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AN EXISTENCE RESULT FOR IMPULSIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

Irene Benedetti

Dipartimento di Matematica "U. Dini"
Università di Firenze
Viale Morgagni 67/A, 50134 – Firenze, Italy
e-mail: benedetti@math.unifi.it

Abstract

We use the topological degree theory for condensing multimaps to present an existence result for impulsive semilinear functional differential inclusions in Banach spaces. Moreover, under some additional assumptions we prove the compactness of the solution set.

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1. Introduction

In this paper, we prove an existence theorem for a system governed by an impulsive semilinear differential inclusion with delay.

More precisely, let E be a real Banach space.

For any $a < b$ let us denote by $\mathcal{C}([a,b],E)$ the space of all piece-wise continuous functions $c : [a,b] \to E$ with a finite number of discontinuity points $\{t_*\} \subset [a,b]$ such that $t_* \neq b$ and all values

$$c(t_*^+) = \lim_{h \to 0^+} c(t_* + h)$$

$$c(t_*^-) = \lim_{h \to 0^-} c(t_* - h)$$

are finite (we consider only $c(t_*^+)$ in the case $t_* = a$).

Let us observe that $\mathcal{C}([a,b],E)$ is a normed space with the norm:

$$\|c\|_{\mathcal{C}} = \sup_{a\le t\le b} \|c(t)\|_E.$$

For $\tau > 0$, let $x \in \mathcal{C}([-\tau,0],E)$ be a given function.

We consider a Cauchy problem for a semilinear functional differential inclusion with impulses at fixed moments. Similarly to the impulsive differential equations (see e.g. [9, 12]) the mathematical model of this problem is the following:

$$y(t) = x(t)\ ,\ t \in [-\tau,0] \tag{1.1}$$

$$y'(t) \in Ay(t) + F(t,y_t)\ ,\ t \in [0,d] \tag{1.2}$$

$$\Delta y\lfloor_{t=t_k} = I_k(y_{t_k^-})\ ,\ k = 0,\dots,m \tag{1.3}$$

where A is the infinitesimal generator of a semigroup, $F : [0,d]\times\mathcal{C}([-\tau,0],E) \multimap E$ is a multivalued map, $y \in \mathcal{C}([-\tau,d],E)$. For $t \in [0,d]$, $y_t \in \mathcal{C}([-\tau,0],E)$ is defined as $y_t(\theta) = y(t+\theta)$, $\theta \in [-\tau,0]$, $0 \le t_1 < \cdots < t_m \le d$ are given points, $\Delta y\lfloor_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-) = y(t_k)$ and $I_k : \mathcal{C}([-\tau,0],E) \to E$, $k = 0,\dots,m$ are given functions.

The jump sizes can be deterministic, as in (1.3) or non-deterministic. This latter case can be formulated considering instead of (1.3)

$$\Delta y\lfloor_{t=t_k} \in \mathcal{I}_k(y_{t_k^-})\ ,\ k = 0,\dots,m \tag{1.4}$$

where $\mathcal{I}_k : \mathcal{C}([-\tau,0],E) \multimap E$ are given multivalued functions.

In this paper, we study the both cases.

Differential inclusions have been the subject of an intensive study of many researchers in the recent decades, see e.g. [2, 6, 7, 8, 14, 15].

Let us note that the existence of mild solutions for semilinear functional differential inclusions in Banach spaces was proved by Obukhovskii ([10]).

It became natural to look for the existence of non-continuous solutions of differential equations and inclusions, in some mathematical models of phenomena studied in physics, population dynamics, biotechnology and economics. Moreover, the model of non-deterministic jump sizes may arise in the case of a control problem, where one wishes to control the jump sizes in order to achieve certain objectives. This problem is related to the study of systems governed by impulsive differential equations and inclusions.

The former have been extensively investigated in finite and infinite-dimensional Banach spaces, see for example [9, 11, 12, 13].

On the other hand, systems governed by impulsive differential inclusions seem to be an interesting new argument of research.

Two of the first authors that have studied this argument are Watson and Ahmed, see [1, 16], moreover, we refer the interested reader to some very recent papers of Benchohra et al, see [3, 4, 5].

We prove the existence theorem using the topological degree theory for condensing multifields and some other methods developed in the book of Kamenskii, Obukhovskii and Zecca ([7]).

The paper is organized as follows. In Section 2, we recall some definitions and results from multivalued analysis which will be used later. In Section 3, we present the existence result for problem (1.1), (1.2), (1.3) and, under an additional assumption on continuity of jumps, we demonstrate the compactness of the solution set.

2. Preliminaries

Let X, Y, be two topological vector spaces.

We denote by $\mathcal{P}(Y)$ the family of all non-empty subsets of Y and by

$$\begin{aligned} \mathcal{K}(Y) &= \{C \in \mathcal{P}(Y), \text{ compact}\}, \\ \mathcal{K}v(Y) &= \{D \in \mathcal{P}(Y), \text{ compact and convex}\}. \end{aligned} \tag{2.1}$$

A multivalued map $F : X \to \mathcal{P}(Y)$ is said to be:

(a) u.s.c. if $F^{-1}(V) = \{x \in X : F(x) \subset V\}$ is an open subset of X for every open $V \subseteq Y$;

(b) closed if its graph $G_F = \{(x, y) \in X \times Y : y \in F(x)\}$ is a closed subset of $X \times Y$.

