

MULTI-VALUED OPERATORS AND FIXED POINT THEOREMS IN BANACH ALGEBRAS

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Abstract

In this paper, two multi-valued versions of the well-known hybrid fixed point theorem of Dhage [6] in Banach algebras are proved. As an application, an existence theorem for a certain differential inclusion in Banach algebras is also proved under the mixed Lipschitz and compactness type conditions.

Keywords: multi-valued operator, fixed point theorem and integral inclusion.

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1. Introduction

Let X be a Banach space and let $P(X)$ denote the class of all subsets of X , called the power set of X . Denote

$$P_f(X) = \{A \subset X \mid A \text{ is non-empty and has a property } f\}.$$

Thus $P_{bd}(X)$, $P_{cl}(X)$, $P_{cv}(X)$, $P_{cp}(X)$, $P_{cl,bd}(X)$, $P_{cp,cv}(X)$ denote respectively the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of X . Similarly, $P_{cl,cv,bd}(X)$ and $P_{cp,cv}(X)$ denote the classes of closed, convex and bounded and compact, convex subsets of X respectively. A mapping $T : X \rightarrow P_f(X)$ is called a multi-valued operator or a multi-valued mapping on X . A point $u \in X$ is called a fixed point

of T if $u \in Tu$. The multi-valued operator T is called lower semi-continuous (in short l.s.c.) if G is any open subset of X , then

$$T^{-1(w)}(G) = \{x \in X \mid Tx \cap G \neq \emptyset\}$$

is an open subset of X . Similarly, the multi-valued operator T is called upper semi-continuous (in short u.s.c.) if the set

$$T^{-1}(G) = \{x \in X \mid Tx \subset G\}$$

is open in X for every open set G in X . Finally T is called continuous if it is lower as well as upper semi-continuous on X . A multi-valued map $T : X \rightarrow P_{cp}(X)$ is called compact if $\overline{T(X)}$ is a compact subset of X . T is called totally bounded if for any bounded subset S of X , $T(S) = \bigcup_{x \in S} Tx$ is a totally bounded subset of X . It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However, the two notions are equivalent on a bounded subset of X . Finally T is called completely continuous if it is upper semi-continuous and totally bounded on X .

For any $A, B \in P_f(X)$, let us denote

$$A \pm B = \{a \pm b \mid a \in A, b \in B\}$$

$$A \cdot B = \{ab \mid a \in A, b \in B\}$$

$$\lambda A = \{\lambda a \mid a \in A\}$$

for $\lambda \in \mathbb{R}$. Similarly denote

$$|A| = \{|a| \mid a \in A\}$$

and

$$\|A\| = \sup\{|a| \mid a \in A\}.$$

Let $A, B \in P_{cl}(X)$ and let $a \in A$. Then by

$$D(a, B) = \inf\{\|a - b\| \mid b \in B\}$$

and

$$\rho(A, B) = \sup\{D(a, B) \mid a \in A\}.$$

The function $H : P_{cl}(X) \times P_{cl}(X) \rightarrow \mathbb{R}^+$ defined by

$$H(A, B) = \max\{\rho(A, B), \rho(B, A)\}$$

is metric and is called the Hausdorff metric on X . It is clear that

$$H(0, C) = \|C\| = \sup\{\|c\| \mid c \in C\}$$

for any $C \in P_{cl}(X)$.

Definition 1.1. Let $T : X \rightarrow P_{cl}(X)$ be a multi-valued operator. Then T is called a multi-valued Lipschitzian if there exists a constant $\lambda > 0$ such that for each $x, y \in X$ we have

$$H(T(x), T(y)) \leq \lambda \|x - y\|.$$

The constant λ is called a Lipschitz constant of T . In particular, if $\lambda < 1$, then T is called a multi-valued contraction with a contraction constant λ .

The following fixed point theorem for multi-valued contraction mappings appears in Covitz and Nadler [4].

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \rightarrow P_{cl}(X)$ be a multi-valued contraction. Then T has a fixed point.*

We shall be interested in the multi-valued analogues of the following hybrid fixed point theorem of the present author involving the product of two operators in Banach algebras.

Theorem 1.2 ([9]). *Let U and \bar{U} be respectively the open-bounded and closed-bounded subset of a Banach algebra X and let $A : X \rightarrow X$ and $B : \bar{U} \rightarrow X$ be two operators such that*

- (a) A is Lipschitz with a Lipschitz constant α ,
- (b) B is completely continuous,
- (c) $\alpha M < 1$, where $M = \sup\{\|Bx\| : x \in X\}$.

Then either

- (i) *the operator equation $AxBx = x$ has a solution, or*
- (ii) *there exists an $u \in \partial U$ such that $\lambda A\left(\frac{u}{\lambda}\right)Bu = u$ for some $0 < \lambda < 1$.*

Note that the above fixed point theorem involves the hypothesis of continuity of the operator T , however, in the case of multi-valued operators we have different types of continuities, namely, lower semi-continuity and upper semi-continuity, etc. In this work, we shall formulate the fixed point theorems for each of these continuity criteria. Some details of the topological fixed point theory in the multi-valued analysis appear in the monographs like Andres and Górniewicz [1], Górniewicz [11] and Granas and Dugundji [12]. Below we give some preliminaries of the multi-valued analysis which will be needed in the sequel.

2. Multi-valued fixed point theory

Before going to the main fixed point theorems, we state two lemmas useful in the sequel.

