrow 0$ and $\Phi(u) \leq 0$. Thus

$$
\frac{1}{2} \int_\Omega a(x) |Du_n|^2 dx - \int_\Omega \int_0^{u_n} f(x,s) ds dx \leq 0.
$$

Dividing with $\|u_n\|^2$ we get

$$
\left| \frac{1}{2} \int_\Omega a(x) |Dy_n|^2 - \frac{1}{\|u_n\|^2} \int_\Omega \int_0^{u_n} f(x,s) ds dx \right| \leq 0, \qquad (3)
$$

where $y_n = \frac{u_n}{\|u_n\|}$. Note that, because of $H(f)$(iii), for $\varepsilon > 0$ there exists $\delta > 0$ such that if $|u| < \delta$, then $F(x,u) \leq (\theta(x)+\varepsilon)|u|^2$. Also, $H(f)$(i) implies that $|F(x,u)| \leq A|u|$ a.e. in $x \in \Omega$, for every $u \in \mathbb{R}$. Thus,

$$|F(x,u)| \leq (\theta(x)+\varepsilon)\,|u|^2 + \beta\,|u|^{2^*}\,, \tag{4}$$

where $\beta \geq A\delta^{1-2^*} - (\|\theta\|_\infty + \varepsilon)\delta^{2-2^*}$. From [3] and [4] we deduce that

$$\begin{aligned} \frac{1}{2}\int_\Omega a(x)\,|Dy_n|^2 &\leq \int_\Omega (\theta(x)+\varepsilon)\,|y_n|^2\,dx + \frac{\beta}{\|u_n\|^2}\int_\Omega |u|^{2^*}\,dx \\ &\leq \int_\Omega (\theta(x)+\varepsilon)\,|y_n|^2\,dx + \beta\,\|u_n\|^{2^*-2}\,, \end{aligned} \tag{5}$$

which, in view of $H(f)$(iii), implies that $\|Dy_n\|_2 \to 0$. Since the sequence $\{y_n\}_{n\in N}$ is bounded in $H^1(\Omega,a)$, there exists $y \in H^1(\Omega,a)$ such that $y_n \to y$ weakly in $H^1(\Omega,a)$. Therefore $\|Dy\|_2 = 0$, i.e., $y(x) = \kappa \in \mathbb{R}$, $\kappa \neq 0$. But then (5) implies that

$$\int_\Omega (\theta(x)+\varepsilon)dx \geq 0 \text{ for every } \varepsilon > 0,$$

a contradiction. ■

We can now state our first existence result.

Theorem 1. *Assume that hypotheses $H(a)$ and $H(f)$ are satisfied Then problem $(*)$ has a solution.*

Proof. We intend to use the mountain pass theorem [7]. In view of the above Lemmas, it remains to show that there exists a point $e \in H^1(\Omega,a)$ such that $\Phi(e) < 0$. Note that if we take $u_\beta(x) := \beta \in \mathbb{R}$ for every $x \in \Omega$, then

$$\Phi(u_\beta) = -\int_\Omega \int_0^{u_\beta} f(x,s)dsdx = -\int_\Omega \int_0^{\beta} f(x,s)dsdx \to -\infty$$

as $\beta \to +\infty$ because of $H(f)$(ii) and the result follows by taking β large enough. ■

If we assume that f depends only on u, then we can remove the hypothesis on its growth. The proof of this result is inspired by [2]. So consider the problem

$$(**)\qquad \begin{cases} -div(a(x)\nabla u) = f(u) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We make the following assumptions:

$Hc(a)$ $a : \mathbb{R} \to \mathbb{R}$ is a function in $L^\infty(\Omega)$ such that $a(x) \geq \sigma > 0$ a.e. in $x \in \Omega$.

$Hc(f)$ $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

(i) $\lim\limits_{|u|\to+\infty} \frac{f(u)}{u} = 0$

(ii) $B = \limsup\limits_{u\to-\infty} G(u) < 0$ and $\Gamma = \liminf\limits_{u\to+\infty} G(u) > 0$, where

$$G(u) = \begin{cases} \dfrac{2}{u}\displaystyle\int_0^u f(s)ds - f(u) & \text{if } u \neq 0 \\ f(0) & \text{if } u = 0. \end{cases}$$

Lemma 3. *Assume that hypotheses $Hc(a)$ and $Hc(f)$ are satisfied. Then $\Phi(.)$ satisfies the Palais-Smale condition.*

