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# CONTROLLABILITY OF EVOLUTION EQUATIONS AND INCLUSIONS DRIVEN BY VECTOR MEASURES

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**Abstract**

In this paper, we consider the question of controllability of a class of linear and semilinear evolution equations on Hilbert space with measures as controls. We present necessary and sufficient conditions for weak and exact (strong) controllability of a linear system. Using this result we prove that exact controllability of the linear system implies exact controllability of a perturbed semilinear system. Controllability problem for the semilinear system is formulated as a fixed point problem on the space of vector measures and is concluded controllability from the existence of a fixed point. Our results cover impulsive controls as well as regular controls.

**Keywords:** controlability, impulsive systems, differential inclusions, Hilbert spaces, vector valued measures, $C_0$ semigroups.

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## 1. Introduction

In control theory the question of controllability is a very important issue. It has been extensively studied in the sixties, through eighties, by Fattorini, Triggiani, Russell and others, see [1–7] and the references therein. These papers deal with ordinary controls and here we use vector measures as controls. We consider systems governed by semilinear evolution equations and inclusions on Hilbert space driven by vector measures [13] as controls. Since the vector measures may be atomic (may contain atoms) our formulation includes impulsive controls. In recent years impulsive systems have been extensively studied developing the existence theory and necessary conditions of optimality [8–12]. But the question of controllability seems to be untouched.

Here we are interested in controllability problems. We present necessary and sufficient conditions for weak and exact (strong) controllability of linear systems. Using this result, we formulate the controllability problem for semilinear systems as a fixed point problem on the space of vector measures. Using the exact controllability result for the linear system, we prove exact controllability of the semilinear system and systems governed by Differential Inclusions.

We formulate the (exact) controllability problems for semilinear systems as a fixed point problem on the space of vector measures. Then we obtain controllability results by proving the existence of fixed points. According to the knowledge of the author, it appears that in the area of systems and control theory this is the first time a fixed point problem on the space of vector measures is encountered. A closely related problem is the viability problem [17–18] and we believe that this can be formulated also as a fixed point problem for multivalued maps.

The paper is organized as follows. In Section 2 some basic notations are presented. In Section 3, we present the basic system dynamics considered in this paper. In Section 4, we discuss the question of weak and exact controllability of linear systems. This section is concluded with some examples of dominating measures and some examples of linear systems admitting exact controllability or lack of it. In Section 5, we present a result on exact controllability of the semilinear system. In Section 6, we present a similar result for Differential Inclusions proving the existence of solutions for multi valued maps on the space of vector measures. The paper is concluded with some comments on open problems.

## 2. Some Notations

**Basic Function Spaces:** Let $H$ be a real separable Hilbert space and $I \equiv [0,T]$ a finite interval and $B_0(I,H)$ the vector space of bounded functions on $I$ with values in $H$. Furnished with the norm topology, given by

$$\parallel z \parallel_0 \equiv \sup\{\parallel z(t) \parallel_H, t \in I\},$$

this is a Banach space. But the elements of this space need not be measurable. Let $\Sigma$ denote the sigma algebra of Borel subsets of the set $I$ and $B(I, H)$ with the space of bounded $\Sigma$ measurable functions on $I$ with values in $H$ equiped the same sup norm topology. This is a closed subspace of $B_0(I, H)$ and hence a Banach space. For convenience of notations we use $\| z \|$ to denote the norm of any element $z \in B(I, H)$.

**Vector Measures:** Let $E$ be any real Hilbert space and $\mathcal{M}_c(I, E)$ the space of bounded countably additive vector measures on the sigma algebra $\Sigma$ with values in the Hilbert space $E$. We furnish this vector space with the (strong) total variation norm,

$$|\nu|_V \equiv |\nu|(I) \equiv \sup_{\pi} \Big\{ \sum_{J \in \pi} \| \nu(J) \|_E \Big\},$$

where the supremum is taken over all partitions $\pi$ of the interval $I$ into a finite number of pair wise disjoint members of $\Sigma$. With respect to this norm topology, $\mathcal{M}_c(I, E)$ is a Banach space. For any $J \in \Sigma$, define the variation of $\nu$ on $J$ by

$$V(\nu)(J) \equiv |\nu|(J).$$

Since $\nu$ is countably additive and bounded, this induces a countably additive bounded positive measure $V(\nu)$ on $\Sigma$.

**Definition 2.1.** Let $\mu$ be any countably additive nonnegative finite measure (having bounded total variation) defined on the sigma field $\Sigma$ and let $\nu$ be a countably additive $E$-valued vector measure on $\Sigma$ having bounded total variation. The vector measure $\nu$ is said to be $\mu$-continuous, written $\nu \ll \mu$, if

$$\lim_{\mu(D) \to 0} \nu(D) = 0, D \in \Sigma.$$

For details on vector measures see Diestel and Uhl [13].

In case $E = R$, the real line, we have the space of real valued countably additive bounded signed measures. We denote this by simply $\mathcal{M}_c(I)$ in place of $\mathcal{M}_c(I, R)$. Clearly for $\nu \in \mathcal{M}_c(I)$, $V(\nu)$ is also a countably additive bounded positive measure. For the uniformity of notation, we may use $\lambda$ to denote the Lebesgue measure $dt$. Given two measures $\mu_1$ and $\mu_2$, we use the notation, $\mu_1 \perp \mu_2$, to indicate that $\mu_1$ is singular with respect to the measure $\mu_2$ and conversely. Similarly, we use, $\mu_1 \prec \mu_2$, to indicate that $\mu_1$ is absolutely continuous with respect to $\mu_2$.

**Lebesgue-Bochner Spaces:** For any Banach space $X$ and any $p \geq 1$ and any countably additive bounded positive measure on $\Sigma$, having bounded total variation on $I \equiv [0, T]$, we let

$$L_p(\mu, X) \equiv \Big\{ f : I \longrightarrow X : \int_I |f(t)|_X^p \mu(dt) < \infty \Big\}$$

denote the equivalence class of $\mu$ measurable Lebesgue-Bochner $p$-th power summable functions. Furnished with the standard norm topology these are Banach spaces.

**Linear Operators:** For any two Banach spaces $X$ and $Y$ we use $\mathcal{L}(X, Y)$ to denote the Banach space of bounded linear operators from $X$ to $Y$. In case $X = Y$, we write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. We introduce other notations as and when required.

