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A UNIFIED TERMINOLOGY IN BLOCK DESIGNS
An Informative Classification

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Abstract

Partially efficiency balanced (PEB) designs with m efficiency classes have been defined by Puri and Nigam [15] as block designs which have simple analysis and, if properly used, allow the important contrasts to be estimated with desired efficiency. Such designs can be made available in varying replications and/or unequal block sizes. However, any block design is a PEB design with m efficiency classes for some $m < v$, where v is the number of treatments in the design. So the term "PEB" itself is not much informative in a statistical sense. More information may be added to this term. In this paper, a unified terminology is suggested, aimed at giving more statistical meaning to the PEB designs, which may or may not be connected. The paper is essentially based on our recent books "BLOCK DESIGNS: A Randomization Approach", Springer Lecture Notes in Statistics, Vol. 150 (2000), Vol. 170 (2003), with some new additions.

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1. Introduction

Any block design can be described by its $v \times b$ incidence matrix $\boldsymbol{N} = [n_{ij}]$, with a row for each treatment and a column for each block, where n_{ij} is the number of experimental units in the jth block receiving the ith treatment ($i = 1, 2, ..., v$; $j = 1, 2, ..., b$). This matrix, together with the vector of block sizes, $\boldsymbol{k} = [k_1, k_2, ..., k_b]' = \boldsymbol{N}'\mathbf{1}_v$, the vector of treatment replications, $\boldsymbol{r} = [r_1, r_2, ..., r_v]' = \boldsymbol{N}\mathbf{1}_b$, and the total number of units, $n = \mathbf{1}_b'\boldsymbol{k} = \mathbf{1}_v'\boldsymbol{r} = \mathbf{1}_v'\boldsymbol{N}\mathbf{1}_b$, where $\mathbf{1}_a$ is an $a \times 1$ vector of ones, is used in defining various matrices that help to understand the statistical properties of the design. In particular, an important role in studying these properties is played by the matrix $\boldsymbol{C} = \boldsymbol{r}^\delta - \boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}'$, where $\boldsymbol{r}^\delta = \mathrm{diag}[r_1, r_2, ..., r_v]$, $\boldsymbol{k}^\delta = \mathrm{diag}[k_1, k_2, ..., k_b]$ and $\boldsymbol{k}^{-\delta} = (\boldsymbol{k}^\delta)^{-1}$. On it, the so-called intra-block analysis of the experimental data is based (see Caliński and Kageyama [4], Section 3.2.1). Of special interest is the spectral decomposition of the matrix $\boldsymbol{C}$, given in the form

$$\boldsymbol{C} = \boldsymbol{r}^\delta \sum_{\beta=0}^{m-1} \varepsilon_\beta \boldsymbol{L}_\beta, \quad \text{with} \quad \boldsymbol{L}_\beta = \sum_{j=1}^{\rho_\beta} \boldsymbol{s}_{\beta j}\boldsymbol{s}_{\beta j}'\boldsymbol{r}^\delta, \tag{1.1}$$

where $\boldsymbol{s}_{\beta 1}, \boldsymbol{s}_{\beta 2}, ..., \boldsymbol{s}_{\beta\rho_\beta}$ are the $\boldsymbol{r}^\delta$-orthonormal eigenvectors of $\boldsymbol{C}$ with respect to $\boldsymbol{r}^\delta$, corresponding to a common eigenvalue ε_β of multiplicity ρ_β, for $\beta = 0, 1, ..., m-1$, and where $m-1$ is the number of distinct, less than 1, nonzero (positive) eigenvalues of $\boldsymbol{C}$ with respect to $\boldsymbol{r}^\delta$, ordered as

$$1 = \varepsilon_0 > \varepsilon_1 > \cdots > \varepsilon_{m-1} > \varepsilon_m = \varepsilon_{m+1} = 0,$$

i.e., $\boldsymbol{C}\boldsymbol{s}_{\beta j} = \varepsilon_\beta \boldsymbol{r}^\delta \boldsymbol{s}_{\beta j}$ for $j = 1, 2, ..., \rho_\beta$, and any of the β's. Notably, ε_β is the *efficiency factor* (also called the "canonical efficiency factor") of the analyzed design for the contrasts $\{\boldsymbol{c}_{\beta j}'\boldsymbol{\tau} = \boldsymbol{s}_{\beta j}'\boldsymbol{r}^\delta\boldsymbol{\tau}, j = 1, 2, ..., \rho_\beta\}$, where $\boldsymbol{\tau} = [\tau_1, \tau_2, ..., \tau_v]'$ is the vector of treatment parameters. These contrasts, for any β ($= 0, 1, ..., m-1$), are called the *basic contrasts* of the design (see Pearce, Caliński and Marshall [12]). The statistical sense of these two concepts is that in the intra-block analysis, to which the efficiency factors refer, the variance of the best linear unbiased estimator (BLUE) of a basic contrast $\boldsymbol{c}_{\beta j}'\boldsymbol{\tau}$ is simply given by

$$\mathrm{Var}(\widehat{\boldsymbol{c}'_{\beta j}\boldsymbol{\tau}}) = \varepsilon_\beta^{-1}\sigma^2, \quad \beta = 0, 1, ..., m-1, \quad j = 1, 2, ..., \rho_\beta,$$

where σ^2 stands for the intra-block variance of a single observation, and the covariance of the BLUEs of any pair of basic contrasts is zero (see Theorem 3.4.1 in Caliński and Kageyama [4]). The basic contrasts will, for convenience, be denoted by the eigenvectors $\{\boldsymbol{s}_{\beta j}\}$ which define them.