Lemma 2.1 ([17]). *Let (X, d) be a complete metric space and $T_1, T_2 : X \rightarrow P_{bd,cl}(X)$ be two multi-valued contractions with the same contraction constant k . Then*

$$\rho(\text{Fix}(T_1), \text{Fix}(T_2)) \leq \frac{1}{1-k} \sup_{x \in X} \rho(T_1(x), T_2(x)).$$

Lemma 2.2. *If $A, B \in P_{bd,cl}(X)$, then $H(AC, BC) \leq H(0, C)H(A, B)$.*

Proof. The proof appears in Dhage [8], but for the sake of completeness we give the details of it. Let $x \in AC$ and $y \in BC$ be arbitrary. Then there exist $a \in A, b \in B$, and $c_1, c_2 \in C$ such that $x = ac_1$ and $y = bc_2$. Now

$$\begin{aligned} D(x, BC) &= \inf\{\|x - y\| \mid y \in BC\} \\ &= \inf\{\|x - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|ac_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|ac_1 - bc_1\| + \|bc_1 - bc_2\| \mid b \in B, c_2 \in C_2\} \\ &\leq \inf\{\|a - b\| \|c_1\| + \|b\| \|c_1 - c_2\| \mid b \in B, c_2 \in C_2\} \\ &= \inf\{\|a - b\| \|c_1\| \mid b \in B\} \\ &= D(a, B)\|c_1\|. \end{aligned}$$

Again

$$\begin{aligned}
\rho(AC, BC) &= \sup\{D(x, BC) \mid x \in A\} \\
&= \sup\{D(a, B)\|c_1\| \mid a \in A, c_1 \in C\} \\
&\leq \sup\{D(a, B)\|C\| \mid a \in A\} \\
&= \rho(A, B)\|C\| \\
&= \rho(A, B)H(0, C).
\end{aligned}$$

Similarly

$$\rho(BC, AC) = \rho(B, A)H(0, C).$$

Hence

$$\begin{aligned}
H(AC, BC) &= \max\{\rho(AC, BC), \rho(BC, AC)\} \\
&\leq \max\{\rho(A, B)H(0, C), \rho(B, A)H(0, C)\} \\
&= H(0, C) \max\{\rho(A, B), \rho(B, A)\} \\
&= H(0, C)H(A, B).
\end{aligned}$$

The proof of the lemma is complete. ■

Now we state a key result which is useful in the sequel.

Theorem 2.1 ([21]). *Let S be a nonempty and closed subset of a Banach space X and let Y be a metric space. Assume that the multi-valued operator $F : S \times Y \rightarrow P_{cl,cv}(S)$ be a multi-valued mapping satisfying*

- (a) $H(F(x_1, y), F(x_2, y)) \leq k\|x_1 - x_2\|$, for each $(x_1, y), (x_2, y) \in S \times Y$,
- (b) for every $x \in S$, $F(x, \cdot)$ is lower semi-continuous (briefly l.s.c.) on Y .

Then there exists a continuous mapping $f : S \times Y \rightarrow S$ such that $f(x, y) \in F(f(x, y), y)$ for each $(x, y) \in S \times Y$.

Theorem 2.2. *Let U and \bar{U} be respectively the open-bounded and closed-bounded subsets of a Banach algebra X and let $A : X \rightarrow P_{bd,cl,cv}(X)$, $B : \bar{U} \rightarrow P_{cp,cv}(X)$ be two multi-valued operators such that*

- (a) A is multi-valued and Lipschitz with a Lipschitz constant k ,

- (b) B is l.s.c. and compact,
- (c) $AxB(y)$ is a convex subset of \bar{U} for each $x, y \in S$,
- (d) $Mk < 1$, where $M = \|B(\bar{U})\| = \sup\{\|B(x)\| \mid x \in \bar{U}\}$.

Then either

- (i) the operator inclusion $x \in AxBx$ has a solution, or
- (ii) there exists a $u \in \partial U$ such that $\lambda u \in A(\lambda u)Bu$ for some $\lambda > 1$, where ∂U is the boundary of U .

Proof. Define a multi-valued operator $T : X \times \bar{U} \rightarrow P_{cl,cv}(X)$ by

$$(2.1) \quad T(x, y) = AxBy,$$

for $x \in X$ and $y \in \bar{U}$. We show that $T(x, y)$ is multi-valued contraction in x for each fixed $y \in \bar{U}$. Let $x_1, x_2 \in X$ be arbitrary. Then by Lemma 2.2,

$$\begin{aligned} H(T(x_1, y), T(x_2, y)) &= H(A(x_1)B(y), A(x_2)B(y)) \\ &\leq H(A(x_1), A(x_2)) H(0, By) \\ &\leq k\|x_1 - x_2\| \|B(\bar{U})\| \\ &\leq kM\|x_1 - x_2\|. \end{aligned}$$

This shows that the multi-valued operator $T(\cdot, y)$ is a contraction on X with a contraction constant kM . Hence an application of Covitz-Nadler fixed point theorem yields that the fixed point set

$$Fix(T_y) = \{x \in X \mid x \in A(x)B(y)\}$$

is a nonempty and closed subset of \bar{U} for each $y \in \bar{U}$.

Now the operator $T(x, y)$ satisfies all the conditions of Theorem 2.1 and hence an application of it yields that there exists a continuous mapping $f : X \times \bar{U} \rightarrow \bar{U}$ such that $f(x, y) \in A(f(x, y))B(y)$. Let us define $C(y) = Fix(T_y)$, $C : \bar{U} \rightarrow P_{cl}(X)$. Let us consider the single-valued operator $c : \bar{U} \rightarrow X$ defined by $c(x) = f(x, x)$, for each $x \in \bar{U}$. Then c is a continuous

mapping having the property that

$$(2.2) \quad c(x) = f(x, x) \in A(f(x, x))B(x) = A(c(x))B(x),$$

for each $x \in \bar{U}$.

