Discussiones Mathematicae
Differential Inclusions, Control and Optimization 24 (2004) 5–11

SOLUTION OF FREDHOLM INTEGRODIFFERENTIAL EQUATION FOR AN INFINITE ELASTIC PLATE

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Abstract

Many authors discussed the problem of an elastic infinite plate with a curvilinear hole, some of them considered this problem in z-plane and the others in the s-plane. They obtained an exact expression for Goursat's functions for the first and second fundamental problem. In this paper, we use the Cauchy integral method to obtain a solution to the first and second fundamental problem by using a new transformation. Some applications are investigated and also some special cases are discussed.

Keywords: integrodifferential equation, Cauchy method, complex variable, infinite plate, curvilinear hole.

2000 Mathematics Subject Classification: 45B05, 45E10.

1. Introduction

Many authors discussed boundary value problems for isotropic homogeneous infinite plates. Some authors [4, 14, 15] used Lauran's theorem to express each complex potential function as a power series, others [1, 3, 5, 6, 8, 9, 10, 11] used the complex variable method of Cauchy's integrals.

From [13] we know that the first and second fundamental problem in the plane theory of elasticity are equivalent to finding two analytic functions ϕ(z) and ψ(z), these function must satisfy the boundary condition

$$k\phi(t) - \overline{t\phi'(t)} - \overline{\psi(t)} = f(t)\,. \tag{1.1}$$

We deal with the first fundamental problem when $k = -1$ and $f(t)$ is a given function of stresses; while in the case of the second fundamental problem when $k = \frac{\lambda+3\mu}{\lambda+\mu} > 1$, $f(t) = 2\mu g(t)$ is a given function of displacement where λ and μ are called Lame's constants.

Muskhellishvili in [13] studied the problem by using conformal mapping $z = cw(z)$, $c > 0$, and $w'(z)$ does not vanish or becomes ∞ for $|\xi| > 1$, then the infinite region is exterior to the unit circle γ. From [4, 13, 15] the two complex functions of potentials are

$$\phi(z)\frac{X+iY}{2\pi(1+\chi)}\ln\xi + c\Gamma\xi + \phi_0(\xi) \tag{1.2}$$

$$\psi(z)\frac{\chi(X+iY)}{2\pi(1+\chi)}\ln\xi + c\Gamma\xi + \psi_0(\xi) \tag{1.3}$$

and also the stress plane components are given by

$$\sigma_{xx} + \sigma_{yy} = 4Re\{\phi'(z)\}\,, \tag{1.4}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = 2\left[z\overline{\phi''(z)} + \overline{\psi'(z)}\right], \tag{1.5}$$

where X, Y being the components of the resultant vector of all external forces. $\Gamma*, \Gamma$ are constants and χ in the Muskhelishvilivi's constant.

2. Mapping function

Many different mapping functions were used in [1, 5, 8, 9, 10, 13] to solve the first and second fundamental problem of an infinite plate with a curvilinear hole. In [3], Abdou and El-Bary used a general transformation with n-poles to solve the same problem. In this paper, we use a more general transformation with many strong poles to solve the same boundary value problem but with different type of singularities as

$$z = cw(\xi) = \frac{\xi + m\xi^{-1}}{(1-n_1\xi^{-1})^2(1-n_2\xi^{-1})}, \qquad n_1 \neq n_2\,. \tag{2.1}$$

3. Method of solution

We use the mapping function (2.1), where m, n_1 and n_2 are real parameters. Now, if $z = cw(\xi)$, therefore $z' = cw'(\xi)$

$$\frac{z}{\bar{z}} = \frac{w(\xi)}{\overline{w'(\xi)}} = \alpha(\xi) + \overline{\beta(\xi)}, \tag{3.1}$$

where $\alpha(\xi)$ is a singular term, while $\beta(\xi)$ is a regular function for $|\xi| > 1$.