The description of block design properties given by (1.1) has led to various definitions of balance related to the efficiency factors. However, in the early papers concerning this subject, the spectral decompositions of the matrix $\boldsymbol{M} = \boldsymbol{r}^{-\delta}\boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}'$ (introduced by Jones [9]) and its modification $\boldsymbol{M}_0 = \boldsymbol{M} - n^{-1}\mathbf{1}_v\boldsymbol{r}'$ (adopted in Caliński [2])were used. In the notation of (1.1), these decompositions can be written in general, including disconnected block designs, as

$$(1.2) \qquad \boldsymbol{M} = \sum_{\beta=0}^{m+1} \mu_\beta \boldsymbol{L}_\beta \quad \text{and} \quad \boldsymbol{M}_0 = \sum_{\beta=0}^{m} \mu_\beta \boldsymbol{L}_\beta, \quad \text{where} \ \ \mu_\beta = 1 - \varepsilon_\beta.$$

Note that $\boldsymbol{L}_\beta^2 = \boldsymbol{L}_\beta$ and $\boldsymbol{L}_\beta\boldsymbol{L}_{\beta'} = \mathbf{O}$ if $\beta \neq \beta'$, for $\beta, \beta' = 0, 1, ..., m+1$, and that $\sum_{\beta=0}^{m} \boldsymbol{L}_\beta = \boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}'$. To be consistent with (1.1), the eigenvalues $\{\mu_\beta\}$ are to be ordered as

$$(1.3) \qquad 0 = \mu_0 < \mu_1 < \cdots < \mu_{m-1} < \mu_m = \mu_{m+1} = 1,$$

with multiplicities ρ_β, for $\beta = 0, 1, ..., m, m+1$, where $\rho_0 = 0$ if no basic contrast is estimated with full efficiency in the intra-block analysis, $\rho_m = 0$ if no basic contrast is totally confounded with blocks, i.e., the design is connected, and $\rho_{m+1} = 1$ always. Thus, $m-1$ is the number of distinct, less than 1, positive eigenvalues in (1.2). As noted by Jones ([9], p. 176), they represent relative losses of information due to partially confounding the relevant contrasts with blocks. Their relation to the efficiency factors is given by $\varepsilon_\beta = 1 - \mu_\beta$, and the multiplicities are obtainable as $\rho_\beta = \mathrm{rank}(\boldsymbol{L}_\beta)$ for $\beta = 0, 1, ..., m, m+1$. It should be noted that $\boldsymbol{L}_0$, appearing in (1.1), is redundant in (1.2), as $\mu_0 = 0$, $\boldsymbol{L}_m = \sum_{j=1}^{\rho_m} \boldsymbol{s}_{mj}\boldsymbol{s}'_{mj}\boldsymbol{r}^\delta$ appears only in a disconnected block design, and $\boldsymbol{L}_{m+1} = n^{-1}\mathbf{1}_v\boldsymbol{r}'$ corresponds to $\mu_{m+1} = 1$.

Also note that the eigenvectors $\{\boldsymbol{s}_{mj}\}$ in $\boldsymbol{L}_m$ represent basic contrasts totally confounded with blocks in a disconnected block design, and so not estimated in the intra-block analysis.

With this notation, a block design is in the terminology of Jones ([9], p. 176) called balanced for a contrast $\boldsymbol{c}'\boldsymbol{\tau} = \boldsymbol{s}'\boldsymbol{r}^{\delta}\boldsymbol{\tau}$ when $\boldsymbol{s}$ is an eigenvector of $\boldsymbol{M}_0$. Thus, in this sense, a block design is balanced for any basic contrast taken separately, but it is also balanced jointly for a set of them, or a subspace spanned by them, if all of them correspond to the same efficiency factor, i.e., to $\varepsilon_\beta = 1-\mu_\beta$, for $\beta = 0, 1, ..., m$ ($m-1$, for a connected design). The number of distinct efficiency factors determines, then, the number of ways in which the design can be considered as balanced. One extreme case is when the matrix $\boldsymbol{M}_0$ of the design has a unique less than 1 eigenvalue, with multiplicity $v-1$. The design is then balanced in the sense of Jones for all possible contrasts, in the same way, and therefore called totally balanced in that sense (see Caliński [2], p. 281). This kind of balance is characterized by

$$(1.4) \qquad \boldsymbol{M}_0 = \mu(\boldsymbol{I}_v - n^{-1}\mathbf{1}_v\boldsymbol{r}') \quad \text{or, equivalently,} \quad \boldsymbol{C} = \varepsilon(\boldsymbol{r}^{\delta} - n^{-1}\boldsymbol{r}\boldsymbol{r}'),$$

where the unique efficiency factor of multiplicity $v-1$ is $\varepsilon = 1-\mu = [n - \mathrm{tr}(\boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}')]/(n - n^{-1}\boldsymbol{r}'\boldsymbol{r}) \le 1$ (see also Caliński and Kageyama [4], Section 2.4.2). An opposite extreme case is when the $v-1$ eigenvalues corresponding to a complete set of basic contrasts of the design are all different. Then, one can only say that the design is balanced for each of the basic contrasts separately. Certainly, many possible situations can exist between these two extremes, as the eigenvalues of $\boldsymbol{M}_0$ may appear in various multiplicities. A design with the property (1.4) was later called *efficiency balanced* by Williams [19] and Puri and Nigam ([13], [14]), and this term is now commonly used (see also Caliński and Kageyama [4], Section 4.1).

The various possibilities of the decompositions (1.2), noted in the early papers, have inspired several people to undertake studies on properties, constructions and classifications of block designs. In particular, Puri and Nigam [15] have introduced the concept of partially efficiency balanced (PEB) designs. In the present notation, their definition can be presented as follows.