Now, we will prove that c is compact on \bar{U} . To do this, it is sufficient to show that C is compact on \bar{U} . Let $\epsilon > 0$. Since B is compact on \bar{U} , $B(\bar{U})$ is compact. Then there exists $Y = \{y_1, \dots, y_n\} \subset \bar{U}$ such that

$$\begin{aligned} B(\bar{U}) &\subset \{w_1, \dots, w_n\} + \mathcal{B}(0, (1 - Mk)\epsilon) \\ &\subset \bigcup_{i=1}^n B(y_i) + \mathcal{B}(0, (1 - Mk)\epsilon), \end{aligned}$$

where $w_i \in B(y_i)$, for each $i = 1, 2, \dots, n$; and $\mathcal{B}(y_i, r)$ is an open ball in X centered at $y_i \in X$ of radius r . By hypothesis (a),

$$\begin{aligned} \|Ax\| &\leq \|A0\| + H(A0, Ax) \\ &\leq \|A0\| + k\|x\| \\ &\leq \delta \end{aligned}$$

for all $x \in \bar{U}$, where

$$(2.3) \quad \delta = \|A0\| + kr < \infty.$$

Therefore it follows that

$$B(y) \subset \bigcup_{i=1}^n B(y_i) + \mathcal{B}\left(0, \frac{1 - Mk}{\delta} \epsilon\right)$$

for each $y \in \bar{U}$, and hence there exists an element $y_k \in Z$ such that

$$\rho(B(y), B(y_k)) < \frac{1 - Mk}{\delta} \epsilon.$$

Then

$$\begin{aligned}
\rho(C(y), C(y_k)) &= \rho(\text{Fix}(T_y, \text{Fix}(T_{y_k}))) \\
&\leq \frac{1}{1 - Mk} \sup_{x \in S} \rho(T_y(x), T_{y_k}(x)) \\
&= \frac{1}{1 - Mk} \sup_{x \in Y} \rho(A(x)B(y), A(x)B(y_k)) \\
&\leq \frac{1}{1 - Mk} \sup_{x \in Y} \rho(0, Ax) \rho(B(y), B(y_k)) \\
&< \frac{\delta}{(1 - Mk)} \frac{(1 - Mk)}{\delta} \epsilon \\
&= \epsilon.
\end{aligned}$$

It follows that for each $u \in C(y)$ there is $v_k \in C(y_k)$ such that $\|u - v_k\| < \epsilon$. Hence, for each $y \in \bar{U}$, $C(y) \subset \bigcup_1^n \mathcal{B}(v_i, \epsilon)$, where $v_i \in C(y_i)$, $i = 1, 2, \dots, n$. Therefore $c(\bar{U}) \subset C(\bar{U}) \subset \bigcup_1^n \mathcal{B}(v_i, \epsilon)$ and so, c is a compact operator on \bar{U} .

Finally, note that the mapping $c : \bar{U} \rightarrow X$ satisfies all the assumptions of nonlinear alternative of Leray-Schauder and hence an application of it yields that either

- (i) the operator equation $x = cx$ has a solution, or
- (ii) there exists a $u \in \partial U$ such that $u = \lambda cu$ for some $0 < \lambda < 1$ where ∂U is a boundary of U .

Further by definition of c this implies that either

- (i) the operator equation $x \in Ax Bx$ has a solution, or
- (ii) there exists a $u \in \partial \bar{U}$ such that $\lambda u \in A(cu)Bu = A(\lambda u)Bu$, $\lambda > 1$ where ∂U is a boundary of U .

This completes the proof. ■

Corollary 2.1. *Let $\mathcal{B}_r(0)$ and $\bar{\mathcal{B}}_r(0)$ denote respectively the open and closed balls centered at the origin of radius r in a Banach algebra X and let $A : X \rightarrow P_{bd,cl,cv}(X)$, $B : \bar{\mathcal{B}}_r(0) \rightarrow P_{cp,cv}(X)$ be two multi-valued operators such that*

- (a) A is a multi-valued contraction with a contraction constant k ,

- (b) B is l.s.c. and compact,
- (c) AxB_y is a convex subset of S for each $x, y \in S$,
- (d) $Mk < 1$, where $M = \|B(\bar{\mathcal{B}}_r(0))\| = \sup\{\|B(x)\| \mid x \in \bar{\mathcal{B}}_r(0)\}$.

Then either

- (i) the operator inclusion $x \in AxBx$ has a solution, or
- (ii) there exists a $u \in X$ with $\|u\| = r$ satisfying $\lambda u \in A(\lambda u)Bu$ for some $\lambda > 1$.

A Hausdorff measure of noncompactness χ of a bounded set S in X is a nonnegative real number $\chi(S)$ defined by

$$(2.4) \quad \chi(S) = \inf \left\{ r > 0 : A \subset \bigcup_{i=1}^n \mathcal{B}(x_i, r), x_i \in X \right\}.$$

The function χ enjoys the following properties:

- (χ_1) $\chi(S) = 0 \iff S$ is precompact.
- (χ_2) $\chi(S) = \chi(\bar{S}) = \chi(\overline{\text{co}}S)$, where \bar{S} and $\overline{\text{co}}S$ denote respectively the closure and the closed convex hull of S .
- (χ_3) $S_1 \subset S_2 \Rightarrow \chi(S_1) \leq \chi(S_2)$.
- (χ_4) $\chi(S_1 \cup S_2) = \max\{\chi(S_1), \chi(S_2)\}$.
- (χ_5) $\chi(\lambda S) = |\lambda|\chi(S), \forall \lambda \in \mathbb{R}$.
- (χ_6) $\alpha(S_1 + S_2) \leq \chi(S_1) + \chi(S_2)$.

The details of measures of noncompactness and their properties may be found in Deimling [5] and Zeidler [22].

Definition 2.1. A mapping $T : X \rightarrow X$ is called condensing if for any bounded subset A of X , $T(A)$ is bounded and

$$\chi(T(S)) < \chi(A), \quad \chi(S) > 0.$$

Note that contraction and completely continuous mappings are condensing but the converse may not be true. The below result is a variant of a fixed point theorem of O'Regan [16].

Theorem 2.3. *Let U and \bar{U} be open and closed subsets of a Banach space X respectively such that $0 \in U$ and let $T : \bar{U} \rightarrow P_{cp,cv}(X)$ be an upper semi-continuous and condensing mapping such that $T(\bar{U})$ is bounded. Then either*

- (i) T has a fixed point, or
- (ii) there exists an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$.

We need the following result in the sequel.