## 3. System dynamics

We wish to study the questions of controllability of the measure driven linear system given by

$$dx(t) = Ax(t)dt + B(t)u(dt), x(0) = x_0,\ t \geq 0, \tag{1}$$

where $A$ is the infinitesimal generator of a $C_0$ semigroup of operatrors $S(t), t \geq 0$, on the Hilbert space $H$ and $u \in \mathcal{M}_c(I, E)$ is a control, vector measure with values in the Hilbert space $E$, and $B$ is a uniformly measurable bounded operator valued function with values in $\mathcal{L}(E, H)$. Next we consider the perturbed system given by the semilinear evolution equation,

$$dx(t) = Ax(t)dt + B(t)u(dt) + f(t, x(t))dt, x(0) = \xi,\ t \in I \tag{2}$$

where $A$ and $B$ are as described for the linear problem, and $f$ is a measurable map from $I \times H$ to $H$ continuous in the second argument. We study the question of exact controllability of the semilinear problem. For basic materials on semigroup theory the reader may consult [15], for vector measures see Diestel and Uhl Jr. [13] and for multivalued maps see [16].

## 4. Controllability of linear system

Let $\mu$ be any fixed nonnegative countably additive real valued bounded measure defined on the sigma field $\Sigma$ having bounded total variation on $I$. For admissible controls, we introduce the following class of vector measures given by

$$\mathcal{U}_\mu \equiv \{u \in \mathcal{M}_c(I, E) : u \ll \mu\} \tag{3}$$

and call $\mu$ the dominating measure. We choose this set as the class of admissible controls.

**Remark.** Since we want to include impulsive forces in our general class of admissible controls, $\mu$ cannot be absolutely continuous with respect to the Lebesgue measure.

Let $PWC(I, H)$ denote the class of piecewise continuous $H$-valued functions on $I$. Considering the system (1) first, we prove that it has a bounded measurable piecewise continuous mild solution. This is stated in the following lemma.

**Lemma 4.1.** *Suppose $A$ is the generator of a $C_0$ semigroup $S(t), t \geq 0$, on $H$ and $B$ is a uniformly measurable bounded operator valued function with values in $\mathcal{L}(E, H)$. Then, for every initial state $x_0 \in H$ and any control $u \in \mathcal{M}_c(I, E)$, the system* (1) *has a unique mild solution $x \in B(I, H)$ and it is given by the expression*

$$x(t) = S(t)x_0 + \int_0^t S(t-s)B(s)u(ds), t \in I. \tag{4}$$

*and further $x \in PWC(I, H)$.*

***Proof.*** The existence and uniqueness of a mild solution follow from the variation of constants formula. Since $S$ is strongly continuous, for fixed $x_0 \in H$ the first term on the right hand side is continuous and hence measurable. By use of Fubin's theorem and strong continuity of $S$ and uniform measurability of $B$, it is easy to see that the function

$$t \longrightarrow \int_0^t S(t-s)B(s)u(ds)$$

is weakly measurable. Since $H$ is separable it follows from well known Petti's measureability theorem that the function is measurable. Hence $x$ given by (4) is measurable. For the boundedness, taking $H$ norm on either side, we obtain

$$\sup\{\| x(t) \|_H, t \in I\} \leq M \| x_0 \|_H + Mb|u|_V < \infty, \tag{5}$$

where

$$M \equiv \sup\{\| S(t) \|_{\mathcal{L}(H)},\ t \in I\} \text{ and } b \equiv \sup\{\| B(t) \|_{\mathcal{L}(E,H)},\ t \in I\}.$$

Thus for any given $u \in \mathcal{M}_c(I, E)$, we have $x \in B(I, H)$. Since the measure $u$ has bounded total variation, it can have at most a countable set of atoms and it is at the atoms that $x$ will have discontinuities. This means that $x \in PWC(I, E) \cap B(I, H)$. This completes the proof. ■

For any $t \geq 0$, the attainable set is given by

$$\mathcal{A}_\mu(t) \equiv \left\{\xi \in H : \xi = S(t)x_0 + \int_0^t S(t-r)B(r)u(dr), u \in \mathcal{U}_\mu\right\}. \tag{6}$$

**Definition 4.2** (Weak and Exact (strong) Controllability)**.** The linear system (1) with admissible controls $\mathcal{U}_\mu$ is said to be weakly globally controllable if, for each $T > 0$, $\mathcal{A}_\mu(T)$ is weakly dense in $H$. It is said to be strongly controllable if $\mathcal{A}_\mu(T) = H$ for a finite $T$.

For any $T > 0$, define the operator

$$C_\mu([0, T]) \equiv C_\mu(T) \equiv \int_0^T S(T-r)B(r)B^*(r)S^*(T-r)\mu(dr), \tag{7}$$

and call this the controllability operator.

**Remark.** It is clear from the above expression that the controllability operator is not only dependent on the system parameters but also on the dominating measure that determines the class of admissible controls. If $\mu$ is a Lebesgue measure, then the controls are classical and clearly do not admit impulsive inputs.

**Theorem 4.3** (Weak Controllability). *Suppose the assumptions of Lemma* 4.1 *hold and let* $\mu$ *be a countably additive nonnegative bounded scalar valued measure having bounded total variation on* $I \equiv [0,T]$; *and let* $\mathcal{U}_\mu$, *as defined by* (3), *denote the class of admissible controls. Then the system* (1) *is weakly globally controllable if and only if the controllability operator* $C_\mu(T)$ *is positive definite in* $H$, *that is,* $(C_\mu(T)h, h) > 0$ *for* $h(\in H) \neq 0$.