We can rewrite equation (3.1) as

$$\frac{w(\xi)}{\overline{w'(\xi)}} = \frac{h_1}{(n_1-\xi)^2} + \frac{h_2}{(n_2-\xi)} + \overline{\beta(\xi)}, \tag{3.2}$$

where h_1 as h_2 are constants and calculated as

$$h_1 = \frac{1}{a\sum_{i=1}^{5} L_i n_1^{i-1}}\left[ab\sum_{i=1}^{5} L_i n_1^{i-1} + c + d - Ea\sum_{i=1}^{4} iL_{i+1} n_1^{i-1} + \sum_{i=1}^{5} L_i n_1^{i-1}\right]$$

and

$$h_2 = \frac{(n_2^4 + mn_2^3)(1-n_1n_2^4)(1-n_2^2)^2}{a^2\sum_{i=1}^{5} L_i n_2^{i-1}},$$

where $L_1 = 1$,

$L_2 = -2(2n_1 + n_2)$,

$L_3 = -m(2n_1 + n_2) + 3n_1(2n_2 + n_1)$,

$L_4 = n_1^2(4m - n_2) + 8n_1n_2m$, and

$L_5 = -3mn_2n_1^2$.

and $a = n_1 - n_2$

$b = (4n_1^3 - 3mn_1^2)(1 - n_1^5)(1 - n_1n_2)^2$,

$c = -4n_1^4(n_1^4 + mn_1^3)(1 - n_1n_2)^2$,

$d = -2n_2(1 - n_1n_2)(1 - n_1^5)(n_1^4 + mn_1^3)$, and

$E = (n_1^4 + mn_1^3)(1 - n_1^5)(1 - n_1n_2)^2$.

By using (3.1) in (1.1) we have

$$k\phi(\sigma) - \frac{w(\sigma)}{\overline{w'(\sigma)}}\overline{\phi'(\sigma)} - \overline{\psi(\sigma)} = f(\sigma)\,. \tag{3.6}$$

Also, using (3.2) in (3.6) we obtain

$$k\phi(\sigma) - \alpha(\sigma)\overline{\phi'(\sigma)} - \psi_*(\sigma) = f_*(\sigma)\,, \tag{3.7}$$

where,

$$\begin{aligned}
&\psi_*(\sigma) = \psi(\xi) + \beta(\xi)\phi'(\xi),\\
&f_*(\xi) = F(\xi) - ck\Gamma\xi + c\overline{\Gamma^*}\xi^{-1} + N(\xi)\left[\alpha(\xi) + \overline{\beta(\xi)}\right],\\
&N(\xi) = C\Gamma - \frac{X - iY}{2\pi(1-\chi)}\xi, \text{ and}\\
&F(\xi) = f(t)\,.
\end{aligned}$$

Multiplying both sides of (3.7) by $\frac{d\sigma}{2\pi(\sigma-\xi)}$ and integrating on γ we get

$$\begin{aligned}
&k\phi(\xi) + \frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \xi}d\sigma\\
&= -c\overline{\Gamma^*}\xi^{-1} - \frac{h_1(X - iY)}{2\pi(1+\chi)(n_1 - \xi)} + \frac{h_1 N(n_1)}{(n_1-\xi)^2} + \frac{h_2 N(n_2)}{n_2 - \xi} - A(\xi)\,,
\end{aligned} \tag{3.8}$$

where $A(\xi) = -\frac{1}{2\pi i}\sum_{\eta=0}^{\infty}\xi^{-\eta-1}\int^n \sigma F(\sigma)d\sigma$.

Using [4, 5, 8, 13], we have

$$\frac{1}{2\pi i}\int_\gamma \frac{\alpha(\sigma)\overline{\phi'(\sigma)}}{\sigma - \xi}d\sigma = ch_1\left[\frac{b_1}{n_1 - \xi} + \frac{b_{11}}{(n_1-\xi)^2} + \frac{ch_2 b_2}{(n_2 - \xi)}\right], \tag{3.9}$$

where b_1, b_{11}, and b_2 are complex constants to be determined.

In [2, 7, 11, 12] authors solved some problems similar to (3.8) with different notations and different kernels.