An arrangement of v treatments into b blocks of sizes k_1, k_2, ..., k_b is said to be a (connected) PEB design with m efficiency classes if

(i) the ith treatment is replicated r_i times, $i = 1, 2, ..., v$;

(ii) the efficiency factor associated with every contrast of the βth class is $1 - \mu_\beta$, where μ_β, $0 \le \mu_\beta < 1$, $\beta = 0, 1, ..., m-1$, are the distinct eigenvalues of the matrix $\boldsymbol{M}_0$, with multiplicities ρ_β (> 0), such that $\sum_{\beta=0}^{m-1} \rho_\beta = v - 1$ (i.e., except the zero eigenvalue corresponding to $\boldsymbol{M}_0 \mathbf{1}_v = \mathbf{0}$);

(iii) the matrix $\boldsymbol{M}_0$ has the spectral decomposition $\boldsymbol{M}_0 = \sum_{\beta=0}^{m-1} \mu_\beta \boldsymbol{L}_\beta$, where $\boldsymbol{L}_\beta, \beta = 0, 1, ..., m-1$, are defined as in (1.1), and $\mu_0 = 0$, as in (1.3). However, if $\rho_0 = 0$, i.e., if $\mu_0 = 0$ does not exist, then the considered arrangement is said to have $m - 1$ efficiency classes.

Note that this definition covers all possible situations of connected block designs between the two extremes mentioned above. In fact, Puri and Nigam [15] regard the case (1.4), i.e., of an efficiency balanced (EB) design, as a *trivial* PEB design with only one efficiency class. Moreover, this definition can be extended to cover also disconnected designs, by noting in (1.2) that in the general case the matrix $\boldsymbol{M}_0$ contains the component $\mu_m \boldsymbol{L}_m$, with $\mu_m = 1$, which corresponds to $\rho_m = \text{rank}(\boldsymbol{L}_m)$ basic contrasts totally confounded with blocks. Thus, any block design, whether connected or disconnected, is PEB in the sense that if it is connected, i.e., $\rho_m = 0$, then all ρ_β basic contrasts of the βth class ($\beta = 0, 1, ..., m-1$) are estimated intra-block with the efficiency $\varepsilon_\beta = 1 - \mu_\beta$ (> 0), but if the design is disconnected, i.e., $\rho_m > 0$, then $\varepsilon_\beta = 0$ for one β ($= m$). Therefore, in a classification of block designs in general, to give merely the number of efficiency classes of a PEB design is not sufficient to indicate the design position.

In the present paper, attention is drawn to a more informative statistical characterization of block designs (introduced in Caliński and Kageyama [3]), and the suitability of this characterization for classifying block designs is discussed and illustrated.

2. Statement

If the experimental problem has its reflection in distinguishing certain subsets of contrasts, ordered according to their importance or interest, the experiment should be designed in such a way that all members of a specified subset receive a common efficiency factor, of the higher value the more important the contrasts of the subset are.

If possible, the design should allow to estimate the most important subset of contrasts with full efficiency, i.e., with $\varepsilon_0 = 1$. In this context, one is interested in knowing how many basic contrasts are estimated with the same efficiency. So the information about the multiplicity ρ_β is essential. It can form a basis for a classification of block designs, as already suggested in Caliński and Kageyama ([3], Section 3.2.2). Thus, *the characterization of a block design by the triples* $(\mu_\beta, \rho_\beta, \boldsymbol{L}_\beta)$ *for* $\beta = 0, 1, ..., m$, i.e., by the idempotent matrices $\{\boldsymbol{L}_\beta\}$ defined in (1.1), their ranks $\{\rho_\beta\}$ and the corresponding eigenvalues $\{\mu_\beta\}$, in the sense of the equality

$$\boldsymbol{M}_0 \boldsymbol{L}_\beta = \mu_\beta \boldsymbol{L}_\beta, \qquad \beta = 0, 1, ..., m,$$

resulting from (1.2), *is very informative.*

First of all, it allows the spectral decomposition (1.1) to be obtained easily. From this, also a generalized inverse (g-inverse) of the matrix $\boldsymbol{C}$ can be obtained in a convenient form, as

$$\sum_{\beta=0}^{m-1} \varepsilon_\beta^{-1} \boldsymbol{L}_\beta \boldsymbol{r}^{-\delta}, \quad \text{where} \quad \varepsilon_\beta = 1 - \mu_\beta. \tag{2.1}$$

Furthermore, the intra-block BLUE of any contrast given by $\boldsymbol{s}'_\beta \boldsymbol{L}'_\beta \boldsymbol{r}^\delta \boldsymbol{\tau}$ for some $\boldsymbol{s}_\beta$, such that $\boldsymbol{L}_\beta \boldsymbol{s}_\beta \neq \boldsymbol{0}$, for $\beta = 0, 1, ..., m-1$, obtains the form

$$\widehat{\boldsymbol{s}'_\beta \boldsymbol{L}'_\beta \boldsymbol{r}^\delta \boldsymbol{\tau}} = \varepsilon_\beta^{-1} \boldsymbol{s}'_\beta \boldsymbol{L}'_\beta (\boldsymbol{\Delta} - \boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{D})\boldsymbol{y}, \tag{2.2}$$

where $\boldsymbol{\Delta}'$ and $\boldsymbol{D}'$ are the design matrices for treatments and blocks, respectively, so that $\boldsymbol{N} = \boldsymbol{\Delta}\boldsymbol{D}'$, and $\boldsymbol{y}$ is an $n \times 1$ vector of observed variables (observations). The variance of (2.2) can then be written as

$$\mathrm{Var}(\widehat{\boldsymbol{s}'_\beta \boldsymbol{L}'_\beta \boldsymbol{r}^\delta \boldsymbol{\tau}}) = \varepsilon_\beta^{-1} \boldsymbol{s}'_\beta \boldsymbol{r}^\delta \boldsymbol{L}_\beta \boldsymbol{s}_\beta \sigma^2. \tag{2.3}$$