(c) compact if its range $F(X)$ is relatively compact in Y, i.e., $\overline{F(X)}$ is compact in Y.

(d) quasicompact if its restriction to any compact subset $M \subset X$ is compact.

It is easy to see that the closeness of a multivalued map F is equivalent to the fact that for any sequences $\{x_n\}_{n=1}^{\infty} \subset X$, $\{y_n\}_{n=1}^{\infty} \subset Y$, if $x_n \to x$ and $y_n \in F(x_n)$, $y_n \to y$, then $y \in F(x)$.

When X, Y are metric spaces the following result can be proved (see [7]).

Theorem 2.1. *Let X and Y be metric spaces and $F : X \to \mathcal{K}(Y)$ a closed quasicompact multimap. Then F is u.s.c.*

Moreover, we have the following property for u.s.c. multimaps (see, e.g. [7]).

Theorem 2.2. *Let $F : X \to \mathcal{K}(Y)$ be an u.s.c. multimap. If $C \subset X$ is a compact set, then its image $F(C)$ is a compact subset of Y.*

Everywhere in this section the symbol $\mathcal{E}$ denotes a real Banach space.

We will use the following important sufficient condition for the weak compactness of a set in $L^1([0,d],\mathcal{E})$ (for the proof see, e.g. [7]).

Proposition 2.3. *Assume that $\Omega \subset L^1([0,d],\mathcal{E})$ is integrably bounded and the sets $\Omega(t) = \{\omega(t) : \omega \in \Omega\}$ are relatively compact for a.e. $t \in [0,d]$. Then Ω is weakly compact in $L^1([0,d],\mathcal{E})$.*

Let us recall some notions (see, e.g. [7] for details).

Definition 2.4. Let $(N,\geq)$ be a partially ordered set, a map $\beta : \mathcal{P}(\mathcal{E}) \to N$ is called a measure of non-compactness (MNC) in $\mathcal{E}$ if

$$\beta(\overline{co}\Omega) = \beta(\Omega)$$

for every $\Omega \in \mathcal{P}(\mathcal{E})$.

A measure of non-compactness β is called:

(i) monotone if Ω_0, $\Omega_1 \in \mathcal{P}(\mathcal{E})$, $\Omega_0 \subseteq \Omega_1$ imply $\beta(\Omega_0) \leq \beta(\Omega_1)$;

(ii) nonsingular if $\beta(\{b\} \cup \Omega) = \beta(\Omega)$ for every $b \in \mathcal{E}$, $\Omega \in \mathcal{P}(\mathcal{E})$;

(iii) algebraically semiadditive if $\beta(\Omega_0 + \Omega_1) \leq \beta(\Omega_0) + \beta(\Omega_1)$ for every $\Omega_0, \Omega_1 \in \mathcal{P}(\mathcal{E})$;

(iv) regular if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

A well known example of measure of non-compactness satisfying all of the above properties is the Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \,\text{has a finite}\ \varepsilon\text{-net}\}.$$

Let us mention the following property.

If L is a bounded linear operator in $\mathcal{E}$, then for every bounded $\Omega \subset \mathcal{E}$

$$\chi(L\Omega) \leq \|L\|\chi(\Omega).$$

Definition 2.5. Let X be a subset of $\mathcal{E}$ and Λ a space of parameters, a multimap $F : X \to \mathcal{K}(\mathcal{E})$, or a family of multimaps $G : \Lambda \times X \to \mathcal{K}(\mathcal{E})$, is called condensing relative to an MNC β, or β-condensing, if for every $\Omega \subseteq X$ that is not relatively compact we have, respectively

$$\beta(F(\Omega)) \not\geq \beta(\Omega) \text{ or } \beta(G(\Lambda \times \Omega)) \not\geq \beta(\Omega).$$

We will need also the following property (cf. [7], Theorem 4.2.2).

Theorem 2.6. *Let $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],\mathcal{E})$ be an integrably bounded sequence of functions such that*

$$\chi(\{f_n\}_{n=1}^{\infty}) \leq q(t)$$

for a.e. $t \in [0,d]$ where $q \in L^1([a,b])$.

If $S : L^1([a,b],\mathcal{E}) \to C([a,b],\mathcal{E})$ is an operator satisfying the following properties:

(S1) *there exists $D \geq 0$ such that*

$$\|Sf(t) - Sg(t)\|_{\mathcal{E}} \leq D \int_a^t \|f(s) - g(s)\|_{\mathcal{E}}\, ds$$

(S2) *for any compact $K \subset \mathcal{E}$ and sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b] \to \mathcal{E}$ such that $\{f_n\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [a,b]$ the weak convergence $f_n \rightharpoonup f_0$ implies $Sf_n \to Sf_0$,* *then*

$$\chi\,(Sf_n(t)_{n=1}^{\infty}) \leq 2D \int_a^t q(s)\,ds \tag{2.2}$$

for all $t \in [a,b]$*, where* $D \geq 0$ *is the constant in condition* (S1)*.*

For the operator that satisfies properties (S1) and (S2) we have the following statement (cf. [7], Theorem 5.1.1).