Use (3.9) in (3.8), we get

$$\begin{aligned} -k\,\phi(\xi) &= A(\xi) + c\overline{\Gamma^*}\xi^{-1} + \frac{h_1}{n_1-\xi}\left[cb_1 + \frac{X-iY}{2\pi(1+\chi)}\right] \\ &+\frac{h_1}{(n_2-\xi)^2}\left[cb_{11}+N(n_1)\right] + \frac{h_2}{n_2-\xi}\left[cb_2+N(n_2)\right]. \end{aligned} \tag{3.10}$$

Differentiating (3.10) with respect to ξ, we have three equations in b_1, b_{11}, and b_2 and by solving this system we obtain

$$b_1 = \frac{1}{H^-}\left[2\nu_1 ReE_2 - \eta_1^+ ReE_1\right] + \frac{i}{H^+}\left[\eta_1^- ImE_1 + 2\nu_1 ImE_2\right]$$

$$b_{11} = \frac{1}{H^-}\left[\eta_2^+ ReE_2 - h_1\nu_1^2 ReE_1\right] + \frac{i}{H^+}\left[\eta_2^- ImE_2 - h_1\nu_1^2 ImE_1\right]$$

$$b_2 = \frac{kE_3 - \nu_2\, h_2\, \overline{E}_2}{c\,\left(k^2 - h_2^2\nu_2^2\right)}, \qquad \nu_j = \frac{n_j}{(1-n_j)^2} \qquad (j=1,2)\,, \tag{3.11}$$

where

$$E_1 = \overline{A''(n_1^{-1})} - 2n_1 c\Gamma^* + \frac{h_1 v(X+iY)}{2\pi(1+\chi)} - 3h_1 v\nu_1 N(n_1)$$

$$E_2 = \overline{A'(n_1^{-1})} + cn_1^2\Gamma^* - \frac{h_1 v(X+iY)}{2\pi(1+\chi)} - 2h_1\nu_1^3 N(n_1)$$

$$E_3 = \overline{A'(n_1^{-1})} + cn_2^2\Gamma^* - ch_2\nu_2 + \frac{h_2\nu_2 n_2(X+iY)}{2\pi(1+\chi)}$$

$$v = 2\nu_1\left[\frac{1}{n_1^2-1} - \nu_1^2\right].$$

Also

$$H^{\pm} = \frac{c}{k}(k \pm 2h_1 \ni_1^3)\left[k \pm \frac{2h_1\nu_1}{(1-n_1^2)}\right] \mp \frac{6ch_1^2\nu_1^3}{k(1-n_1^2)^2}$$

and

$$\eta_i^{\pm} = c\left[1 \pm \frac{2h_i}{k}\nu_i^2\right], \qquad (i = 1, 2)\,. \tag{3.12}$$

Finally, we have

$$-k\phi(\xi) = A(\xi) + c\Gamma^* \xi^{-1} + \frac{T_1}{n_1 - \xi} + \frac{T_2}{(n_1 - \xi)^2} + \frac{T_3}{n_2 - \xi}\,, \tag{3.13}$$

where

$$T_1 = h_1\left[cb_1 + \frac{X - iY}{2\pi(1+\chi)}\right], \qquad T_2 = h_1[cb_{11} + N(n_1)]$$

and

$$T_3 = h_2[cb_2 + N(n_2)]\,.$$

From the boundary condition we calculate $\psi(\xi)$ as

$$\begin{aligned} \psi(\xi) = {} & \frac{ck\overline{\Gamma}}{\xi} - \frac{w(\xi^{-1})}{w'(\xi)}\phi_*(\xi) - \frac{h_1\xi}{(1 - n_1\xi)}\phi_*(n_1^{-1}) \\ & + \frac{h_1\xi^2}{(1 - n_1\xi)^2}\phi_*(n_1^{-1}) - \frac{h_2\xi}{(1 - n_2\xi)}\phi_*(n_2^{-1}) + B(\xi) - B\,, \end{aligned} \tag{3.14}$$

where

$$\phi_*(\xi) = \phi'(\xi) + \overline{N(\xi)},$$

$$B(\xi) = \frac{1}{2\pi i}\int \frac{\overline{F(\sigma)}}{\sigma - \xi}\,d\sigma, \qquad \text{and}$$

$$B = \frac{1}{2\pi i}\int \frac{\overline{F(\sigma)}}{\sigma}\,d\sigma\,.$$

The stress components are calculated by using (3.13) and (3.14) in (1.4) and (1.5).

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Received 16 January 2004