***Proof.*** Suppose the system is weakly globally controllable. We must show that the operator $C_\mu(T)$ is positive definite. Clearly the operator $C_\mu(T)$ is symmetric and positive. Being bounded, it is a self adjoint positive operator in $H$. Since the system is globally weakly controllable, the attainable set $\mathcal{A}_\mu(T)$ is dense in $H$. Thus, $S(T)x_0$ being fixed, the set

$$\mathcal{R}_\mu(T) \equiv \left\{ \xi \in H : \xi = \int_0^T S(T-r)B(r)u(dr), u \in \mathcal{U}_\mu \right\}$$

must be dense in $H$. This means that

$$(z, h) = 0 \ \forall \ z \in \mathcal{R}_\mu(T) \Longrightarrow h = 0.$$

This is equivalent to saying that

$$\int_0^T < B^*(r)S^*(T-r)h, u(dr) >_E = 0 \ \forall \ u \in \mathcal{U}_\mu \Longrightarrow h = 0. \tag{8}$$

Since $u \ll \mu$ and every Hilbert space, (in general, every reflexive Banach space), satisfies RNP (Radon-Nikodym Property) [13], for every $u \in \mathcal{U}_\mu$ there exists a $g_u \in L_1(\mu, E)$ such that $du = g_u d\mu$. Hence

$$\int_0^T < B^*(r)S^*(T-r)h, g(r) >_E \mu(dt) = 0 \ \forall \ g \in i(\mathcal{U}_\mu) \Longrightarrow h = 0, \tag{9}$$

where

$$i(\mathcal{U}_\mu) = \left\{ g \in L_1(\mu, E) : du = g d\mu, u \in \mathcal{U}_\mu \right\} \subset L_1(\mu, E).$$

In fact, it is evident that we have an isometric isomorphism,

$$i(\mathcal{U}_\mu) \cong L_1(\mu, E).$$

Letting $L$ denote the linear operator,

$$Lu \equiv \int_0^T S(T-r)B(r)u(dr), \tag{10}$$

from $\mathcal{M}_c(I,E)$ to $H$, it follows from (9) that the adjoint of the operator $L$ has the trivial null space, that is,

$$KerL^* = \{0\}.$$

This is equivalent to the statement that

$$(L^*h)(r) \equiv B^*(r)S^*(T-r)h = 0, \quad \mu \ a.e \Longrightarrow h = 0.$$

Since $B(\cdot)$ is a uniformly measurable bounded operator valued function with values in $\mathcal{L}(E,H)$ so is also its adjoint $B^*(\cdot)$. So it follows from strong continuity of the semigroup $S(\cdot)$ that the function

$$r \longrightarrow (L^*h)(r) \equiv g^*(r), r \in I$$

is a bounded strongly measurable function with values in $E$. That is $g^* \in B(I,E)$. Clearly, this implies that $g^* \in L_\infty(\mu,E) \subset L_2(\mu,E) \subset L_1(\mu,E)$ and hence $g^* \in i(\mathcal{U}_\mu)$. Substituting $g^*$ for $g$ in equation (9), it follows that

$$\int_0^T < B^*(r)S^*(T-r)h, B^*(r)S^*(T-r)h >_E \mu(dt) = 0 \ \Longrightarrow h = 0. \tag{11}$$

This is identical to the statement,

$$(C_\mu(T)h,h) = 0 \Longrightarrow h = 0. \tag{12}$$

Thus $C_\mu(T)$ is a positive definite self adjoint operator in the Hilbert space $H$.

For the converse, suppose the controllability operator $C_\mu(T)$ is positive definite, but the system is not weakly controllable. Then the reachable set $\mathcal{R}_\mu(T)$ is not dense in $H$. This means that the weak closure of the reachable set $\mathcal{R}_\mu(T)$ is a proper subset of $H$. That is,

$$\overline{\mathcal{R}_\mu(T)}^w \subsetneq H$$

and hence there is an element $\zeta \in H$ which is not in the weak closure of the reachable set $\mathcal{R}_\mu(T)$. Then it follows from the Hahn-Banach separation theorem that there exists a linear functional $\ell$ on $H$ or equivalently an element $\ell_{h^*} = h^* \neq 0 \in H$ such that

$$\ell_{h^*}(\zeta) = (h^*, \zeta) > 0 \text{ and } \ell_{h^*}(\mathcal{R}_\mu(T)) = \{(h^*, z)\ z \in \mathcal{R}_\mu(T)\} = 0.$$

This means that

$$\int_0^T < B^*(r)S^*(T-r)h^*, u(dr) >_E = 0\ \forall\ u \in \mathcal{U}_\mu. \tag{13}$$

Since the function

$$r \longrightarrow B^*(r)S^*(T-r)h^* \equiv \tilde{g}(r)$$

is uniformly bounded and strongly measurable, it belongs to $L_1(\mu, E)$ as well as $L_\infty(\mu, E)$ and hence the measure $\nu$, defined by $d\nu = \tilde{g}d\mu$, is an element of the admissible class $\mathcal{U}_\mu$. Thus it follows from (13) that

$$\begin{aligned} 0 &= \int_0^T < B^*(r)S^*(T-r)h^*, \nu(dr) >_E \\ &= \int_0^T < B^*(r)S^*(T-r)h^*, \tilde{g}(r) >_E \mu(dr) \\ &= \int_0^T < B^*(r)S^*(T-r)h^*, B^*(r)S^*(T-r)h^* >_E \mu(dr) \\ &= (C_\mu(T)h^*, h^*)_H. \end{aligned}$$

Since by our hypothesis the operator $C_\mu(T)$ is positive definite in $H$, we must have $h^* = 0$. This is a contradiction and hence $\mathcal{R}_\mu(T)$ is weakly dense in $H$ proving the theorem. ■

We have so far considered weak controllability or equivalently approximate controllability. Some applied problems may require exact contollability, that is, hitting the target exactly.

**Theorem 4.4** (Exact Controllability)**.** *Consider the linear system* (1) *and suppose* $\{A, B(\cdot)\}$ *satisfy the assumptions of Lemma* 4.1. *Then the system* (1), *with admissible controls* $\mathcal{U}_\mu$ *given by* (3), *is exactly controllable if and only if the controllability operator* $C_\mu$ *given by* (7) *has a bounded inverse in* $H$.