Lemma 2.3 ([3]). *If $S_1, S_2 \in P_{bd}(X)$, then*

$$\chi(S_1.S_2) \leq \chi(S_1)\|S_2\| + \chi(S_2)\|S_1\|.$$

Theorem 2.4. *Let U and \bar{U} be the open-bounded and closed-bounded subsets of a Banach algebra X such that $0 \in U$ and let $A : X \rightarrow P_{bd,cl,cv}(X)$, $B : \bar{U} \rightarrow P_{cl,cv}(X)$ be two multi-valued operators satisfying*

- (a) A is Lipschitz with a Lipschitz constant k ,
- (b) B is compact and upper semi-continuous,
- (c) $AxBx$ is a convex subset of X for each $x \in \bar{U}$,
- (d) $M\phi(r) < r$ whenever $r > 0$ with $M = \|B(\bar{U})\|$.

Then either

- (i) the operator inclusion $x \in AxBx$ has a solution, or
- (ii) there exists a $u \in \partial U$ such that $\lambda u \in AuBu$ for some $\lambda > 1$ where ∂U is a boundary of U .

Proof. Define a mapping $T : \bar{U} \rightarrow P_{cl}(X)$ by

$$(2.5) \quad Tx = AxBx, x \in \bar{U}.$$

We shall show that T satisfies all the conditions of Theorem 2.2 on \bar{U} .

Step I. First we claim that T defines a multi-valued map $T : \bar{U} \rightarrow P_{cp,cv}(X)$. Obviously Tx is convex subset of X for each $x \in \bar{U}$. Next from Lemma 2.3 it follows that

$$\chi(Tx) = \chi(Ax \cdot Bx) \leq \chi(Ax) \cdot \|B(x)\| + \chi(Bx) \cdot \|A(x)\| = 0$$

for every $x \in \bar{U}$ and the claim follows.

Step II. Now we shall show that the mapping T is an upper semi-continuous on \bar{U} . Let $\{x_n\}$ be a sequence in \bar{U} converging to the point $x^* \in \bar{U}$ and let $\{y_n\}$ be a sequence defined by $y_n \in Tx_n$ converging to the point y^* . It is enough to prove that $y^* \in Tx^*$. Now for any $x, y \in \bar{U}$ we have

$$\begin{aligned} H(Tx, Ty) &= H(AxBx, AyBy) \\ &\leq H(AxBx, AyBx) + H(AyBx, AyBy) \\ (2.6) \quad &\leq H(Ax, Ay)H(0, Bx) + H(0, Ay)H(Bx, By) \\ &\leq k\|x - y\| \|B(\bar{U})\| + \|Ay\| H(Bx, By) \\ &\leq Mk\|x - y\| + \|Ay\| H(Bx, By). \end{aligned}$$

Since B is u.s.c., it is H -upper semi-continuous and consequently

$$H(Bx_n, Bx^*) \rightarrow 0 \quad \text{whenever} \quad x_n \rightarrow x^*.$$

Therefore

$$\begin{aligned} D(y^*, Tx^*) &\leq \lim_{n \rightarrow \infty} D(y_n, Tx^*) \\ &\leq H(Tx_n, Tx^*) \\ &\leq Mk\|x_n - x^*\| + \|Ay_n\| H(Bx_n, Bx^*) \\ &\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty. \end{aligned}$$

This shows that the multi-valued T is upper semi-continuous on X .

Step III. Finally, we show that that T is χ -condensing on X . Since \bar{U} is a bounded subset of X , then there is a real number $r > 0$ such that $\|x\| \leq r$ for all $x \in \bar{U}$. Then we have the following estimate concerning the multi-valued operators A and B . By hypothesis (a),

$$\|Ax\| \leq \delta$$

for all $x \in \bar{U}$, where $\delta = \|A0\| + kr < \infty$ for all $x \in \bar{U}$. Hence $A(\bar{U})$ is a bounded subset of X . Since B is compact, $B(\bar{U})$ is a precompact and consequently a bounded subset of X . As $T(\bar{U}) \subset A(\bar{U})B(\bar{U})$, we have that $T(\bar{U})$ is a bounded subset of X . Let S be a subset of \bar{U} . Then $T(S) \subset T(\bar{U})$ and $T(S)$ is bounded. Let $\epsilon > 0$ be given. Then there exists $Z = \{x_1, \dots, x_n\}$ in X such that

$$\begin{aligned} B(S) &\subset \bigcup_{i=1}^n \mathcal{B}\left(y_i, \frac{\epsilon}{\delta}\right) \\ &\subset \{y_1, \dots, y_n\} + \mathcal{B}\left(0, \frac{\epsilon}{\delta}\right) \\ &\subset \bigcup_{i=1}^n \mathcal{B}\left(Bx_i, \frac{\epsilon}{\delta}\right) \end{aligned}$$

for some $y_i \in Bx_i$ for $i = 1, \dots, n$. Therefore for every $x \in S$, there exists an $x_i \in Z$ such that

$$\rho(Bx, Bx_i) < \frac{\epsilon}{\delta}.$$

Let $\chi(S) = r$. Then we have

$$S \subset \bigcup_{i=1}^m \mathcal{B}(x_i, r + \epsilon).$$

Now for any $x \in S$ we have

$$\begin{aligned} \rho(Tx, Tx_i) &\leq H(Tx, Tx_i) \\ &\leq Mk\|x - x_i\| + \delta H(Bx, Bx_i) \\ (2.7) \quad &< Mk\|x - x_i\| + \delta \frac{\epsilon}{\delta} \\ &\leq Mk(r + \epsilon) + \epsilon \end{aligned}$$

for each $i = 1, \dots, n$. Again each Tx_i is compact for each $i = 1, \dots, n$, there are $y_1^i, \dots, y_{n(i)}^i$ in Tx_i such that

$$Tx_i \subset \bigcup_{j=1}^{n(i)} \mathcal{B}\left(y_j^i, \frac{\epsilon}{2}\right).$$

Now from (2.6) it follows that

$$T(S) \subset \bigcup_{i=1}^n \left\{ \bigcup_{j=1}^{n(i)} \{ \mathcal{B}(y_j^i, Mk(r + \epsilon) + \epsilon) \} \right\}.$$

Therefore

$$\chi(T(S)) < Mk(r + \epsilon) + \epsilon.$$

Since ϵ is arbitrary, one has

$$\chi(T(S)) \leq Mkr = Mk\chi(S) < \chi(S)$$

whenever $\chi(S) > 0$. This shows that T is χ -condensing on \bar{U} into itself. Now an application of Theorem 2.3 yields that either

- (i) the operator inclusion $x \in Tx$ has a solution, or
- (ii) there exists an element $u \in \partial U$ such that $\lambda u \in Tu$ for some $\lambda > 1$.