Theorem 2.7. *Let* $S : L^1([a,b],\mathcal{E}) \to C([a,b],\mathcal{E})$ *be an operator satisfying the conditions* (S1) *and* (S2)*. Then for every integrably bounded sequence* $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];\mathcal{E})$ *such that the set* $\{f_n(t)\}_{n=1}^{\infty}$ *is relatively compact for almost every* $t \in [a,b]$*, the sequence* $\{Sf_n\}_{n=1}^{\infty}$ *is relatively compact in* $C([a,b],\mathcal{E})$ *and moreover, if* $f_n \rightharpoonup f_0$*, then* $Sf_n \to Sf_0$*.*

Remark 2.8. An example of an operator that satisfies properties (S1), (S2) is given by the integral operator $\Gamma : L^1([a,b];\mathcal{E}) \to C([a,b];\mathcal{E})$ defined as:

$$\Gamma f(t) = \int_a^t T(t-s)f(s)\,ds,$$

where $T(\theta)$ is a C_0-semigroup.

Let $W \subset \mathcal{E}$ be an open set, $K \subseteq \mathcal{E}$ a closed convex subset; β a monotone MNC in $\mathcal{E}$ and $F : \overline{W}_K \to \mathcal{K}v(K)$ a u.s.c, β-condensing multimap such that $x \notin F(x)$ for all $x \in \partial W_K$, where $\overline{W}_K$ and ∂W_K denote the closure and the boundary of the set $W_K = W \cap K$ in the relative topology of the space K. In such a setting, the relative topological degree $\deg_K(F, \overline{W}_K)$ satisfying the standard properties is defined (see [7], Chapter 3).

Moreover, the following properties of the fixed points set of F can be proved (see [7]).

Theorem 2.9. *Let* M *be a closed bounded subset of* $\mathcal{E}$*,* $F : M \to \mathcal{K}(\mathcal{E})$ *a closed,* β*-condensing multimap, where* β *is a monotone MNC defined on* $\mathcal{E}$*. Then the fixed point set* $\operatorname{Fix} F = \{x \in M : x \in F(x)\}$ *is compact.*

Theorem 2.10. *Let M be a closed subset of $\mathcal{E}$, β a monotone MNC on $\mathcal{E}$, Λ a metric space and $G : \Lambda \times M \to \mathcal{K}(\mathcal{E})$ a closed multimap which is β-condensing in the second variable and such that*

$$\mathcal{F}(\lambda) := \operatorname{Fix} G(\lambda, \cdot) \neq \emptyset,$$

for every $\lambda \in \Lambda$. Then the multimap $\mathcal{F} : \Lambda \to \mathcal{P}(\mathcal{E})$ is u.s.c.

3. An existence theorem for a semilinear functional differential inclusion

In order to give an existence result for the problem (1.1), (1.2), (1.3), let us assume the following hypotheses:

(A) The linear operator $A : D(A) \subseteq E \to E$ is the infinitesimal generator of a C_0-semigroup e^{At};

The multivalued map $F : [0, d] \times \mathcal{C}([-\tau, 0], E) \to \mathcal{K}v(E)$ is such that:

(F1) The multifunction $F(\cdot, c) : [0, d] \to \mathcal{K}v(E)$ has a strongly measurable selection for every $c \in \mathcal{C}([-\tau, 0], E)$, i.e., there exists a strongly measurable function $f : [0, d] \to E$ such that $f(t) \in F(t, c)$ for a.e. $t \in [0, d]$;

(F2) The multimap $F(t, \cdot) : \mathcal{C}([-\tau, 0], E) \to \mathcal{K}v(E)$ is u.s.c. for a.e. $t \in [0, d]$;

(F3) There exists a function $\alpha \in L^1_+([0, d])$ such that:

$$\|F(t, c)\| \leq \alpha(t)(1 + \|c\|_{\mathcal{C}}) \text{ for a.e. } t \in [0, d];$$

(F4) There exists a function $\mu \in L^1_+([0, d])$ such that:

$$\chi(F(t, D)) \leq \mu(t)\varphi(D) \text{ for a.e. } t \in [0, d],$$

for every bounded $D \subset \mathcal{C}([-\tau, 0], E)$ where χ is the Hausdorff MNC in E and $\varphi(D) = \sup\limits_{-\tau \leq t \leq 0} \chi(D(t))$.

Definition 3.1. A function $y \in \mathcal{C}([-\tau, d], E)$ is said to be a mild solution to the problem (1.1), (1.2), (1.3) if there exists a function $f \in L^1([0,d], E)$ such that $f(t) \in F(t, y_t)$ for a.e. $t \in [0,d]$ and

$$y(t) = x(t)\ ,\ t \in [-\tau, 0];$$

$$y(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(s)\, ds + \sum_{0<t_k<t} e^{A(t-t_k)} I_k(y_{t_k})\ ,\ t \in [0,d].$$

Remark 3.2. Let us observe that the superposition multioperator $P_F : \mathcal{C}([-\tau, d], E) \to \mathcal{P}(L^1([0,d], E))$ generated by F, that assigns to every function $y \in \mathcal{C}([-\tau, d], E)$ the set of all strongly measurable selections of the multifunction $t \to F(t, y_t)$, $t \in [0,d]$, is well defined (see Theorem 1.3.5 [7]). (Notice that from (F3) it follows that all such selections are summable).