Proof. Suppose that $\{u_n\}_{n\in N}$ is a sequence in $H^1(\Omega, a)$ such that $|\Phi(u_n)| \leq c$, $c \in \mathbb{R}$, and $\Phi'(u_n) \to 0$. As in the proof of Lemma 1 we will show first that $\{u_n\}_{n\in N}$ is bounded. So assume that $\|u_n\| \to +\infty$ and let $y_n = \frac{u_n}{\|u_n\|}$. By passing to a subsequence if necessary, we may assume that $y_n \to y$ weakly in $H^1(\Omega, a)$, $y_n \to y$ strongly in $L^2(\Omega)$ and $y_n(x) \to y(x)$ a.e. Since $|\Phi(u_n)| \leq c$ we have that

$$\left| \frac{1}{2}\int_\Omega a(x)\left|Du_n\right|^2 dx - \int_\Omega \int_0^{u_n} f(s)dsdx \right| \leq c.$$

By dividing this inequality with $\|u_n\|^2$ we get

$$\left| \frac{1}{2}\int_\Omega a(x)\left|Dy_n\right|^2 dx - \frac{1}{\|u_n\|^2}\int_\Omega \int_0^{u_n} f(s)dsdx \right| \leq \frac{c}{\|u_n\|^2}.$$

In view of $Hc(f)$(i),

$$\int_\Omega a(x)\,|Dy_n|^2 \to 0 \text{ as} n \to +\infty,$$

which implies that $y = \xi \in \mathbb{R}$, $\xi \neq 0$. Consequently, $|u_n(x)| \to +\infty$ as $n \to +\infty$ a.e. in Ω. So for ε, $\delta > 0$, by Egoroff's theorem, there exists a measurable subset Σ of Ω and $n_0 \in N$ such that

$$\mu(\Omega\backslash\Sigma) < \delta \text{ and } |y_n(x) - \xi| < \varepsilon \text{ for } x \in \Sigma \text{ and } n > n_0. \tag{6}$$

Hence, for any $\zeta \in \mathbb{R}$, we have

$$\begin{aligned}
&\mu\{x \in \Omega : |u_n(x)| \leq \zeta\} \\
&\quad = \mu\{x \in \Omega\backslash\Sigma : |u_n(x)| \leq \zeta\} + \mu\{x \in \Sigma : |u_n(x)| \leq \zeta\} \\
&\quad \leq \delta + \mu\{x \in \Sigma : |u_n(x)| \leq \zeta\},
\end{aligned}$$

which combined with [6] yields

$$\lim_{n\to+\infty} \mu\{x \in \Omega : |u_n(x)| \leq \zeta\} = 0.$$

Because of our hypotheses on $\{u_n\}_{n\in N}$ there holds

$$\lim_{n\to+\infty} \frac{\langle \Phi'(u_n), u_n\rangle - 2\Phi(u_n)}{\|u_n\|} = 0.$$

Thus

$$\lim_{n\to+\infty} \frac{2\int_\Omega \int_0^{u_n} f(s)dsdx - \int_\Omega f(u_n)u_n}{\|u_n\|} dx$$

$$= \lim_{n\to+\infty} \int_\Omega \left(\frac{2}{u_n} \int_0^{u_n} f(s)ds - f(u_n) \right) \frac{u_n}{\|u_n\|} dx = \lim_{n\to+\infty} \int_\Omega G(u_n) \frac{u_n}{\|u_n\|} dx = 0.$$

Since G is continuous, for $\varepsilon > 0$ small enough there exists $\zeta > 0$ and $\eta > 0$

such that

$$G(u) \geq \Gamma - \varepsilon = \Gamma_\varepsilon \text{ if } u > \zeta, G(u) \leq B + \varepsilon = B_\varepsilon \text{ if } u < -\zeta$$
$$|G(u)| \leq \eta \text{ if } |u| \leq \zeta,$$

where

$$B_\varepsilon = \begin{cases} \limsup\limits_{u \to -\infty} G(u) - \varepsilon & if \;\; \limsup\limits_{u \to -\infty} G(u) > -\infty \\ -\frac{1}{\varepsilon} & \text{otherwise,} \end{cases}$$

and

$$\Gamma_\varepsilon = \begin{cases} \liminf\limits_{u \to +\infty} G(u) + \varepsilon & if \;\; \liminf\limits_{u \to +\infty} G(u) < +\infty \\ \frac{1}{\varepsilon} & \text{otherwise.} \end{cases}$$

Assume first that $\xi > 0$. Note that

$$\int_\Omega G(u_n) \frac{u_n}{\|u_n\|} dx$$
$$= \int_{|u_n| \leq \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx + \int_{u_n > \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx + \int_{u_n < -\zeta} G(u_n) \frac{u_n}{\|u_n\|} dx.$$