This further by definition of T implies that

- (i) the operator inclusion $x \in AxBx$ has a solution, or
- (ii) there exists an element $u \in \partial U$ such that $\lambda u \in AuBu$ for some $\lambda > 1$.

This completes the proof. ■

Corollary 2.2. *Let $\mathcal{B}_r(0)$ and $\bar{\mathcal{B}}_r(0)$ denote respectively the open and closed balls centered at the origin of radius r in a Banach algebra X and let $A : \bar{\mathcal{B}}_r(0) \rightarrow P_{bd,cl,cv}(X)$, $B : \bar{\mathcal{B}}_r(0) \rightarrow P_{cp,cv}(X)$ be two multi-valued operators such that*

- (a) A is a multi-valued and Lipschitz with a Lipschitz constant k ,
- (b) B is u.s.c. and compact,
- (c) $AxBx$ is a convex subset of X for each $x, y \in \bar{\mathcal{B}}_r(0)$,
- (d) $Mk < 1$, where $M = \|B(\bar{\mathcal{B}}_r(0))\| = \sup\{\|B(x)\| \mid x \in \bar{\mathcal{B}}_r(0)\}$.

Then either

- (i) the operator inclusion $x \in AxBx$ has a solution, or
- (ii) there exists a $u \in X$ with $\|u\| = r$ such that $\lambda u \in AuBu$ for some $\lambda > 1$.

3. Functional integral inclusions

In this section, we prove the existence theorems for the integral inclusions in Banach algebras by the applications of the abstract results of the previous section under generalized Lipschitz and Carathéodory conditions.

Given a closed and bounded interval $J = [0, a]$ in \mathbb{R} for some $a \in \mathbb{R}$, $a > 0$, consider the integral inclusion

$$(3.1) \quad x(t) \in [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} G(s, x(\eta(s))) ds \right), \quad t \in J$$

where $f : J \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$ is continuous, $G : J \times \mathbb{R} \rightarrow P_{cp,cv}(\mathbb{R})$ and $\theta, \sigma, \eta : J \rightarrow J$.

By a solution to the integral (3.1) we mean a function $x \in BM(J, \mathbb{R})$ that satisfies $x(t) = [f(t, x(\theta(t)))](q(t) + \int_0^{\sigma(t)} v(\eta(s)) ds)$, $t \in J$ for some $v \in L^1(J, \mathbb{R})$ satisfying $v(\eta(t)) \in G(t, x(\eta(t)))$ a.e. $t \in J$, where $BM(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J .

The integral inclusion (3.1) is new to the theory of integral inclusions and the special cases of it have been discussed in the literature extensively. For example, if $G(t, x) = \{g(t, x)\}$, then the integral inclusion (3.1) reduces to the integral equation

$$(3.2) \quad x(t) = [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} g(s, x(\eta(s))) ds \right), \quad t \in J.$$

A considerable amount of work has been done and presented in the literature for the functional integral equation (3.2). See Dhage and Regan [10], Dhage [6, 7] and the references therein. Therefore it is of interest to discuss the functional integral inclusion (3.1) in various aspects of its solution under suitable conditions. In this section, we shall prove the existence of the solution of the integral inclusion (3.1) under the mixed generalized Lipschitz and Carathéodory conditions.

Define a norm $\|\cdot\|$ in $BM(J, \mathbb{R})$ and the multiplication “ \cdot ” by

$$\|x\| = \sup_{t \in J} |x(t)|$$

and

$$(x \cdot y)(t) = x(t)y(t) \quad \forall \quad t \in J$$

respectively. Then $BM(J, \mathbb{R})$ is a Banach algebra with the above norm and multiplication in it.

We need the following definitions in the sequel.

Definition 3.1. A multi-valued map $F : J \rightarrow P_f(\mathbb{R})$ is said to be measurable if for any $y \in X$, the function $t \rightarrow d(y, F(t)) = \inf\{|y - x| : x \in F(t)\}$ is measurable.

Definition 3.2. A measurable multi-valued function $F : J \rightarrow P_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(J, \mathbb{R})$ such that $\|v\| \leq h(t)$ a.e. $t \in J$ for all $v \in F(t)$.

Remark 3.1. It is known that if $F : J \rightarrow \mathbb{R}$ is an integrably bounded multi-function, then the set S_F^1 of all Lebesgue integrable selections of F is closed and non-empty. See Covitz and Nadler [4].

Definition 3.3. A multi-valued function $\beta : J \times \mathbb{R} \rightarrow P_{bd,cl}(\mathbb{R})$ is called Carathéodory if

- (i) $t \mapsto \beta(t, x)$ is measurable for each $x \in E$, and
- (ii) $x \mapsto \beta(t, x)$ is an upper semi-continuous almost everywhere for $t \in J$.

Definition 3.4. A Carathéodory multi-function $\beta(t, x)$ is called L^1 -Carathéodory if there exists a function $h_r \in L^1(J, \mathbb{R})$ such that

$$\|\beta(t, x)\| \leq h_r(t) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$ with $|x| \leq r$.

Denote

$$S_\beta^1(x) = \{v \in L^1(J, E) \mid v(t) \in \beta(t, x(t)) \text{ a.e. } t \in J\},$$

where $\eta : J \rightarrow J$ is continuous. Then we have the following lemmas due to Lasota and Opial [16].