Here, ε_β is the common efficiency factor of the design for all contrasts of the considered type, i.e., contrasts generated by $\boldsymbol{L}_\beta$. The variance (2.3) is reduced to $\varepsilon_\beta^{-1}\sigma^2$ if $\boldsymbol{s}_\beta$ represents one of the basic contrasts corresponding to the common efficiency factor ε_β. In particular, for $\beta = 0$, a contrast $\boldsymbol{s}_0'\boldsymbol{L}_0'\boldsymbol{r}^\delta\boldsymbol{\tau}$ obtains the BLUE of the form

$$\boldsymbol{s}_0'\widehat{\boldsymbol{L}_0'\boldsymbol{r}^\delta}\boldsymbol{\tau} = \boldsymbol{s}_0'\boldsymbol{L}_0'\boldsymbol{\Delta}\boldsymbol{y},$$

with the variance

$$\mathrm{Var}(\boldsymbol{s}_0'\widehat{\boldsymbol{L}_0'\boldsymbol{r}^\delta}\boldsymbol{\tau}) = \boldsymbol{s}_0'\boldsymbol{r}^\delta\boldsymbol{L}_0\boldsymbol{s}_0\sigma^2,$$

which reduces to σ^2 if $\boldsymbol{s}_0$ represents one of the basic contrasts corresponding to the efficiency factor $\varepsilon_0 = 1$ (see Caliński and Kageyama [4], Section 4.4.3). More generally, for any set of contrasts $\boldsymbol{U}'\boldsymbol{\tau}$, where $\boldsymbol{U} = \boldsymbol{C}\boldsymbol{S}$ for some matrix $\boldsymbol{S}$ of v rows, such that $\boldsymbol{C}\boldsymbol{S} \neq \mathbf{O}$, the intra-block BLUEs are of the form

$$\widehat{\boldsymbol{U}'\boldsymbol{\tau}} = \boldsymbol{U}'\sum_{\beta=0}^{m-1}\varepsilon_\beta^{-1}\boldsymbol{L}_\beta\boldsymbol{r}^{-\delta}(\boldsymbol{\Delta} - \boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{D})\boldsymbol{y}, \tag{2.4}$$

and their covariance (dispersion) matrix is of the form

$$\mathrm{Cov}(\widehat{\boldsymbol{U}'\boldsymbol{\tau}}) = \boldsymbol{U}'\sum_{\beta=0}^{m-1}\varepsilon_\beta^{-1}\boldsymbol{L}_\beta\boldsymbol{r}^{-\delta}\boldsymbol{U}\sigma^2, \tag{2.5}$$

as it follows from (1.1), (2.1) and the results in Section 3.4 of Caliński and Kageyama [4].

With the present notation, the following modification of the original (Puri and Nigam [15]) definition of partial efficiency balance for any connected block design can be given (modifying also the definition of Ceranka and Mejza [7]).

Definition 2.1. A connected block design is said to be $(\rho_0; \rho_1, ..., \rho_{m-1})$-efficiency balanced (EB) if a complete set of its $v-1$ basic contrasts can be partitioned into at most m disjoint and nonempty subsets such that all the ρ_β basic contrasts of the βth subset correspond to a common efficiency factor $\varepsilon_\beta = 1 - \mu_\beta$, different for different $\beta = 0, 1, ..., m-1$, i.e., so that the matrix $\boldsymbol{M}_0$ $(= \boldsymbol{r}^{-\delta}\boldsymbol{N}\boldsymbol{k}^{-\delta}\boldsymbol{N}' - n^{-1}\mathbf{1}_v\boldsymbol{r}')$ has the spectral decomposition

$$\boldsymbol{M}_0 = \sum_{\beta=0}^{m-1} \mu_\beta \boldsymbol{L}_\beta,$$

where the distinct eigenvalues $0 = \mu_0 < \mu_1 < \cdots < \mu_{m-1} < 1$ have the multiplicities $\rho_0 \geq 0$, $\rho_1 \geq 1$, ..., $\rho_{m-1} \geq 1$, respectively, and the matrices $\{\boldsymbol{L}_\beta\}$ are defined as in (1.1).

The parameters of a $(\rho_0; \rho_1, ..., \rho_{m-1})$-EB design can be written as v, b, $\boldsymbol{r}$, $\boldsymbol{k}$, $\varepsilon_\beta = 1 - \mu_\beta$, ρ_β, $\boldsymbol{L}_\beta$ for $\beta = 0, 1, ..., m-1$. Note that $\rho_0 = 0$ if no basic contrast is estimated in the intra-block analysis with full efficiency, i.e., the design is not orthogonal for any contrasts estimated in this analysis.

Definition 2.1 can be extended so to cover also disconnected block designs. For this recall (Caliński and Kageyama [4], Definition 2.2.6a) that a block design is said to be disconnected of degree $g-1$ if its $v \times b$ incidence matrix $\boldsymbol{N}$, after an appropriate ordering of rows and columns, can be written as $\boldsymbol{N} = \text{diag}[\boldsymbol{N}_1 : \boldsymbol{N}_2 : \cdots : \boldsymbol{N}_g]$, where $\boldsymbol{N}_\ell$, $\ell = 1, 2, ..., g$, are $v_\ell \times b_\ell$ incidence matrices of connected subdesigns corresponding to some partitions $v = v_1 + \cdots + v_g$ and $b = b_1 + \cdots + b_g$ ($g = 1$ meaning that the design is connected).