Moreover, the following statement, that in [7] is given for a continuous function, is still valid in the space $\mathcal{C}([-\tau, d], E)$.

Lemma 3.3. *Assume the sequences*

$$\{x_n\}_{n=1}^{\infty} \subset \mathcal{C}([-\tau, d], E),\ \{f_n\}_{n=1}^{\infty} \subset L^1([0,d], E)$$

$f_n \in P_F(x_n)$, $n \geq 1$ *are such that* $x_n \to x_0$, $f_n \rightharpoonup f_0$. *Then* $f_0 \in P_F(x_0)$.

Theorem 3.4. *Under assumptions* (A), (F1)–(F4) *the problem* (1.1), (1.2), (1.3) *has a mild solution on* $[-\tau, d]$, *moreover, if* I_k, $k = 0, \ldots, m$ *are continuous functions, the solution set is a compact subset of the space* $\mathcal{C}([-\tau, d], E)$.

We divide the construction of the solution of the problem (1.1), (1.2), (1.3) into steps. We solve the problem in the interval $[-\tau, t_1]$, then in the interval $[t_1, t_2]$ and so on until the final interval $[t_m, d]$. More precisely, we proceed in the following way.

Consider the set $Q^0 \subset \mathcal{C}([-\tau, t_1], E)$ defined as:

$$Q^0 = \{y : y(t) = x(t),\ t \in [-\tau, 0],\ y \,\text{is continuous on}\, [0, t_1]\}\,.$$

It is clear that Q^0 is a closed, convex subset of the normed space $\mathcal{C}([-\tau, t_1], E)$.

Given $f \in L^1([0,t_1],E)$ we define the function $p_f^0 : [-\tau,t_1] \to E$ as:

$$p_f^0(t) = \begin{cases} x(t)\ ,\ t \in [-\tau,0]; \\ e^{At}x(0) + \displaystyle\int_0^t e^{A(t-s)} f(s)\, ds\ ,\ t \in (0,t_1]. \end{cases}$$

Now consider the multioperator $P^0 : Q^0 \to \mathcal{P}(Q^0)$ defined as:

$$P^0(y^0) = \left\{ p_f^0 \in Q^0 :\ f \in L^1([0,t_1],E),\ f(s) \in F(s,y_s^0) \right\}. \tag{3.1}$$

A fixed point of the multioperator P^0, $y^0 \in P^0(y^0)$ gives a mild solution of (1.1), (1.2), (1.3) on the interval $[-\tau,t_1]$ (it is clear that $y^0 = x$ in the case $t_1 = 0$).

Further we continue the inductive process in the following way. If a function $y^{(k-1)}$ is constructed on the interval $[-\tau,t_k]$, then we define a convex, closed subset $Q^{(k)} \subset \mathcal{C}([-\tau,t_{k+1}],E)$ as:

$$Q^{(k)} = \left\{ y : \begin{array}{l} y(t) = y^{(k-1)}(t),\ t \in [-\tau,t_k], \\ y \text{ is continuous on } (t_k,t_{k+1}],\ y(t_k^+) = y^{k-1}(t_k) + I_k\big(y_{t_k}^{(k-1)}\big) \end{array} \right\}.$$

Given $f \in L^1((t_k,t_{k+1}),E)$, we define the function $p_f^{(k)} : [-\tau,t_{k+1}] \to E$ as:

$$p_f^{(k)}(t) = \begin{cases} y^{(k-1)}(t)\ ,\ t \in [-\tau,t_k]; \\ e^{A(t-t_k)}\big(y^{(k-1)}(t_k) + I_k\big(y_{t_k}^{(k-1)}\big)\big) + \displaystyle\int_{t_k}^t e^{A(t-s)} f(s)\, ds,\ t \in (t_k,t_{k+1}]. \end{cases}$$

Now define a multioperator $P^k : Q^{(k)} \to \mathcal{P}(Q^{(k)})$ in the following way:

$$P^{(k)}(y^{(k)}) = \left\{ p_f^{(k)} \in Q^{(k)},\ f \in L^1((t_k,t_{k+1}),E),\ f(s) \in F\big(s,y_s^{(k)}\big) \right\}. \tag{3.2}$$

As above if $y^{(k)} \in P^{(k)}(y^{(k)})$, then $y^{(k)}$ is a mild solution of (1.1), (1.2), (1.3) on $[-\tau,t_{k+1}]$. We have to prove that the multioperator $P^{(k)}$ has at least a fixed point. To this aim we split the proof proving three technical lemmata.

Lemma 3.5. *The multioperator $P^{(k)}$ is u.s.c. with compact, convex values.*

Proof. Let $\{y_n^{(k)}\}_{n=1}^{\infty}$, $\{p_n^{(k)}\}_{n=1}^{\infty} \subset Q^{(k)}$, $y_n^{(k)} \to y_0^{(k)}$, $p_n^{(k)} \in P^{(k)}(y_n^{(k)})$, $n \geq 1$ and $p_n^{(k)} \to p_0^{(k)}$.