Lemma 3.1. *Let E be a Banach space. If $\dim(E) < \infty$ and $\beta : J \times E \rightarrow P_{bd,cl}(E)$ is L^1 -Carathéodory, then $S_\beta^1(x) \neq \emptyset$ for each $x \in E$.*

Lemma 3.2. *Let E be a Banach space, β a Carathéodory multi-map with $S_\beta^1 \neq \emptyset$ and let $\mathcal{L} : L^1(J, E) \rightarrow C(J, E)$ be a continuous linear mapping. Then the operator*

$$\mathcal{L} \circ S_\beta^1 : C(J, E) \rightarrow P_{bd,cl}(C(J, E))$$

is a closed graph operator on $C(J, E) \times C(J, E)$.

We consider the following hypotheses in the sequel.

- (H₀) The functions $\theta, \sigma, \eta : J \rightarrow J$ are continuous.
- (H₁) The function $q : J \rightarrow \mathbb{R}$ is continuous.
- (H₂) The function $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a bounded function $\ell : J \rightarrow \mathbb{R}$ with a bound $\|\ell\|$ satisfying

$$|f(t, x) - f(t, y)| \leq \ell(t)|x - y| \quad \text{a.e. } t \in J$$

for all $x, y \in \mathbb{R}$.

- (H₃) The multi-function $G : J \times \mathbb{R} \rightarrow P_{cp,cv}(\mathbb{R})$ is L^1 -Carathéodory.
- (H₄) There exists a function $\gamma \in L^1(J, \mathbb{R})$ with $\gamma(t) > 0$ a.e. $t \in J$ and a nondecreasing function $\psi : \mathbb{R}^+ \rightarrow (0, \infty)$ such that

$$\|G(t, x)\| \leq \gamma(t)\psi(|x|) \quad \text{a.e. } t \in J$$

for all $x \in \mathbb{R}$.

Theorem 3.1. *Assume that the hypotheses (H₀)–(H₄) hold. Suppose that there exists a real number $r > 0$ such that*

$$(3.3) \quad r > \frac{F(\|q\| + \|\gamma\|_{L^1}\psi(r))}{1 - \|\ell\|(\|q\| + \|\gamma\|_{L^1}\psi(r))}$$

where $\|\ell\|(\|q\| + \|\gamma\|_{L^1}\psi(r)) < 1$ and $F = \sup_{t \in J} |f(t, 0)|$. Then the integral inclusion (3.1) has a solution on J .

Proof. Let $X = BM(J, \mathbb{R})$. Define an open ball $\mathcal{B}(0, r)$ in X centered at the origin of radius r , where r satisfies the inequality (3.3) and consider the two multi-valued mappings A and B on X defined by

$$(3.4) \quad Ax(t) = f(t, x(\theta(t)))$$

and

$$(3.5) \quad Bx(t) = \left\{ u \in X \mid u(t) = q(t) + \int_0^{\sigma(t)} v(s) ds, \quad v \in S_G^1(x(\eta)) \right\}$$

for all $t \in J$.

Then the integral inclusion (3.1) is equivalent to the operator inclusion

$$(3.6) \quad x(t) \in Ax(t)Bx(t), \quad t \in J.$$

We shall show that the multi-valued operators A and B satisfy all the conditions of Corollary 2.2. Clearly, the operator B is well defined since $S_G^1(x(\eta)) \neq \emptyset$ for each $x \in X$ in view of Lemma 3.1.

Step I. We first show that the operators A and B define the multi-valued operators $A, B : \mathcal{B}[0, r] \rightarrow P_{cp,cv}(X)$. The case of A is obvious since it is a single-valued operator on $\mathcal{B}[0, r]$. We only prove the claim for the operator B . Let $\{u_n\}$ be a sequence in Bx converging to a point u . Then there is a sequence $\{v_n\} \subset S_G^1(x(\eta))$ such that

$$u_n(t) = q(t) + \int_0^{\sigma(t)} v_n(s) ds$$

and $v_n \rightarrow v$. Since $G(t, x)$ is closed for each $(t, x) \in J \times \mathbb{R}$, we have $v \in S_G^1(x(\eta))$. As a result

$$u(t) = q(t) + \int_0^{\sigma(t)} v(s) ds \in Bx(t), \quad \forall t \in J.$$

Hence B has closed values on X . Again let $u_1, u_2 \in Bx$. Then there are $v_1, v_2 \in S_G^1(x(\eta))$ such that

$$u_1(t) = q(t) + \int_0^{\sigma(t)} v_1(s) ds, \quad t \in J,$$

and

$$u_2(t) = q(t) + \int_0^{\sigma(t)} v_2(s) ds, \quad t \in J.$$

Now for any $\gamma \in [0, 1]$,

$$\begin{aligned} \gamma u_1(t) + (1 - \gamma)u_2(t) &= \gamma \left(q(t) + \int_0^{\sigma(t)} v_1(s) ds \right) \\ &\quad + (1 - \gamma) \left(q(t) + \int_0^{\sigma(t)} v_2(s) ds \right) \\ &= q(t) + \int_0^t [\gamma v_1(s) + (1 - \gamma)v_2(s)] ds \\ &= q(t) + \int_0^{\sigma(t)} v(s) ds \end{aligned}$$

where $v(t) = \gamma v_1(t) + (1 - \gamma)v_2(s) \in G(t, x(\eta(t)))$ for all $t \in J$. Hence $\gamma u_1 + (1 - \gamma)u_2 \in Bx$ and consequently Bx is convex for each $x \in \mathcal{B}[0, r]$. As a result A defines a multi-valued operator $B : X \rightarrow P_{bd,cl,cv}(X)$. Again let $t, \tau \in J$. Then for any $u \in Bx$ we have

$$\begin{aligned} |u(t) - u(\tau)| &\leq \left| \int_0^{\sigma(t)} v(s) ds - \int_0^{\sigma(\tau)} v(s) ds \right| \\ &\leq \left| \int_{\sigma(\tau)}^{\sigma(t)} |v(s)| ds \right| \\ &\leq |p(t) - p(\tau)| \end{aligned}$$

where $p(t) = \int_0^{\sigma(t)} h(s) ds$.