Definition 2.2. A block design with disconnectedness of degree $g-1$ (connected when $g = 1$) is said to be $(\rho_0; \rho_1, ..., \rho_{m-1}; \rho_m)$-EB if a complete set of its $v-1$ basic contrasts can be partitioned into at most $m+1$ disjoint and nonempty subsets such that all ρ_β basic contrasts of the βth subset correspond to a common efficiency factor $\varepsilon_\beta = 1 - \mu_\beta$, different for different $\beta = 0, 1, ..., m-1, m$, i.e., so that the matrix $\boldsymbol{M}_0$ has the spectral decomposition

$$\boldsymbol{M}_0 = \sum_{\beta=0}^{m} \mu_\beta \boldsymbol{L}_\beta,$$

where the distinct eigenvalues $0 = \mu_0 < \mu_1 < \cdots < \mu_{m-1} < \mu_m = 1$ have the multiplicities $\rho_0 \geq 0$, $\rho_1 \geq 1, ..., \rho_{m-1} \geq 1$, $\rho_m = g - 1 \geq 0$, respectively, and the matrices $\{\boldsymbol{L}_\beta\}$ are defined as in (1.1).

The parameters of a $(\rho_0; \rho_1, ..., \rho_{m-1}; \rho_m)$-EB design can be written as v, b, $\boldsymbol{r}$, $\boldsymbol{k}$, $\varepsilon_\beta = 1-\mu_\beta$, ρ_β, $\boldsymbol{L}_\beta$ for $\beta = 0, 1, ..., m-1, m$. Hence (i) ρ_0 shows the number of basic contrasts estimated with full efficiency, i.e., not confounded with blocks, (ii) $\{\rho_1, ..., \rho_{m-1}\}$ give the numbers of basic contrasts estimated in the intra-block analysis with efficiencies $\{\varepsilon_\beta, \beta = 1, ..., m-1\}$ less than 1, i.e., partially confounded with blocks, and (iii) ρ_m gives the number of basic contrasts with zero efficiency, i.e., totally confounded with blocks. In a connected design $\rho_m = 0$, i.e., no basic contrast is totally confounded with blocks.

It should be noted that because the decompositions (1.2) hold for any block design, whether connected, i.e., with $\rho_m = 0$, or disconnected, i.e., with $\rho_m \geq 1$, any block design satisfies the condition of Definition 2.2. Therefore, for a classification of the designs the specification of the multiplicities $\{\rho_\beta\}$ appearing in the definition is essential. To see it better, it may be interesting to indicate some special classes of block designs specified from the general Definition 2.2 point of view.

A block design belongs to the class of $(0; v-1; 0)$-EB designs if it is connected and totally balanced in the sense of Jones [9] (see Definition 2.4.5 in Caliński and Kageyama [4]), or EB designs in the terminology of Williams [19], and Puri and Nigam ([13], [14]), i.e., satisfies the conditions

$$\boldsymbol{M}_0 = \mu_1 \boldsymbol{L}_1, \quad \text{with} \quad \boldsymbol{L}_1 = \boldsymbol{I}_v - n^{-1}\boldsymbol{1}_v\boldsymbol{r}', \quad \text{and} \quad \boldsymbol{C} = \varepsilon_1(\boldsymbol{r}^\delta - n^{-1}\boldsymbol{r}\boldsymbol{r}').$$

In particular, any balanced incomplete block (BIB) design belongs to this class.

A block design belongs to the class of $(0; v-g; g-1)$-EB designs if it is disconnected of degree $g-1$ and is balanced in the sense of Jones [9] (see Definition 2.4.6 in Caliński and Kageyama [4]), i.e., satisfies the conditions

$$\boldsymbol{M}_0 = \mu_1 \boldsymbol{L}_1 + \boldsymbol{L}_2 \quad \text{and} \quad \boldsymbol{C} = \varepsilon_1 \boldsymbol{r}^\delta \boldsymbol{L}_1.$$

A block design belongs to the class of $(v-1;0;0)$-EB designs if it is connected and orthogonal, i.e., satisfies the conditions

$$\boldsymbol{M}_0 = \mathbf{O} \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{r}^{\delta} - n^{-1}\boldsymbol{r}\boldsymbol{r}'.$$

A block design belongs to the class of $(v-g;0;g-1)$-EB designs if it is disconnected of degree $g-1$ and orthogonal, i.e., satisfies the conditions

$$\boldsymbol{M}_0 = \mu_1\boldsymbol{L}_1 \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{r}^{\delta}\boldsymbol{L}_0.$$

A block design belongs to the class of $(\rho_0; v-1-\rho_0; 0)$-EB designs if it is a simple PEB, or PEB(s), design in the terminology of Puri and Nigam [15], or belongs to the class D_1 of designs considered by Shah [18], or to the class of C-designs according to Saha [17], i.e., satisfies the conditions

$$\boldsymbol{M}_0 = \mu_1\boldsymbol{L}_1 \quad \text{and} \quad \boldsymbol{C} = \boldsymbol{r}^{\delta}(\varepsilon_0\boldsymbol{L}_0 + \varepsilon_1\boldsymbol{L}_1),$$

with at least one of the eigenvalues $\varepsilon_0, \varepsilon_1$ present. In particular, it can be seen that any connected 2-associate partially balanced incomplete block (PBIB) design, with its incidence matrix $\boldsymbol{N}$ of rank less than v, belongs to this class. There are many such designs in the classes of group divisible, triangular and Latin-square type designs (see Raghavarao [16]). For example, any singular group divisible and any semi-regular group divisible design with $v = mn$ treatments is $(v-m; m-1; 0)$-EB and $(m-1; v-m; 0)$-EB, respectively.

On the other hand, any connected 2-associate PBIB design, with its incidence matrix $\boldsymbol{N}$ of the rank equal to v, belongs to the class of $(0; \rho_1, \rho_2; 0)$-EB designs. For example, any regular group divisible design is $(0; m-1, v-m; 0)$-EB.