Consider a sequence $\{f_n^{(k)}\}_{n=1}^{\infty} \subset L^1((t_k, t_{k+1}), E)$ such that $f_n^{(k)} \in P_F(y_n^{(k)})$, $n \geq 1$, $p_n^{(k)} = G^{(k)}(f_n^{(k)})$, where for $f \in L^1((t_k, t_{k+1}), E)$, $G^{(k)}(f) = p_f^{(k)}$. From assumption (F3) it follows that the sequence $\{f_n^{(k)}\}_{n=1}^{\infty}$ is integrably bounded.

Hypothesis (F4) implies that:

$$\chi\left(\{f_n^{(k)}(t)\}_{n=1}^{\infty}\right) \leq \mu(t)\varphi\left(\{(y_n^{(k)})_t\}_{n=1}^{\infty}\right) = 0,$$

for a.e. $t \in [t_k, t_{k+1}]$, i.e., the set $\{f_n^{(k)}(t)\}_{n=1}^{\infty}$ is relatively compact for a.e. $t \in [t_k, t_{k+1}]$.

From Proposition 2.3 it follows that the sequence $\{f_n^{(k)}\}_{n=1}^{\infty}$ is weakly compact in $L^1((t_k, t_{k+1}), E)$, so we can assume, without loss of generality that $f_n^{(k)} \rightharpoonup f_0^{(k)}$.

Applying Theorem 2.7 and Remark 2.8 we conclude that $p_n^{(k)} = G^{(k)} f_n^{(k)} \to G^{(k)} f_0^{(k)} = p_0^{(k)}$. Moreover, by Lemma 3.3 we have $f_0^{(k)} \in P_F(y_0^{(k)})$, therefore $p_0^{(k)} \in G^{(k)} \circ P_F(y_0^{(k)}) = P^{(k)}(y_0^{(k)})$, demonstrating that the multioperator $P^{(k)}$ is closed.

In particular for any convergent sequence $\{y_n^{(k)}\}_{n=1}^{\infty} \subset Q^{(k)}$, a sequence $\{f_n^{(k)}\}_{n=1}^{\infty} \subset L^1((t_k, t_{k+1}), E)$, with $f_n^{(k)} \in P_F(y_n^{(k)})$, $n \geq 1$, is integrably bounded and the set $\{f_n^{(k)}(t)\}_{n=1}^{\infty}$ is relatively compact for almost every $t \in [t_k, t_{k+1}]$ and, by Theorem 2.7, the corresponding sequence $\{G^{(k)}(f_n^{(k)})\}_{n=1}^{\infty} \subset Q^{(k)}$ is relatively compact, therefore the multioperator $P^{(k)} = G^{(k)} \circ P_F$ is quasicompact, then we obtain that it is u.s.c. by Theorem 2.1.

The quasicompactness property obviously implies the compactness of values of $G^{(k)} \circ P_F$. Finally, its convexity follows easily from the fact that F has convex values. ∎

It easy to verify that the function ν defined on bounded sets $\Omega \subset Q^{(k)}$ with values in $(\mathbb{R}^2, \geq)$ as:

$$\nu(\Omega) = \max_{D \in \mathcal{D}(\Omega)} (\gamma(D), \delta(D)) \tag{3.3}$$

where $\mathcal{D}(\Omega)$ is the collection of all denumerable subsets of Ω and, for a given constant $L > 0$,

$$\gamma(D) = \sup_{t_k \le t \le t_{k+1}} e^{-Lt}\chi(D(t)) \tag{3.4}$$

$$\delta(D) = \operatorname{mod}_C \left(D\lfloor_{[t_k,t_{k+1}]}\right),$$

is a monotone, non singular, regular MNC.

Lemma 3.6. *The multioperator $P^{(k)}$ is ν-condensing on bounded subsets of $Q^{(k)}$, where ν is the MNC defined in* (3.3), (3.4).

Proof. Let $\Omega \subset Q^{(k)}$ be a bounded set such that

$$\nu\big(P^{(k)}(\Omega)\big) \ge \nu(\Omega). \tag{3.5}$$

Let the maximum in $\nu(P^{(k)}(\Omega))$ be achieved for the countable set $D' = \{p_n^{(k)}\}_{n=1}^{\infty}$, where $p_n^{(k)} = G^{(k)}(f_n^{(k)})$, $f_n^{(k)} \in P_F(y_n^{(k)})$, $n \ge 1$ and $\{y_n^{(k)}\}_{n=1}^{\infty} \subset \Omega$. From (3.5) we have:

$$\gamma\left(\{p_n^{(k)}\}_{n=1}^{\infty}\right) \ge \gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right). \tag{3.6}$$