Since p is continuous on the compact interval J , it is uniformly continuous. Hence Bx is compact by Arzela-Ascoli theorem. Thus we have $B : \mathcal{B}[0, r] \rightarrow P_{cp,cv}(X)$. Hence $A, B : \mathcal{B}[0, r] \rightarrow P_{cp,cv}(X)$.

Step II. We show that A is a contraction on X . Let $x, y \in X$. Then

$$\begin{aligned} \|Ax - Ay\| &= \sup_{t \in J} |Ax(t) - Ay(t)| \\ &= \sup_{t \in J} |f(t, x(\theta(t))) - f(t, y(\theta(t)))| \\ &\leq \sup_{t \in J} \ell(t) |x(\theta(t)) - y(\theta(t))| \\ &\leq \|\ell\| \|x - y\|, \end{aligned}$$

showing that A is Lipschitz on X with a Lipschitz constant $\|\ell\|$.

Step III. Next we show that B is compact and upper semi-continuous on X . First we prove that $B(\mathcal{B}[0, r])$ is totally bounded on X . To do this, it is enough to prove that $B(\mathcal{B}[0, r])$ is a uniformly bounded and equi-continuous set in X . To see this, let $u \in B(\mathcal{B}[0, r])$ be arbitrary. Then there is a $v \in S_G^1(x(\eta))$ such that

$$u(t) = q(t) + \int_0^{\sigma(t)} v(s) ds$$

for some $x \in \mathcal{B}[0, r]$. Hence

$$\begin{aligned} |u(t)| &\leq |q(t) + \int_0^{\sigma(t)} |v(s)| ds \\ &\leq \|q\| + \int_0^{\sigma(t)} \|G(s, x(s))\| ds \\ &\leq \int_0^1 h_r(s) ds \\ &= \|h_r\|_{L^1} \end{aligned}$$

for all $x \in \mathcal{B}[0, r]$ and so $B(\mathcal{B}[0, r])$ is a uniformly bounded set in X . Again as in Step I, it is proved that

$$|u(t) - u(\tau)| \leq |p(t) - p(\tau)|$$

where $p(t) = \int_0^{\sigma(t)} h(s) ds$.

Notice that p is a continuous function on J , so it is uniformly continuous on J . As a result we have that

$$|u(t) - u(\tau)| \rightarrow 0 \text{ as } t \rightarrow \tau.$$

This shows that $B(\overline{U})$ is an equi-continuous set in X . Next we show that B is an upper semi-continuous multi-valued mapping on $\mathcal{B}[0, r]$. Let $\{x_n\}$ be a sequence in $\mathcal{B}[0, r]$ such that $x_n \rightarrow x_*$. Let $\{y_n\}$ be a sequence such that $y_n \in Bx_n$ and $y_n \rightarrow y_*$. We shall show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S_G^1(x_n(\eta))$ such that

$$y_n(t) = q(t) + \int_0^{\sigma(t)} v_n(s) ds, \quad t \in J.$$

We must prove that there is a $v_* \in S_G^1(x_*)$ such that

$$y_*(t) = q(t) + \int_0^{\sigma(t)} v_*(s) ds, \quad t \in J.$$

Consider the continuous linear operator $\mathcal{K} : L^1(J, \mathbb{R}) \rightarrow C(J, E)$ defined by

$$\mathcal{K}y(t) = \int_0^{\sigma(t)} v(s) ds, \quad t \in J.$$

Now $\|(y_n - q(t)) - (y_* - q(t))\| \rightarrow 0$ as $n \rightarrow \infty$.

From Lemma 3.2, it follows that $\mathcal{K} \circ S_G^1$ is a closed graph operator. Also from the definition of \mathcal{K} we have

$$y_n(t) - q(t) \in \mathcal{K} \circ S_G^1(x_n(\eta)).$$

Since $y_n \rightarrow y_*$, there is a point $v_* \in S_G^1(x_*(\eta))$ such that

$$y_*(t) = q(t) + \int_0^{\sigma(t)} v_*(s) ds, \quad t \in J.$$

This shows that B is an upper semi-continuous operator on $\mathcal{B}[0, r]$. Thus B is an upper semi-continuous and compact operator on \overline{U} .

Step IV. Here we show that $AxBx$ is a convex subset of X for each $x \in \mathcal{B}[0, r]$. Let $x \in \mathcal{B}[0, r]$ be arbitrary and let $w, y \in AxBx$. Then there are $u, v \in S_G^1(x(\eta))$ such that

$$w(t) = [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} u(s) ds \right)$$

and

$$y(t) = [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} v(s) ds \right).$$

Now for any $\lambda \in [0, 1]$,

$$\begin{aligned} & \lambda y(t) + (1 - \lambda)w(t) \\ &= \lambda [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} v(s) ds \right) \\ & \quad + (1 - \lambda) [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} u(s) ds \right) \\ &= [f(t, x(\theta(t)))] \left(\lambda q(t) + \int_0^{\sigma(t)} \lambda v(s) ds \right) \\ & \quad + [f(t, x(\theta(t)))] \left((1 - \lambda)q(t) + \int_0^{\sigma(t)} (1 - \lambda)u(s) ds \right) \\ &= [f(t, x(\theta(t)))] \left(q(t) + \int_0^{\sigma(t)} [\lambda v(s) + (1 - \lambda)u(s)] ds \right). \end{aligned}$$

Since $G(t, x(\eta(t)))$ is convex, $z(t) = \lambda y(t) + (1 - \lambda)w(t) \in G(t, x(\eta(t)))$ for all $t \in J$ and so $z \in S_G^1(x(\eta))$. As a result $\lambda y + (1 - \lambda)w \in AxBx$. Hence $AxBx$ is a convex subset of X .