***Proof.*** Clearly the operator $C_\mu = C_\mu(T)$ is a bounded selfadjoint positive operator in the Hilbert space $H$. Suppose it has a bounded inverse, that is, $C_\mu^{-1} \in \mathcal{L}(H)$. Take any target $h \in H$ and note that the control $u^o$ given by the vector measure,

$$u^o(\sigma) \equiv \int_\sigma B^*(r)S^*(T-r)C_\mu^{-1}\big(h - S(T)x_0\big)\mu(dr), \sigma \in \Sigma,$$

is in the admissible class $\mathcal{U}_\mu$. Substituting this control in (4) one can easily verify that this drives the system from the initial state $x_0 \in H$ to the target state $h \in H$ at time $T$, where $T$ is any finite positive number. Thus the system is exactly (globally) controllable. This proves sufficiency. Now we prove that this condition is also necessary. Suppose the system is exactly controllable. Then, clearly the range $R(L)$ of the linear operator $L$, as defined in Theorem 4.3, is all of $H$. Thus it follows from a corollary to the closed range theorem [14, Corollary 1, p. 208] that $L^*$ has a continuous inverse. Hence there exists a constant $c > 0$ such that

$$(C_\mu h, h) \equiv \int_I |(L^*h)(r)|_E^2 \mu(dr) \geq c|h|_H^2 \ \forall\ h \in H$$

and consequently $|C_\mu h|_H \geq c|h|_H$. Hence the operator $C_\mu$ admits a continuous inverse (on its range) and it follows from the same corollary that $R(C_\mu^*) = H$. Since $C_\mu$ is self adjoint $R(C_\mu) = R(C_\mu^*) = H$. Thus $C_\mu$ is bijective and hence $C_\mu^{-1}$ is also a continuous linear operator in $H$. This completes the proof. ∎

**Remark.** Note, in the finite dimensional case, $(dim(H), dim(E) < \infty)$, strict positivity of $C_\mu$ (that is $C_\mu(T)$ positive definite) implies nonsingularity and hence invertibility which in turn implies exact controllability. This is false in the infinite dimensional case since $C_\mu > 0$ does not imply that $C_\mu$ has a continuous inverse.

**Two Examples of Dominating Measures.** Here we present two simple examples of dominating measures.

(1): Consider the purely atomic measure given by

$$\mu(dt) \equiv \sum_{i=1}^{\infty} p_i \delta_{t_i}(dt), p_i \geq 0, \sum_{i=1}^{\infty} p_i < \infty, \tag{14}$$

for $t_i \in I$ where $I \subset [0,T]$ any finite interval. Clearly, this is a countably additive bounded positive measure defined on $\Sigma$ having a bounded total variation on $I$. Appending the $\mu$-null sets, we may consider $(I, \Sigma, \mu)$ a complete measure space. Define the linear operator $\Gamma_\mu$ from $C(I,E)$ to $E$ by

$$\Gamma_\mu(g) \equiv \int_I g(t)\mu(dt).$$

Clearly, this is a bounded linear operator with the bound $\| \Gamma_\mu \| = |\mu|_V = \mu(I)$. Thus it has a bounded linear extension to $B(I,E)$, which we denote again by $\Gamma_\mu$. Now consider the class of vector measures given by

$$\mathcal{U}_\mu^0 \equiv \left\{ \nu : \nu(\sigma) \equiv \int_\sigma g(t)\mu(dt), \sigma \in \Sigma, g \in B(I,E) \right\} \tag{15}$$

as the class of admissible controls for the system (1). In other words, the system is subject to purely impulsive controls. Clearly, corresponding to this class of control measures, the controllability operator $C_\mu$ is given by

$$C_\mu(T) \equiv \sum_{i=1}^{\infty} p_i \bigg( S(T-t_i)B(t_i)B^*(t_i)S^*(T-t_i) \bigg). \tag{16}$$

Thus by our controllability theorem, the system (1) with admissible controls given by (15) is weakly controllable if and only if

$$\bigcup_{i=1}^{\infty} Ker\{p_i B^*(t_i)S^*(T-t_i)\} = \{0\}. \tag{17}$$

It is also interesting to observe that the system is weakly controllable with respect to the given class of admissible controls if for any subinterval $J \subset I$ with $J \in \Sigma$,

$$\bigcup_{t_i \in J} Ker\{p_i B^*(t_i)S^*(T-t_i)\} = \{0\}. \tag{18}$$

(2): Consider the positive measure given by $\tilde{\mu} = \mu + \beta$ where $\mu$ is purely atomic as in the preceding example, and $\beta$ is any positive measure absolutely continuous with respect to the Lebesgue measure. It is easy to verify that $\tilde{\mu}$ is also a countably additive bounded positive measure having a bounded total variation on bounded subsets. Again including $\tilde{\mu}$ null sets, we may consider $(I, \Sigma, \tilde{\mu})$ a complete measure space. For admissible controls one may choose $\tilde{\mu}$ as the dominating measure with $\mathcal{U}_{\tilde{\mu}}$ being the admissible controls comprised of $\tilde{\mu}$ continuous $E$ valued vector measures. Again by virtue of RNP, we have the controllability operator $C_{\tilde{\mu}}(T)$ given by

$$\begin{aligned} C_{\tilde{\mu}}(T) &= C_{\mu}(T) + C_{\beta}(T) \\ &= \sum_{i=1}^{\infty} p_i \Big( S(T-t_i)B(t_i)B^*(t_i)S^*(T-t_i) \Big) \\ &\quad + \int_0^T \Big( S(T-r)B(r)B^*(r)S^*(T-r) \Big) \beta(dr). \end{aligned}$$

It is clear from these examples that for some nontrivial dominating measures the system may be exactly controllable while for others it may not be.

**Some Examples**

**(E1)** (Exact Controllability) Consider the system

$$dx = Axdt + b(t)u(dt), x(0) = x_0 \in H, t \in I.$$

Suppose $A$ generates a unitary group $S(t), t \in R$ in $H$ and $b(\not\equiv 0) \in L_2(I, \mu)$ with $\parallel b \parallel_{\mu}^2 \equiv \int_0^T b^2(t)\mu(dt) > 0$. Note that here $E = H$. The controllability operator is defined

$$C_{\mu}(T) \equiv \int_0^T b^2(t) S(T-t) S^*(T-t) \mu(dt) = \Big( \int_0^T b^2(t)\mu(dt) \Big) I_d$$

where $I_d$ denotes the identity operator in $H$. Clearly, this is invertible and the system is exactly controllable in time $T$. For any target $z \in H$, the control

$$u(dt) = b(t) S^*(T-t) C_{\mu}^{-1}(T)(z - S(T)x_0)\mu(dt)$$

drives the system to the target $z$.