From (F4) we have for $s \in (t_k, t_{k+1}]$:

$$\chi\left(\{f_n^{(k)}(s)\}_{n=1}^{\infty}\right) \le \mu(s) \sup_{-\tau\le\theta\le 0} \chi\left(\{y_n^{(k)}(s+\theta)\}_{n=1}^{\infty}\right)$$

$$= e^{Ls}\mu(s)e^{-Ls} \sup_{-\tau\le\theta\le 0} \chi\left(\{y_n^{(k)}(s+\theta)\}_{n=1}^{\infty}\right)$$

$$= e^{Ls}\mu(s)e^{-Ls} \sup_{t_k\le\sigma\le s} \chi\left(\{y_n^{(k)}(\sigma)\}_{n=1}^{\infty}\right)$$

$$\le e^{Ls}\mu(s) \sup_{t_k\le\sigma\le t_{k+1}} e^{-L\sigma}\chi\left(\{y_n^{(k)}(\sigma)\}_{n=1}^{\infty}\right)$$

$$= e^{Ls}\mu(s)\gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right).$$

Moreover, from properties of MNC χ we have for $t \in (t_k, t_{k+1}]$:

$$\begin{aligned}
&\chi\left(\{p_n^{(k)}(t)\}_{n=1}^{\infty}\right) \\
&= \chi\left(e^{A(t-t_k)}\left(y^{(k-1)}(t_k) + I_k\left(y_{t_k}^{(k-1)}\right)\right) + \int_{t_k}^{t} e^{A(t-s)}\{f_n^{(k)}(s)\}_{n=1}^{\infty}\, ds\right) \\
&\leq \chi\left(e^{A(t-t_k)}\left(y^{(k-1)}(t_k) + I_k\left(y_{t_k}^{(k-1)}\right)\right)\right) + \chi\left(\int_{t_k}^{t} e^{A(t-s)}\{f_n^{(k)}(s)\}_{n=1}^{\infty}\, ds\right) \\
&= \chi\left(\int_{t_k}^{t} e^{A(t-s)}\{f_n^{(k)}(s)\}_{n=1}^{\infty}\, ds\right).
\end{aligned}$$

Applying Theorem 2.6 we have

$$\begin{aligned}
&e^{-Lt}\chi\left(\{p_n^{(k)}(t)\}_{n=1}^{\infty}\right) \leq e^{-Lt} 2M \int_{t_k}^{t} e^{Ls}\mu(s)\, ds \;\cdot \gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right) \\
&\leq 2M \sup_{t\in[t_k,t_{k+1}]} e^{-Lt}\int_{t_k}^{t} e^{Ls}\mu(s)\, ds \cdot \gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right),
\end{aligned}$$

where $M > 0$ is a constant such that:

$$\|e^{At}\|_E \leq M\ ,\ t \in [0, d]. \tag{3.7}$$

Then

$$\gamma\left(\{p_n^{(k)}\}_{n=1}^{\infty}\right) \leq \sup_{t\in[t_k,t_{k+1}]} e^{-Lt} 2M \int_{t_k}^{t} e^{Ls}\mu(s)\, ds \cdot \gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right). \tag{3.8}$$

We can choose the constant $L > 0$ so that

$$\sup_{t\in[t_k,t_{k+1}]}\left[2M\int_{t_k}^{t} e^{-L(t-s)}\mu(s)\, ds\right] < 1.$$

Then from (3.6) and (3.8) we have

$$\gamma\left(\{p_n^{(k)}\}_{n=1}^{\infty}\right) = \gamma\left(\{y_n^{(k)}\}_{n=1}^{\infty}\right) = 0$$

and hence

$$\chi\left(\{y_n^{(k)}(t)\}_{n=1}^{\infty}\right)=0,$$

for all $t\in(t_k,t_{k+1}]$.

With the same arguments used in Lemma 3.5 we obtain that $\{p_n^{(k)}\}_{n=1}^{\infty}$ is a relatively compact sequence, therefore

$$\delta\left(\{p_n^{(k)}\}_{n=1}^{\infty}\right)=0,$$

i.e., $\nu\left(P^{(k)}(\Omega)\right)=(0,0)$ and from (3.5) $\nu(\Omega)=(0,0)$, then Ω is a relatively compact set. ■

Let the function $\tilde{y}^{(k)}\in Q^{(k)}$ be defined by $\tilde{y}^{(k)}(t)\equiv e^{A(t-t_k)}(y^{(k-1)}(t_k)+I_k(y_{t_k}^{(k-1)}))$, $t\in(t_k,t_{k+1}]$. Consider the following family of multimaps $\Phi_k: Q^{(k)}\times[0,1]\to\mathcal{K}v(Q^{(k)})$ given by:

$$\Phi_k(y^{(k)},\lambda)=\left\{w^{(k)}\in Q^{(k)}|\begin{array}{l} w^{(k)}(t)=\tilde{y}^{(k)}(t)+\lambda\displaystyle\int_{t_k}^{t}e^{A(t-s)}f(s)\,ds,\\ t\in(t_k,t_{k+1}]: f\in P_F(y^{(k)})\end{array}\right\}. \tag{3.9}$$

Lemma 3.7. *The set of fixed points of Φ_k, i.e.,*

$$\operatorname{Fix}\Phi_k=\{y\in\Phi_k(y,\lambda)\,\textit{for some}\,\lambda\in[0,1]\},$$

is a priori bounded.