Considering designs with supplemented balance (see Pearce [11], Caliński [2]), i.e., designs obtained from a block design by adding to each block one or more supplementary treatments, it may be interesting to note that an orthogonally supplemented BIB design (in the sense of Caliński [2]) belongs to the class of $(\rho_0; \rho_1; 0)$-EB designs. On the other hand, an orthogonally supplemented connected 2-associate PBIB design, with an incidence matrix $\boldsymbol{N}$, belongs either to the class of $(\rho_0; \rho_1, \rho_2; 0)$-EB or $(\rho_0; \rho_1; 0)$-EB designs, depending on whether both of the distinct eigenvalues of $\boldsymbol{N}\boldsymbol{N}'$ are positive or one of them is zero.

Thus, the present terminology, using the multiplicities ρ_β, may be much more suitable for indicating the statistical advantage and utility of a block design, than the original reference to a PEB design with m efficiency classes. Note, particularly, the difference between a $(0; \rho_1, \rho_2; 0)$-EB design and a $(\rho_0; \rho_1; 0)$-EB design, both being PEB designs with two efficiency classes according to the definition of PEB designs stated in Section 1.

3. Illustration

Many examples of block designs belonging to the various classes of the proposed classification, described in Definition 2.2, can be found in Caliński and Kageyama [5], where also their constructional aspects are discussed. Here two examples of designs illustrating the classes of $(0; v-1; 0)$-EB and $(\rho_0; v-1-\rho_0; 0)$-EB designs will be presented.

Example 3.1. Ceranka and Kaczmarek [6] have analyzed data from a plant-breeding field experiment carried out in a BIB design with $v = 28$ progenies (three-line hybrids) obtained from the triallel crossing between $p = 7$ lines (of a group P), $q = 2$ testers (of a group Q) and $r^* = 2$ varieties (of a group R) of barley. (Here the notation r^* is used to distinguish from r, the number of treatment replications.) The $v = 28$ progenies were allocated in $b = 36$ blocks of size $k = 7$, i.e., on $n = 252$ plots in total, each progeny in $r = 9$ replications, every two progenies concurring in exactly $\lambda = 2$ blocks. Thus, the BIB design chosen for the experiment is that of No. 76 in Table 8.2 given in Caliński and Kageyama [5]. The efficiency factor of the design, common for all contrasts, is $\varepsilon_1 = 8/9 = 0.8889$, with the multiplicity $\rho_1 = v - 1 = 27$. Of course, the design belongs to the class of $(0; \rho_1; 0)$-EB designs, according to Definition 2.2, with the parameters as already given, and with $\boldsymbol{M}_0 = (1/9)\boldsymbol{L}_1$ and $\boldsymbol{C} = 8\boldsymbol{L}_1$, where $\boldsymbol{L}_1 = \boldsymbol{I}_{28} - (1/28)\mathbf{1}_{28}\mathbf{1}_{28}'$.

In the context of this experiment, of interest are the following contrasts of progeny parameters:

$\boldsymbol{g}^P$, $\boldsymbol{g}^Q$ and $\boldsymbol{g}^R$ – the general effects of parental lines from the groups P, Q, and R, respectively,

$\boldsymbol{s}^{PQ}$, $\boldsymbol{s}^{PR}$ and $\boldsymbol{s}^{QR}$ – the two-line specific effects from the pairs $\{P, Q\}$, $\{P, R\}$, and $\{Q, R\}$, respectively,

$\boldsymbol{s}^{PQR}$ – the three-line specific effects from the triple $\{P, Q, R\}$.

Denoting the progeny parameters by $\{\tau_{ijh}\}$, with $i = 1, 2, ..., 7$ (for P), $j = 1, 2$ (for Q) and $h = 1, 2$ (for R), and arraying them in the 28×1 vector $\boldsymbol{\tau}$ according to lexical order, one can write the above contrasts as

$$\boldsymbol{g}^P = \left[g_1^P, g_2^P, ..., g_7^P\right]' = \left[(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}_7') \otimes (1/2)\mathbf{1}_2' \otimes (1/2)\mathbf{1}_2'\right] \boldsymbol{\tau} = \boldsymbol{U}_P'\boldsymbol{\tau},$$

$$\boldsymbol{g}^Q = \left[g_1^Q, g_2^Q\right]' = \left[(1/7)\mathbf{1}_7' \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2') \otimes (1/2)\mathbf{1}_2'\right] \boldsymbol{\tau} = \boldsymbol{U}_Q'\boldsymbol{\tau},$$

$$\boldsymbol{g}^R = \left[g_1^R, g_2^R\right]' = \left[(1/7)\mathbf{1}_7' \otimes (1/2)\mathbf{1}_2' \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2')\right] \boldsymbol{\tau} = \boldsymbol{U}_R'\boldsymbol{\tau},$$

$$\begin{aligned} \boldsymbol{s}^{PQ} &= \left[s_{11}^{PQ}, s_{12}^{PQ}, ..., s_{72}^{PQ}\right]' \\ &= \left[(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}_7') \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2') \otimes (1/2)\mathbf{1}_2'\right] \boldsymbol{\tau} = \boldsymbol{U}_{PQ}'\boldsymbol{\tau}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{s}^{PR} &= \left[s_{11}^{PR}, s_{12}^{PR}, ..., s_{72}^{PR}\right]' \\ &= \left[(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}_7') \otimes (1/2)\mathbf{1}_2' \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2')\right] \boldsymbol{\tau} = \boldsymbol{U}_{PR}'\boldsymbol{\tau}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{s}^{QR} &= \left[s_{11}^{QR}, s_{12}^{QR}, s_{21}^{QR}, s_{22}^{QR}\right]' \\ &= \left[(1/7)\mathbf{1}_7' \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2') \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2')\right] \boldsymbol{\tau} = \boldsymbol{U}_{QR}'\boldsymbol{\tau}, \end{aligned}$$

$$\begin{aligned} \boldsymbol{s}^{PQR} &= \left[s_{111}^{PQR}, s_{112}^{PQR}, ..., s_{722}^{PQR}\right]' \\ &= \left[(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}_7') \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2') \otimes (\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}_2')\right] \boldsymbol{\tau} = \boldsymbol{U}_{PQR}'\boldsymbol{\tau}. \end{aligned}$$