**(E2)** (Exact Controllability) Consider the same system with $A$ generating a group $S(t), t \in R$. In this case the system is exactly controllable with control given by

$$u(dt) = (1/ \parallel b \parallel_{\mu}^{2})b(t)S(t-T)(z - S(T)x_0)\mu(dt).$$

**(E3)** (Exact Controllability) Consider the same system (1) with $A$ generating a group $S(t), t \in R$, in $H$ and suppose there exists $J \subset I$ with $\mu(J) > 0$ such that $KerB^*(t) = \{0\}$ for $\mu$-a.a $t \in J$. Then the controllability operator $C_\mu(T)$ given by

$$C_\mu(T) = \int_0^T S(T-t)B(t)B^*(t)S^*(T-t)\mu(dt)$$

has a bounded inverse on H and hence the system is exactly controllable.

**(E4)** (Lack of Exact Controllability) Consider the system (1) with $A$ generating a compact semigroup $S(t), t > 0$, in $H$, and suppose $\mu$ has no atom at $\{T\}$. Then it is not difficult to verify that $C_\mu(T)$ is a compact operator in $H$. Let $B_r$ denote the closed ball of radius $r > 0$ in $H$. Since the controllability operator is bounded, there exists $r > 0$ such that

$$\parallel C_\mu(T) \parallel_{\mathcal{L}(H)} \leq r.$$

Then $\overline{C_\mu(T)B_1}$, being compact, is a proper subset of the ball $B_r$ and therefore the operator $C_\mu(T)$ is not invertible and hence the system (1) is not exactly controllable.

**(E5)** (Lack of Exact Controllability) In case $A$ generates an analytic semigroup, it is easy to see that $C_\mu(T)H \subset D(A)$ and hence the system (1) is not exactly controllable.

## 5. Cotrollability of semilinear systems

Now we consider the semilinear system (2) given by

$$dx(t) = Ax(t)dt + B(t)u(dt) + f(t, x(t))dt, x(0) = \xi, \, t \in I. \tag{19}$$

We study the question of exact controllability of the semilinear problem. First we prove that the evolution equation has a unique solution in $B(I,H)$ for every $u \in \mathcal{U}_\mu$ and that the control to the solution map is continuous.

**Lemma 5.1** (Existence and Regularity)**.** *Suppose $A$ and $B$ satisfy the assumptions of Lemma* 3.1 *and the nonlinear operator $f : I \times H \longrightarrow H$ is Borel measurable and there exists a $K \in L_1^+(I)$ such that*

$$\| f(t,x) \|_H \leq K(t)[1+ \| x \|_H], \textit{for } x \in H \tag{20}$$

*and it is locally Lipschitz, that is, for every $R > 0$, there exists a $K_R \in L_1^+(I)$ such that*

$$\| f(t,x) - f(t,y) \|_H \leq K_R(t)[\| x - y \|_H] \textit{ for } x, y \in B_R \subset H, \tag{21}$$

*where $B_R$ denotes the open ball of radius $R$ around the origin. Then, for every $x_0 \in H$ and $u \in \mathcal{U}_\mu$, equation* (19) *has a unique mild solution $x \in B(I,H)$ and the map*

$$u \longrightarrow x(\cdot, u)$$

*is Lipschitz continuous with respect to the strong topologies on $\mathcal{M}_c(I,E)$* (*induced by the total variation norm*)*, and $B(I,H)$* (*induced by the sup norm*)*.*

***Proof.*** For a given $u \in \mathcal{M}_c(I,E)$ and $x_0 \in H$, the proof of the existence and uniquness is classical. It follows from the integral equation

$$x(t) = (Gx)(t) = h(t) + \int_0^t S(t-r)f(r,x(r))dr,\ t \in I \tag{22}$$

where $h$ is given by the following expression,

$$h(t) \equiv S(t)x_0 + \int_0^t S(t-r)B(r)u(dr),\ t \in I.$$

It is clear that $h \in B(I,H)$ and hence $\sup\{\| h(t) \|_H, t \in I\} \equiv \bar{h}_u < \infty$. Using the integral equation (22) and the growth condition satisfied by $f$, one can easily derive a priori bound. In fact, this bound is given by

$$R_u \equiv \bar{h}_u \ Exp\left\{ M \int_I K(t)dt \right\}.$$

Further, for any $y \in B(I,H)$, it is easy to see that $Gy \in B(I,H)$. Thus following the standard procedure, involving the Banach fixed point theorem, one can verify the existence and uniqueness of a solution $x \in B(I,H)$. We prove the Lipschitz continuity of the control to the solution map. Let $x(\cdot,u), x(\cdot,v)$ denote the unique solutions corresponding to control measures $u, v \in \mathcal{U}_\mu$ respectively. By the a priori bounds, there exists an $R \equiv R_u \vee R_v$ so that $x(t,u), x(t,v) \in B_R \subset H$ for all $t \in I$. Then it follows from the integral equations corresponding to $u$ and $v$ respectively, that

$$\begin{aligned} \| x(t,u) - x(t,v) \|_H \leq M \int_0^t \| B(r) \|_{\mathcal{L}(E,H)} \, |u-v|(dr) \\ + M \int_0^t K_R(r) \| x(r,u) - x(r,v) \|_H \, dr, \end{aligned} \tag{23}$$

where $M \equiv \sup\{\| S(t) \|, t \in I\}$. Since $B$ is uniformly measurable and bounded, there exists a number $b > 0$ such that $\sup\{\| B(r) \|, r \in I\} \leq b$. Then it follows from (23) and Gronwall inequality that

$$\begin{aligned} \| x(u) - x(v) \|_{B(I,H)} &\equiv \sup\{\| x(t,u) - x(t,v) \|_H, \, t \in I\} \\ &\leq C|u-v|_V \end{aligned} \tag{24}$$

where $C \equiv (Mb) \, exp\{M \int_0^T K_R(s)ds\} < \infty$. Thus we have proved Lipschitz continuity of the control to the solution map. ■

Now we show that the exact controllability problem of the semilinear system (19) is equivalent to a fixed point problem in the space of vector measures $\mathcal{M}_c(I,E)$. We have seen in Lemma 5.1 that for any $v \in \mathcal{M}_c(I,E)$, the system (19) has a unique (mild) solution denoted by $x(\cdot,v)$. For any $z \in H$, and for any such solution, define the mapping

$$\eta_z : I \times \mathcal{M}_c(I,E) \longrightarrow H$$

by

$$\eta_z(t,v) \equiv z - \int_0^t S(t-r) f(r, x(r,v)) dr, t \in [0,T], v \in \mathcal{M}_c(I,E). \tag{25}$$