Proof. Let $y^{(k)}\in\operatorname{Fix}\Phi_k$, $N_{k-1}=\sup\limits_{-\tau\le t\le t_k}|y^{(k-1)}(t)|$, then we have for $t\in[t_k,t_{k+1}]$:

$$\begin{aligned}
&|y^{(k)}(t)|\le\|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\int_{t_k}^{t}|f(s)|\,ds\\
&\le\|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\int_{t_k}^{t}\alpha(s)\left(1+\|y_s^{(k)}\|_{\mathcal{C}([-\tau,0],E)}\right)ds\\
&\le\|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}+M\int_{t_k}^{t}\alpha(s)\|y_s^{(k)}\|_{\mathcal{C}([-\tau,0],E)}\,ds\\
&\le\|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}
\end{aligned}$$

$$+M\int_{t_k}^{t}\alpha(s)\left(\sup_{-\tau\le\sigma\le t_k}\left|y^{(k)}(\sigma)\right|+\sup_{t_k<\sigma\le s}\left|y^{(k)}(\sigma)\right|\right)ds$$

$$\le \|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}\sup_{-\tau<\sigma\le t_k}\left|y^{(k-1)}(\sigma)\right|$$

$$+M\int_{t_k}^{t}\alpha(s)\sup_{t_k<\sigma\le s}\left|y^{(k)}(\sigma)\right|ds$$

$$\le \|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}(1+N_{k-1})$$

$$+M\int_{t_k}^{t}\alpha(s)\sup_{t_k<\sigma\le s}\left|y^{(k)}(\sigma)\right|ds.$$

The right hand side is a an increasing function in t, so we have the same estimate for all $t_k<r\le t$, i.e.

$$\sup_{t_k<r\le t}\left|y^{(k)}(r)\right|\le L_k+M\int_{t_k}^{t}\alpha(s)\sup_{t_k<\sigma\le s}\left|y^{(k)}(\sigma)\right|ds,$$

where $L_k=\|\tilde{y}^{(k)}\|_{\mathcal{C}([-\tau,t_{k+1}],E)}+M\|\alpha\|_{L^1_+[t_k,t_{k+1}]}(1+N_{k-1})$.

Since $y^{(k)}$ is continuous on $(t_k,t_{k+1}]$, the function $\psi(t)=\sup_{t_k<r\le t}|y^{(k)}(r)|$ is also continuous, so

$$\psi(t)\le L_k+M\int_{t_k}^{t}\alpha(s)\psi(s)\,ds$$

by Gronwall-Bellmann inequality:

$$\psi(t)\le L_k\exp\left\{M\int_{t_k}^{t}\alpha(s)\,ds\right\}\le L_k\exp\left\{M\,\|\alpha\|_{L^1_+[t_k,t_{k+1}]}\right\}.$$

■

***Proof of Theorem* 3.4.** Using the same arguments as before we may verify that the family Φ_k defined in (3.9) is u.s.c. and ν condensing on every bounded set $\Omega\subset Q^{(k)}$.

Now we take an open bounded set $U\in\mathcal{C}([-\tau,t_{k+1}],E)$ containing the set $\operatorname{Fix}\Phi_k$. The family Φ_k is fixed point free on the relative boundary $\partial U_{Q^{(k)}}$ and hence it determines an homotopy between the multifield $i-P^{(k)}$ and the multifield $i - \tilde{y}^{(k)}$. Taking into account that $\tilde{y}^{(k)} \in Q^{(k)}$ and using the homotopy and normalization properties of the topological degree for condensing multifields (see [7]), we obtain that $\deg_{Q^{(k)}}(i - P^{(k)}, \overline{U}_{Q^{(k)}}) = \deg_{Q^{(k)}}(i - \tilde{y}^{(k)}, \overline{U}_{Q^{(k)}}) = 1$ and therefore

$$\emptyset \neq \operatorname{Fix} P^{(k)} \subset U_{Q^{(k)}}.$$

Now assume that functions I_k are continuous.

Applying Theorem 2.9 we obtain that the fixed points set of $P^{(0)}$ is compact. This set formes a solution set $\Sigma^{(0)}(x)$ to the problem (1.1), (1.2), (1.3) on the interval $[-\tau, t_1]$. Starting from each solution $y^{(0)} \in \Sigma^{(0)}(x)$ we obtain, again from Theorem 2.9, the compactness of the fixed points set of $P^{(1)}$, and hence the compactness of the set $\Sigma^{(1)}(y^{(0)})$ of all solutions $y^{(1)}$ on the interval $[-\tau, t_2]$ with a given restriction $y^{(0)}$ on $[-\tau, t_1]$.

Using the fact that the function I_1 is continuous, and applying Theorem 2.10, we conclude that the multimap $y^{(0)} \multimap \Sigma^{(1)}(y^{(0)})$ defined on the compact set $\Sigma^{(0)}(x)$ is u.s.c.

Then from Theorem 2.2 it follows that the solution set on the interval $[-\tau, t_2]$, $\Sigma^{(1)} = \Sigma^{(1)}(\Sigma^{(0)}(x))$, is a compact set.

Iterating this process we obtain the compactness of the solutions set on the whole interval $[-\tau, d]$. ∎

Now we consider the case with the jump sizes not deterministic, i.e., we give an existence theorem for the problem (1.1), (1.2), (1.4).