Note that $\boldsymbol{U}'_P\boldsymbol{L}_1 = \boldsymbol{U}'_P$, and this, on account of (2.4) and (2.5), allows the intra-block BLUEs of the contrasts $\boldsymbol{U}'_P\boldsymbol{\tau}$ to be written as

$$\widehat{\boldsymbol{U}'_P\boldsymbol{\tau}} = (\varepsilon_1 r)^{-1}\boldsymbol{U}'_P(\boldsymbol{\Delta} - k^{-1}\boldsymbol{N}\boldsymbol{D})\boldsymbol{y} = (1/8)\boldsymbol{U}'_P(\boldsymbol{\Delta} - (1/7)\boldsymbol{N}\boldsymbol{D})\boldsymbol{y},$$

and their covariance matrix as

$$\mathrm{Cov}(\widehat{\boldsymbol{U}'_P\boldsymbol{\tau}}) = (\varepsilon_1 r)^{-1}\boldsymbol{U}'_P\boldsymbol{U}_P\sigma^2 = (1/32)(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}'_7)\sigma^2.$$

In a similar way one can estimate, in the intra-block analysis, any other of the above sets of contrasts. In particular, for the three-line specific effects, the covariance matrix of the intra-block BLUEs is

$$\begin{aligned}\mathrm{Cov}(\widehat{\boldsymbol{U}'_{PQR}\boldsymbol{\tau}}) &= (\varepsilon_1 r)^{-1}\boldsymbol{U}'_{PQR}\boldsymbol{U}_{PQR}\sigma^2\\ &= \frac{1}{8}\left[(\boldsymbol{I}_7 - (1/7)\mathbf{1}_7\mathbf{1}'_7)\otimes(\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}'_2)\otimes(\boldsymbol{I}_2 - (1/2)\mathbf{1}_2\mathbf{1}'_2)\right]\sigma^2.\end{aligned}$$

This simplicity of the intra-block estimation of contrasts of treatment parameters is an evident advantage of the class of $(0; v-1; 0)$-EB designs.

Although in this plant-breeding experiment the interest has been in the general and specific combining ability effects, this example could also be considered as a factorial experiment with three factors, P, Q and R, applied at 7, 2 and 2 levels, respectively. Then, one would be interested in contrasts among the main effects, for each of the three factors, contrasts among the PQ, PR and QR interactions and, finally, among the PQR interactions. The estimation procedure would then be the same as above.

Example 3.2. Patterson and Silvey [10] have advocated the use of generalized lattice designs in crop variety trials conducted for testing the performance of new varieties of agricultural and vegetable crops. In this broad class of designs, of particular interest are the affine resolvable proper block designs with parameters $v = sk$, $b = sr$, r and k. Their construction and optimality properties have been considered by Bailey, Monod and Morgan [1]. As noted in Caliński and Kageyama ([5], p. 258), these designs belong to the class of $(\rho_0; \rho_1; 0)$-EB designs with $\varepsilon_0 = 1$, $\varepsilon_1 = (r-1)/r$, $\rho_0 = v - 1 - r(s-1)$ and $\rho_1 = r(s-1)$.

Hence, the average (harmonic mean) efficiency factor of any such design is $\varepsilon = (v-1)/(\rho_0+\varepsilon_1^{-1}\rho_1) = (v-1)k(r-1)/(vrk+vr-vk+k-2rk)$. (See also Theorem 3.1 in Bailey *et al.* [1].) Moreover, the matrices $\boldsymbol{L}_\beta$, for $\beta = 0, 1$, of any of these designs are (see Caliński and Kageyama [5], Section 9.3.1) of the form

$$\text{(3.1)} \qquad \boldsymbol{L}_0 = \boldsymbol{I}_v - v^{-1}\boldsymbol{1}_v\boldsymbol{1}'_v - \boldsymbol{L}_1 \quad \text{and} \quad \boldsymbol{L}_1 = k^{-1}\boldsymbol{N}\boldsymbol{N}' - v^{-1}r\boldsymbol{1}_v\boldsymbol{1}'_v,$$

from which the matrix (2.1) gets the form

$$\text{(3.2)} \qquad \begin{aligned} & r^{-1}(\boldsymbol{L}_0 + \varepsilon_1^{-1}\boldsymbol{L}_1) \\ & = r^{-1}\{\boldsymbol{I}_v + [(r-1)k]^{-1}\boldsymbol{N}\boldsymbol{N}' - [v(r-1)]^{-1}(2r-1)\boldsymbol{1}_v\boldsymbol{1}'_v\}. \end{aligned}$$