By virtue of Lemma 5.1, this is well defined and it is readily verified that, for each $z \in H$, $t \in I$, and $v \in \mathcal{U}_\mu \subset \mathcal{M}_c(I,E)$, it takes values from the Hilbert space $H$. Now suppose that the linear system (1) is globally exactly controllable. Then considering $\eta_z(\tau, v)$ as the target for the linear system, it is clear that for any given $z \in H$ there exists a control $u^o \in \mathcal{U}_\mu$ such that

$$\eta_z(\tau, v) = S(\tau)x_0 + \int_0^\tau S(\tau - r)B(r)u^o(dr) \tag{26}$$

for some $\tau \in [0, T]$. Exact controllability of the linear system (1) however implies that the (controllability) operator $C_\mu(\tau)$ has a bounded inverse in $H$. Hence a control that drives the linear system to the target $\eta_z(\tau, v)$ is given by the vector measure

$$u^o(\sigma) \equiv \int_\sigma B^*(r)S^*(\tau-r)C_\mu^{-1}(\tau)[\eta_z(\tau, v)-S(\tau)x_0]\mu(dr), \sigma \in \Sigma_\tau \subset \Sigma, \tag{27}$$

where $\Sigma_\tau$ denotes the sigma algebra of Borel sets of the interval $I_\tau \equiv [0, \tau]$. This is easily verified by direct substitution of (27) into (26).

Define the operator

$$F : \mathcal{M}_c([0, \tau], E) \longrightarrow \mathcal{M}_c([0, \tau], E)$$

with values $(Fv)(J))$ given by

$$(Fv)(J) \equiv \int_J B^*(s)S^*(\tau - s)C_\mu^{-1}(\tau)\big[\eta_z(\tau, v) - S(\tau)x_0\big]\mu(ds), \tag{28}$$

for $J \in \Sigma_\tau \subset \Sigma$. Now we can state the following lemma.

**Lemma 5.2** (Cotrollability $\cong$ Fixed Points)**.** *Consider the system* (19) *and suppose* $\{A, B(\cdot), f\}$ *satisfy the assumptions of Lemma* 5.1 *and let* $\mathcal{U}_\mu$ *denote the class of admissible controls as defined in* (3)*. Suppose the linear system* (1) *is globally exactly controllable. Then exact controllability of the nonlinear system* (19) *is equivalent to the existence of a fixed point of the operator* $F$ *on* $\mathcal{U}_\mu \subset \mathcal{M}_c(I, E)$*, that is, a* $u^* \in \mathcal{U}_\mu$ *such that* $u^* = Fu^*$.

***Proof.*** Suppose the system (19) is globally exactly controllable. Let $z \in H$ be the target, fixed but arbitrary, and let $u^o \in \mathcal{U}_\mu$ be the control that drives the system from state $x_0 \in H$ to the target $z$ at time $\tau$. Let $x(\cdot, u^o)$ denote the corresponding mild solution of (19) or equivalently the solution of the integral equation

(29) $x(t) = S(t)x_0 + \int_0^t S(t-r)B(r)u^o(dr) + \int_0^t S(t-r)f(r, x(r))dr,\ t \in I.$

Since $x(\tau) \equiv x(\tau, u^o) = z$, it is clear that $\eta_z$ as defined by (25), takes the value

$$\eta_z(\tau, u^o) \equiv z - \int_0^\tau S(\tau - r)f(r, x(r, u^o))dr, \tag{30}$$

and that $\eta_z(\tau, u^o) \in H$. By our assumption, the linear system is globally exactly controllable and hence $C_\mu(\tau)$ has a bounded inverse. Thus the control $u^o$, given by

$$u^o(dr) \equiv B^*(r)S^*(\tau - r)C_\mu^{-1}(\tau)[\eta_z(\tau, u^0) - S(\tau)x_0]\mu(dr), \tag{31}$$

is a well defined element of the admissible class $\mathcal{U}_\mu$ and further it drives the linear system to the target $\eta_z(\tau, u^o)$, that is

$$S(\tau)x_0 + \int_0^\tau S(\tau - r)B(r)u^o(dr) = \eta_z(\tau, u^o). \tag{32}$$

Thus it follows from (31) and (28) that, for each $\sigma \in \Sigma$,

$$u^o(\sigma) = \int_\sigma B^*(r)S^*(\tau - r)C_\mu^{-1}(\tau)[\eta(\tau, u^o) - S(\tau)x_0]\mu(dr) \equiv (Fu^o)(\sigma).$$

Hence $u^o$ is a fixed point of the operator $F$.

For the reverse implication, suppose $u^* \in \mathcal{U}_\mu$ is a fixed point of the operator $F$, that is,

$$u^*(\sigma) = (Fu^*)(\sigma), \sigma \in \Sigma. \tag{33}$$

We must show that $u^*$ drives the nonlinear system to the target $z$. Let $x(\cdot, u^*)$ denote the (mild) solution of the evolution equation (19) corresponding to the control $u^*$. Then clearly,

$$\begin{aligned} x(\tau, u^*) = S(\tau)x_0 &+ \int_0^\tau S(\tau - r)B(r)u^*(dr) \\ &+ \int_0^\tau S(\tau - r)f(r, x(r, u^*))dr\,. \end{aligned} \tag{34}$$

Recalling the definition of the operator $F$ and substituting the control $u^*$ given by (33), as the fixed point of the operator $F$, into the second term of equation (34) one can easily verify that $x(\tau, u^*) = z$. Hence $u^*$ drives the system (19) to the target. This proves the equivalence. ■

Now we are ready to prove the main result of this section. We prove that (global) exact controllability of the linear system (1) implies (global) exact controllability of the nonlinear system (19).