In this case a mild solution is defined in the following way:

Definition 3.8. A function $y \in \mathcal{C}([-\tau, d], E)$ is said to be a mild solution of the problem (1.1), (1.2), (1.4), if there exists a function $f \in L^1([0, d], E)$ such that $f(t) \in F(t, y_t)$ for a.e. $t \in [0, d]$ and

$$y(t) = x(t)\ ,\ t \in [-\tau, 0];$$
$$y(t) = e^{At}x(0) + \int_0^t e^{A(t-s)} f(s)\, ds + \sum_{0<t_k<t} e^{A(t-t_k)} z_k\ ,\ t \in [0, d];$$

with $z_k \in \mathcal{I}_k(y_{t_k})$.

Theorem 3.9. *Under assumption* (A), (F1)–(F4), *the problem* (1.1), (1.2), (1.4) *has a mild solution on* $[-\tau, d]$, *moreover if* $\mathcal{I}_k$, $k = 0, \ldots, m$ *are u.s.c. multimaps with compact values, then the solution set is compact in the space* $\mathcal{C}([-\tau, d], E)$.

Proof. We proceed as in the previous case and we define the sets $\overline{Q}^{(k)} \subset \mathcal{C}(-\tau, t_{k+1})$ and the multioperators $\overline{P}^{(k)} : \overline{Q}^{(k)} \to \mathcal{P}(\overline{Q}^{(k)})$, $k = 0, \ldots, m$, in the following way.

$$\overline{Q}^0 = Q^0;$$

$$\overline{Q}^{(k)} = \left\{ y : \begin{array}{l} y(t) = y^{(k-1)}(t),\ t \in [-\tau, t_k], \\ y \,\text{is continuous on}\, (t_k, t_{k+1}],\ y(t_k^+) = z_k, \\ z_k \in y^{(k-1)}(t_k) + \mathcal{I}_k(y_{t_k}^{(k-1)}) \,\text{is a given point} \end{array} \right\}.$$

Given $f \in L^1((t_k, t_{k+1}), E)$ and $z_k \in y^{(k-1)}(t_k) + \mathcal{I}_k(y_{t_k}^{(k-1)})$, we define functions $p_f^{(k)} : [-\tau, t_{k+1}] \to E$ as:

$$\overline{p}_f^{(k)}(t) = \begin{cases} y^{(k-1)}(t)\ ,\ t \in [-\tau, t_k]; \\ e^{A(t-t_k)} z_k + \displaystyle\int_{t_k}^{t} e^{A(t-s)} f(s)\, ds\ ,\ t \in (t_k, t_{k+1}]; \end{cases}$$

and we define

$$\overline{P}^0 \equiv P^0;$$

$$\overline{P}^{(k)}(y^{(k)}) = \left\{ \overline{p}_f^{(k)} \in Q^{(k)},\ f \in L^1((t_k, t_{k+1}), E),\ f(s) \in F(s, y_s^{(k)}) \right\};$$

With arguments very much similar to those used in Lemmata 3.5, 3.6, 3.7 we can prove that multioperators $P^{(k)}$, $k = 0, \ldots, m$ are u.s.c. with compact convex values and ν-condensing.

Let $\overline{y}^{(k)} \in \overline{Q}^{(k)}$ be the function defined by $\overline{y}^{(k)}(t) \equiv e^{A(t-t_k)} z_k$, $t \in [t_k, t_{k+1}]$.

As in the deterministic case we can construct the homotopy $\overline{\Phi}_k : \overline{Q}^{(k)} \times [0,1] \to \mathcal{K}v(Q^{(k)})$ in the following way:

$$\overline{\Phi}_k(y^{(k)}, \lambda) = \left\{ w^{(k)} \in Q^{(k)} \middle| \begin{array}{c} w^{(k)}(t) = \overline{y}^{(k)}(t) + \lambda \displaystyle\int_{t_k}^{t} e^{A(t-s)} f(s)\, ds, \\ t \in (t_k, t_{k+1}] : f \in P_F(y^{(k)}) \end{array} \right\}.$$

As in Lemma 3.7 we can prove that the fixed points set of $\overline{\Phi}_k$ is a bounded set.

Now we take an open bounded set $U \subset \overline{Q}^{(k)}$ containing the set $\operatorname{Fix} \overline{\Phi}_k$, then from the homotopy and normalization properties of the relative topological degree we have that $\deg_{\overline{Q}^{(k)}}(i - \overline{P}^{(k)}, \overline{U}_{\overline{Q}^{(k)}}) = \deg_{\overline{Q}^{(k)}}(i - \overline{y}^{(k)}, \overline{U}_{\overline{Q}^{(k)}}) = 1$, then

$$\emptyset \neq \operatorname{Fix} \overline{P}^{(k)} \subset U_{Q^{(k)}}.$$

The similar reasoning as before and the assumption that $\mathcal{I}_k$, $k = 0, \ldots, m$ are u.s.c. with compact values allow us to prove, by using Theorem 2.2, that the solutions set on the whole interval $[-\tau, d]$ is a compact set. ■

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