Since

$$\left| \int_{|u_n| \leq \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \right| \leq \frac{\eta \zeta \mu \{x \in \Omega : |u_n(x)| \leq \zeta\}}{\|u_n\|} \to 0,$$

$$\liminf_{n \to +\infty} \int_{u_n < -\zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \geq 0$$

and

$$\liminf_{n \to +\infty} \int_{u_n > \zeta} G(u_n) \frac{u_n}{\|u_n\|} dx \geq \int_{u_n > \zeta} \liminf_{n \to +\infty} G(u_n) \frac{u_n}{\|u_n\|} dx$$
$$= \liminf_{u \to +\infty} G(u) \mu(\Omega) \xi > 0,$$

we get

$$0 = \liminf_{n\to+\infty} \int_\Omega G(u_n)\frac{u_n}{\|u_n\|}dx \geq \liminf_{n\to+\infty} \int_{u_n>\zeta} G(u_n)\frac{u_n}{\|u_n\|}dx > 0,$$

a contradiction. Similarly for $\xi < 0$. Thus $\{u_n\}$ is bounded. We can now proceed as in the previous theorem. ■

We denote by X_1 the subspace of $H^1(\Omega, a)$ consisting of the constant functions and by $X_2 = \{u \in H^1(\Omega, a)$: there exists $v \in H^1(\Omega, a)$ such that $u(x) = v(x) - \frac{1}{\mu(\Omega)}\int_\Omega v d\mu\}$ its complement. Then $H^1(\Omega, a) = X_1 \oplus X_2$. For $u \in H^1(\Omega, a)$ let $\overline{u} = \frac{1}{\mu(\Omega)}\int_\Omega u d\mu$.

Lemma 4. (i) $\Phi(h) \to -\infty$ *as* $|h| \to +\infty$ *for* $h \in X_1$, *and*
(ii) Φ *is bounded from below in* X_2.

Proof. (i) Let us assume that there exists a sequence $\{\gamma_n\}_{n\in N}$ of real numbers such that $|\gamma_n| \to +\infty$ and $|\Phi(\gamma_n)| \leq c$ for some $c \in \mathbb{R}$. Suppose first that $\gamma_n \to +\infty$. As in [2] we can show that

$$\frac{F(u)}{u} \geq \Gamma_\varepsilon$$

for $u > \zeta$. Thus

$$\begin{aligned} 0 &= \limsup_{n\to\infty}\frac{\Phi(\gamma_n)}{\gamma_n} = -\liminf_{n\to\infty}\int_\Omega \frac{F(\gamma_n)}{\gamma_n}dx \\ &\leq -\int_\Omega \Gamma_\varepsilon dx, \end{aligned}$$

a contradiction. Similarly for $\gamma_n \to -\infty$. So (i) holds.
To prove (ii) we proceed as follows

$$\begin{aligned} &\Phi(u-\overline{u}) - \int_\Omega a(x)\left|\nabla u\right|^2 dx = -\int_\Omega\int_0^{u-\overline{u}} f(s)dsdx \\ &= -\int_\Omega\int_0^{\zeta} f(s)dsdx - \int_\Omega\int_\zeta^{u-\overline{u}} f(s)dsdx \\ &\geq C(\zeta,\eta) - \varepsilon\int_\Omega |u-\overline{u}|\,dx \ \ (\text{by } Hcf(\text{i})) \\ &\geq C(\zeta,\eta) - d_1\left\|u-\overline{u}\right\|_2, \end{aligned}$$

where $C(\zeta,\eta)$ is a constant which depends only on ζ, η and d_1 is a constant which depends only on ε and Ω. Thus, by the Poincare-Wirtinger inequality

$$\Phi(u-\overline{u}) \geq \int_\Omega a(x)\,|\nabla u|^2\,dx + C(\zeta,\eta) - d_2\left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2},$$

where d_2 is a constant which depends only on ε, a and Ω, proving (ii). ■

We can now apply the saddle point theorem, see [7], to show the following

Theorem 2. *Suppose that hypotheses $Hc(a)$ and $Hc(f)$ hold. Then $(**)$ has a solution.*

References

[1] D. Arcoya and L. Orsina, *Landesman-Laser conditions and quasilinear elliptic equations*, Nonlin. Anal. TMA **28** (1997), 1623–1632.

[2] J. Bouchala and P. Drabek, *Strong resonance for some quasilinear elliptic equations*, J. Math. Anal. Appl. **245** (2000), 7–19.

[3] P. Caldiroli and R. Musina, *On a variational degenerate elliptic problem*, NoDEA **7** (2000), 187–199.

[4] F. Cîrstea, D. Motreanu and V. Rădulescu, *Weak solutions of quasilinear problems with nonlinear boundary condition*, Nonlin. Anal. **43** (2001), 623–636.

[5] P. Drabek, A. Kufner and F. Nicolosi, Quasilinear Elliptic Equations with Degenerations and Singulaties, W. De Gruyter 1997.

[6] W. Li and H. Zhen, *The applications of sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator*, Nonlin. Anal. TMA **24** (2) (1995), 185–193.

[7] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, Amer. Math. Soc. Prividence, 1976.

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