**Theorem 5.3** (Exact Controllability). *Consider the system* (19) *and suppose* $\{A, B, f\}$ *satisfy the assumptions of Lemma* 5.2 *and let* $\mathcal{U}_\mu$ *denote the class of admissible controls as defined in* (3). *Suppose the linear system* (1) *is globally exactly controllable. Then the nonlinear system* (19) *is also globally exactly controllable.*

***Proof.*** Consider the operator $F$ as defined by (28). By virtue of Lemma 5.2, it suffices to prove the existence of a fixed point of this operator on $\mathcal{U}_\mu$. Since, by Lemma 5.1, for every $u \in \mathcal{U}_\mu$ the nonlinear system (19) has a unique mild solution $x(\cdot, u) \in B(I, H)$, it is clear from the definition (25) that, for every $z \in H$, $\eta_z(\tau, u) \in H$ and hence $\eta_z(\tau, u) - S(\tau)x_0 \in H$. Since the linear system (1) is (globally) exactly controllable, the controllability operator $C_\mu(\tau)$ has a bounded inverse. Thus by uniform measureability and boundedness of $B$, it follows from the expression for the integrand on the right hand side of (28) that, for any fixed $\tau \in [0, T]$, the integrand is Borel measurable and uniformly bounded, and hence it belongs to $L_\infty(\mu, E) \subset L_1(\mu, E)$. Thus for $u \in \mathcal{U}_\mu$, $Fu \in \mathcal{U}_\mu$ and therefore $F$ maps $\mathcal{U}_\mu$ into itself. We prove that for some $\tau > 0$, $F$ is a contraction on $\mathcal{U}_\mu$. For $u, v \in \mathcal{U}_\mu$, it follows from the definition of the operator $F$ that for any $\sigma \in \Sigma_\tau$,

$$
\begin{aligned}
&(Fu)(\sigma) - (Fv)(\sigma) \\
&= \int_\sigma B^*(r)S^*(\tau - r)C_\mu^{-1}(\tau)(\eta_z(\tau, u) - \eta_z(\tau, v))\mu(dr).
\end{aligned} \tag{35}
$$

Since the operators $B(\cdot), S(\cdot), C_\mu^{-1}(\tau)$ are uniformly bounded on $[0, \tau], \tau \in [0, T]$, there exists a finite positive number $d$ such that

$$
\sup\{\| B^*(r)S^*(\tau - r)C_\mu^{-1}(\tau) \|_{\mathcal{L}(H,E)}, r \in [0, \tau]\} \le d < \infty. \tag{36}
$$

Using the defintion of $\eta_z(\tau,\cdot)$ given by (25) and the estimate (24), we obtain

$$\| \eta_z(\tau,u) - \eta_z(\tau,v) \|_H \leq MC\biggl(\int_0^\tau K(s)ds\biggr)|u-v|_V. \tag{37}$$

Now computing the $E$ norm of either side of (35), it follows from (36)–(37) that

$$\| (Fu)(\sigma) - (Fv)(\sigma) \|_E \leq (MCd)\biggl(\int_0^\tau K(s)ds\biggr)\mu(\sigma)|u-v|_V. \tag{38}$$

Since $\mu$ is a countably additive nonnegative measure having a bounded total variation, it follows from (38) that

$$|(Fu) - (Fv)|_V \leq (MCd)\mu([0,\tau])\biggl(\int_0^\tau K(s)ds\biggr)|u-v|_V. \tag{39}$$

Clearly, $MCd\ \mu([0,\tau]) < \infty$ as $\mu$ has a bounded variation. Since $K \in L_1^+(I)$, it follows from this inequality that there exists a $\tau^o > 0$ such that

$$\rho \equiv (MCd)\mu([0,\tau^o])\biggl(\int_0^{\tau^o} K(s)ds\biggr) < 1,$$

and hence

$$|(Fu) - (Fv)|_V \leq \rho|u-v|_V. \tag{40}$$

Thus by Banach fixed point theorem, $F$ has a unique fixed point on $\mathcal{U}_\mu$ restricted to the sigma algebra $\Sigma_{\tau^o}$. This proves the exact controllability of the nonlinear system (19). ■

**Remark.** By examining the proof of the previous theorem, we observe that the conclusions remain valid for any starting time $t_0 \geq 0$ and not just for $t_0 = 0$. However, since the operator $B$ is time dependent, the controllability operator $C_\mu(J)$ may be invertible for some intervals of time $J \subset [0,\infty)$ and may fail for others implying exact controllability or lack of it respectively. Clearly, if $C_\mu(J) > 0$ for any interval $J \subset [0,\infty)$, it is positive for all subsets of $[0,\infty)$ containing $J$, hence controllability over $J$ implies controllability over all intervals containing $J$.

## 6. Controllability of differential inclusions

The exact controllability result presented above can be extended to differential inclusions of the form

$$dx \in Axdt + B(t)u(dt) + Q(t,x)dt, t \in I, \tag{41}$$

where $Q$ is a suitable multivalued map.

**Definition 6.1.** The system (41) is exactly controllable if for any initial state $x_0 \in H$ and target $z \in H$, there exists a control $u^* \in \mathcal{U}_\mu$ so that $z \in X(T, u^*)$ where $X(\cdot, u^*)$ denotes the set of solutions of the evolution inclusion corresponding to the control $u^*$.

We can prove a controllability result similar to that of Theorem 5.3. Consider the evolution equation associated with (41) given by

$$dx = Axdt + B(t)u(dt) + f(t)dt, t \in I \equiv [0,T], \tag{42}$$

for any $f \in L_1(I,H)$. Clearly, for $x_0 \in H$ and $u \in \mathcal{U}_\mu$, this system has a unique (mild) solution $x \in B(I,H)$, given by

$$\begin{aligned} x(t) = R_t^u(f) \equiv S(t)x_0 &+ \int_0^t S(t-r)B(r)u(dr) \\ &+ \int_0^t S(t-r)f(r)dr, t \in I. \end{aligned} \tag{43}$$

Thus, for $x_0 \in H$ and $u \in \mathcal{U}_\mu$, an element $x \in B(I,H)$ is a solution of the differential inclusion (41), if there exists an $f \in L_1(I,H)$ such that $x$ is given by (43) and $f(t) \in Q(t,x(t)) = Q(t, R_t^u(f))$ *a.e.* For any $u \in \mathcal{U}_\mu$, define the multivalued map $\hat{Q}_u : L_1(I,H) \longrightarrow 2^{L_1(I,H)}$ given by

$$\hat{Q}_u(f) \equiv \big\{g \in L_1(I,H) : g(t) \in Q(t, R_t^u f),\ a.a\ t \in I\big\} \tag{44}$$

for $f \in L_1(I,H)$. Under certain assumptions such as measureability, closure, convexity and Lipschitz property (in the second argument) with respect to the Hausdorff metric, it can be shown that the set $\hat{Q}_u$ has a nonempty set of fixed points in $L_1(I,H)$, that is,

$$Fix(\hat{Q}_u) \equiv \{f \in L_1(I,H) : f \in \hat{Q}_u(f)\} \neq \emptyset.$$