Step V. Finally from condition (3.3) it follows that

$$Mk = \|\ell\| \left(\|q\| + \|\gamma\|_{L^1} \psi(r) \right) < 1.$$

Thus all the conditions of Corollary 2.2 are satisfied and hence a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in \mathcal{E}$ be arbitrary. Then we have, for any $\lambda > 1$,

$$\begin{aligned} \lambda u(t) &\in Au(t)Bu(t) \\ &= [f(t, u(\sigma(t))) \left(q(t) + \int_0^{\sigma(t)} G(s, u(\eta(s))) ds \right)], \quad t \in J \end{aligned}$$

for some real number $\lambda > 1$. Therefore

$$\lambda u(t) \in [f(t, u(\theta(t))) \left(q(t) + \int_0^{\sigma(t)} G(s, u(\eta(s))) ds \right)]$$

or

$$u(t) = \lambda^{-1} [f(t, u(\theta(t))) \left(q(t) + \int_0^{\sigma(t)} v(s) ds \right)]$$

for some $v \in S_G^1(u(\eta))$.

Now

$$\begin{aligned} |u(t)| &= \left| \lambda^{-1} [f(t, u(\theta(t)))] \right| \left| \left(q(t) + \int_0^{\sigma(t)} v(s) ds \right) \right| \\ &\leq \left| [f(t, u(\theta(t)))] \right| \left(|q(t)| + \int_0^{\sigma(t)} |v(s)| ds \right) \\ &\leq \left[|f(t, u(\theta(t))) - f(t, 0)| + |f(t, 0)| \right] \\ &\quad \times \left(|q(t)| + \int_0^{\sigma(t)} \|G(s, u(\eta(s)))\| ds \right) \\ &\leq |\ell(t)| |u(\theta(t))| \left(|q(t)| + \int_0^{\sigma(t)} \gamma(s) \psi(|u(\eta(s))|) ds \right) \\ &\quad + F \left(|q(t)| + \int_0^{\sigma(t)} \gamma(s) \psi(|u(\eta(s))|) ds \right) \end{aligned}$$

$$\begin{aligned}
&\leq \|\ell\| \|u\| (\|q\| + \|\gamma\|_{L^1} \psi(\|u\|)) \\
&\quad + F(\|q\| + \|\gamma\|_{L^1} \psi(\|u\|)) \\
&\leq \frac{F(\|q\| + \|\gamma\|_{L^1} \psi(\|u\|))}{1 - \|\ell\| (\|q\| + \|\gamma\|_{L^1} \psi(\|u\|))}.
\end{aligned}$$

Taking the supremum over t , we obtain

$$\|u\| \leq \frac{F(\|q\| + \|\gamma\|_{L^1} \psi(\|u\|))}{1 - \|\ell\| (\|q\| + \|\gamma\|_{L^1} \psi(\|u\|))}.$$

Substituting $\|u\| = r$ in the above inequality yields

$$r \leq \frac{F(\|q\| + \|\gamma\|_{L^1} \psi(r))}{1 - \|\ell\| (\|q\| + \|\gamma\|_{L^1} \psi(r))}.$$

This is a contradiction to (3.3) and hence the conclusion (ii) of Corollary 2.2 does not hold. Therefore the operator inclusion $x \in AxBx$ and consequently the functional integral inclusion (3.1) has a solution on J . This completes the proof. \blacksquare

4. An application

In this section, we prove the existence theorem for the differential inclusions in Banach algebras by the application of the abstract result of the previous section.

Given a closed and bounded interval $J = [0, 1]$ in \mathbb{R} , consider the differential inclusion (in short DI)

$$(4.1) \quad \begin{cases} \left(\frac{x(t)}{f(t, x(\theta(t)))} \right)' \in G(t, x(\eta(t))) & \text{a.e. } t \in J \\ x(0) = x_0 \in \mathbb{R} \end{cases}$$

where $f : J \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$ is continuous, $G : J \times \mathbb{R} \rightarrow P_{cp,cv}(\mathbb{R})$ and $\theta, \eta : J \rightarrow J$ are continuous with $\theta(0) = 0$.

By a solution to DI (4.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

$$\left(\frac{x(t)}{f(t, x(\theta(t)))} \right)' = v(t), \quad t \in J$$

for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in G(t, x(\eta(t)))$ a.e. $t \in J$, where $AC(J, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on J .

The DI (4.1) is new to the theory of differential inclusions and the special cases of it have been discussed in the literature extensively.

Theorem 4.1. *Assume that hypotheses (H₂)–(H₅) hold. Suppose that there exists a real number $r > 0$ such that*

$$(4.2) \quad r > \frac{F \left(\left| \frac{x_0}{f(0, x_0)} \right| + \|\gamma\|_{L^1} \psi(r) \right)}{1 - \|\ell\| \left(\left| \frac{x_0}{f(0, x_0)} \right| + \|\gamma\|_{L^1} \psi(r) \right)}$$

where $\|\ell\| \left(\left| \frac{x_0}{f(0, x_0)} \right| + \|\gamma\|_{L^1} \psi(r) \right) < 1$ and $F = \sup_{t \in J} |f(t, 0)|$. Then the differential inclusion (4.1) has a solution on J .

Proof. Let $X = C(J, \mathbb{R})$, the space of continuous real-valued functions on J , and define an open ball $\mathcal{B}_r(0)$ in X centered at the origin of radius r , where the real number r satisfies the inequalities in (4.2). Define a norm $\|\cdot\|$ and the multiplication “ \cdot ” in X by $\|x\| = \sup_{t \in J} |x(t)|$ and $(x \cdot y)(t) = x(t)y(t) \forall t \in J$. Then X is a Banach algebra with respect this norm and the multiplication in X .

Now the differential inclusion (4.1) is equivalent to the functional integral inclusion

$$(4.3) \quad x(t) \in [f(t, x(\theta(t)))] \left(\frac{x_0}{f(0, x_0)} + \int_0^t G(s, x(\eta(s))) ds \right)$$

for all $t \in J$. Since $C(J, \mathbb{R}) \subseteq BM(J, \mathbb{R})$, the desired conclusion follows by the application of Theorem 3.1. This completes the proof. ■

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