For any $x_0 \in H$ and $u \in \mathcal{U}_\mu$ we claim that the system governed by the differential inclusion (41) has a nonempty set of solutions given by

$$X(u) \equiv \big\{x \in B(I,H), x(0) = x_0, x(t) = R_t^u(f) \quad \text{for } t \in I, f \in Fix(\hat{Q}_u)\big\}, \tag{45}$$

with values $X(t,u) = \{x(t), x \in X(u)\}$. Now we return to the controllability problem. For any target $z \in H$ and $u \in \mathcal{U}_\mu$ define the multivalued map

$$\Lambda_z(T,u) \equiv \left\{\xi \in H : \xi = z - \int_0^T S(T-r)f(r)dr, f \in Fix(\hat{Q}_u)\right\}. \tag{46}$$

Then define the multivalued map $F$,

$$F : \mathcal{M}_c(I,E) \longrightarrow 2^{\mathcal{M}_c(I,E)} \setminus \emptyset$$

taking values $(Fu)(\sigma), \sigma \in \Sigma$, given by

$$\begin{aligned}(Fu)(\sigma) &\equiv \int_\sigma B^*(r)S^*(T-r)C_\mu^{-1}(T)\Big(\Lambda_z(T,u) - S(T)x_0\Big)\mu(d\theta),\\ &= \left\{\int_\sigma B^*(r)S^*(T-r)C_\mu^{-1}(T)\Big(y - S(T)x_0\Big)\mu(d\theta), y \in \Lambda_z(T,u)\right\}.\end{aligned} \tag{47}$$

Again following similar arguments as in the previous section, we can verify that exact controllability of the evolution inclusion (41) is equivalent to the existence of fixed points in $\mathcal{U}_\mu$ of the multifunction $F$ given by (47) that is, $u^* \in F(u^*)$, for some $u^* \in \mathcal{U}_\mu$.

In fact, we can prove the following result.

**Theorem 6.2** (Exact Controllability)**.** *Consider the system* (41) *with initial state $x_0 \in H$ and suppose $\{A,B\}$ satisfy the assumptions of Lemma* 5.2 *and let $\mathcal{U}_\mu$ denote the class of admissible controls as defined in* (3)*. Let $Q$ be a measurable multivalued map $Q : I \times H \longrightarrow 2^H \setminus \emptyset$ with closed convex values and suppose there exists $K \in L_1^+(I)$, and, for each $r \in [0,\infty)$, a $K_r \in L_1^+(I)$, such that*

(Q1): $\sup\{|\zeta|_H, \zeta \in Q(t,x)\} \le K(t)(1 + |x|_H)$

(Q2): $\inf\{|\zeta|_H, \zeta \in Q(t,x)\} \le K_r(t), \forall\ x \in B_r \equiv \{x \in H : |x|_H \le r\}$,

(Q3): $d_H(Q(t,x), Q(t,y)) \le K_r(t) \parallel x - y \parallel_H, x, y \in B_r$

*where $d_H$ denotes the Hausdorff metric defined on closed subsets of $H$. Then, if the linear system* (1) *is exactly controllable, the nonlinear system* (41) *is exactly controllable in the sense of Definition* 6.1.

***Proof.*** (A Brief Outline) Let $x_0 \in H$ be the initial state and $z \in H$ be the target. We prove that there exists a control $u^o \in \mathcal{U}_\mu$ such that $z \in X(T, u^o)$. The proof consists of three steps.

**Step 1.** We show that, for any $f \in L_1(I, H)$ and $u \in \mathcal{U}_\mu$, the multifunction

$$t \longrightarrow Q(t, R_t^u(f))$$

has $L_1(I, H)$ selections and use this to prove that $Fix(\hat{Q}_u) \neq \emptyset$. Assumption (Q1) is used to establish an a priori bound which is then used along with assumption (Q2) to prove the existence of $L_1(I, H)$ selections of the multifunction defined above. Then we use measureability, convexity, closedness and Lipschitz property (Q3) to prove the existence of fixed points of the multifunction $\hat{Q}_u$ proving $Fix(\hat{Q}_u) \neq \emptyset$. From this result follows the existence of (mild) solutions of the evolution inclusion (41), that is, $X(u) \neq \emptyset$, and nonemptyness of the set $\Lambda_z(T, u)$ given by (46).

**Step 2.** Given the above results and the exact controllability of the linear system (1), we prove, following similar arguments as in Theorem 5.3, the existence of fixed points on $\mathcal{U}_\mu$ of the multivalued map $F$ given by (47). This gives us a set of fixed points of $F$ contained in $\mathcal{U}_\mu$.

**Step 3.** Using any of these fixed points, say, $u^o$, in equation (43) with $f = f^o \in Fix(\hat{Q}_{u^o})$, we verify that that $z \in X(T, u^o)$. This ends the brief outline of our proof. ■

**Some Comments and Open Questions**

(C1): From the results of Section 4, it is clear that a system which is not controllable with ordinary controls (strongly measurable $E$-valued vector functions) may be controllable with $E$-valued vector measures.

(C2): It would be interesting to study controllability problems with constraints on the vector measures such as

$$\mathcal{U}_\mu^1 \equiv \{u \in \mathcal{M}_c(I,E) : u \ll \mu \;\&\; |u|_V \le b\} \subset \mathcal{U}_\mu$$
$$\mathcal{U}_\mu^2 \equiv \{u \in \mathcal{M}_c(I,E) : u \ll \mu \;\&\; u(\sigma) \in U \subset E \;\forall\, \sigma \in \Sigma\} \subset \mathcal{U}_\mu$$

where $U$ is a weakly compact subset of $E$.

(C3): We expect that similar results can be proved for stochastic systems naturally driven by stochastic vector measures.

(C4): Closely related to the controllability problem is the viability problem. It would be interesting to study such problems for impulsive systems driven by vector measures as controls. For a recent comprehensive survey on viability for stochastic inclusions see [17–18].

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