This allows the intra-block BLUE of any contrast $\boldsymbol{c}'\boldsymbol{\tau}$ to be written, applying (2.4), as

$$\widehat{\boldsymbol{c}'\boldsymbol{\tau}} = r^{-1}\boldsymbol{c}'[\boldsymbol{I}_v + (r-1)^{-1}k^{-1}\boldsymbol{N}\boldsymbol{N}'](\boldsymbol{\Delta} - k^{-1}\boldsymbol{N}\boldsymbol{D})\boldsymbol{y},$$

and its variance, on account of (2.5), as

$$\mathrm{Var}(\widehat{\boldsymbol{c}'\boldsymbol{\tau}}) = r^{-1}\boldsymbol{c}'[\boldsymbol{I}_v + (r-1)^{-1}k^{-1}\boldsymbol{N}\boldsymbol{N}']\boldsymbol{c}\sigma^2,$$

for any affine resolvable proper block design. In the special case of $\boldsymbol{c} = r\boldsymbol{L}_\beta\boldsymbol{s}_\beta$, for some $\boldsymbol{s}_\beta$ (not nullifying $\boldsymbol{L}_\beta$), $\beta = 0$ or 1, the variance is of the form (2.3), here equal to

$$\mathrm{Var}(\widehat{\boldsymbol{c}'\boldsymbol{\tau}}) = \varepsilon_\beta^{-1}r\boldsymbol{s}'_\beta\boldsymbol{L}_\beta\boldsymbol{s}_\beta\sigma^2, \quad \beta = 0, 1.$$

However, in many cases, particularly in variety trials, of interest are elementary contrasts, $\tau_i - \tau_{i'}$, $i, i' = 1, 2, ..., v$ $(i \neq i')$. For them, the variance gets the form

$$\text{(3.3)} \qquad \mathrm{Var}(\widehat{\tau_i - \tau_{i'}}) = \frac{2[r - \lambda_{ii'} + k(r-1)]}{kr(r-1)}\sigma^2,$$

where $\lambda_{ii'}$ is the number of blocks in which the ith and the i'th treatments "concur", i.e., are both present. This can be seen by noting that if the vector $\boldsymbol{c}$ is such that $\boldsymbol{c}'\boldsymbol{\tau} = \tau_i - \tau_{i'}$, then $\boldsymbol{c}'\boldsymbol{N}\boldsymbol{N}'\boldsymbol{c} = 2(r - \lambda_{ii'})$. (See also Theorem 3.6 in Bailey *et al.* [1].)

The affine resolvable proper block designs are often used by the Research Centre for Cultivar Testing (Słupia Wielka) in variety trials conducted at their experimental stations in Poland. Let one of these designs be presented here. It is a design based on the incidence matrix

$$\boldsymbol{N} = [\boldsymbol{N}_1 : \boldsymbol{N}_2 : \boldsymbol{N}_3 : \boldsymbol{N}_4],$$

where

$$\boldsymbol{N}_1' = \begin{bmatrix} 1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1 \end{bmatrix},$$

$$\boldsymbol{N}_2' = \begin{bmatrix} 1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1 \end{bmatrix},$$

$$\boldsymbol{N}_3' = \begin{bmatrix} 1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0 \\ 0&0&0&0&1&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0 \end{bmatrix},$$

$$\boldsymbol{N}_4' = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0 \\ 0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&0&1&0&0&1&0&0&0 \end{bmatrix},$$

It can be seen that this 25×20 incidence matrix represents, according to a traditional terminology, a square lattice design (a subclass of the affine resolvable proper block designs), with $v = s^2, b = sr, r, k = s$ and the average efficiency factor $\varepsilon = (s+1)(r-1)/[(s+1)(r-1)+r]$ (see John [8], Section 3.4.2). For the present example, the parameters are $v = 5^2 = 25, b = 5 \times 4 = 20, r = 4$ and $k = 5$. It is a $(\rho_0; \rho_1; 0)$-EB design with $\varepsilon_0 = 1$, $\varepsilon_1 = 3/4$, $\rho_0 = 8$ and $\rho_1 = 16$. Its average efficiency factor is $\varepsilon = 0.8182$. The design can be constructed either as described in Caliński and Kageyama ([5], Section 9.6) or by taking the dual of a semi-regular group divisible design with the parameters $v^* = 4 \times 5 = 20$ (treatments divided into $m^* = 4$ groups of $n^* = 5$ treatments each), $b^* = 25, r^* = 5, k^* = 4, \lambda_1^* = 0$ and $\lambda_2^* = 1$ (see Caliński and Kageyama [5], Section 9.3.1). The design can also be seen as resulting from the balanced lattice design for $s^2 = 25$ treatments in $r = s + 1 = 6$ replications (given originally by Yates [20], Table VIII) when deleting 2 out of the 6 replications (superblocks).

The matrices (3.1) for this experiment are

$$\boldsymbol{L}_0 = \boldsymbol{I}_{25} - (1/25)\mathbf{1}_{25}\mathbf{1}'_{25} - \boldsymbol{L}_1 \quad \text{and} \quad \boldsymbol{L}_1 = (1/5)\boldsymbol{N}\boldsymbol{N}' - (4/25)\mathbf{1}_{25}\mathbf{1}'_{25}.$$

With them, one obtains (3.2) in the form

$$r^{-1}(\boldsymbol{L}_0 + \varepsilon_1^{-1}\boldsymbol{L}_1) = (1/4)[\boldsymbol{I}_{25} + (1/15)\boldsymbol{N}\boldsymbol{N}' - (7/75)\mathbf{1}_{25}\mathbf{1}'_{25}].$$

Hence, for any contrast $\boldsymbol{c}'\boldsymbol{\tau}$ the intra-block BLUE is

$$\widehat{\boldsymbol{c}'\boldsymbol{\tau}} = (1/4)\boldsymbol{c}'[\boldsymbol{I}_{25} + (1/15)\boldsymbol{N}\boldsymbol{N}'](\boldsymbol{\Delta} - (1/5)\boldsymbol{N}\boldsymbol{D})\boldsymbol{y},$$

and its variance is

$$\mathrm{Var}(\widehat{\boldsymbol{c}'\boldsymbol{\tau}}) = (1/4)\boldsymbol{c}'[\boldsymbol{I}_{25} + (1/15)\boldsymbol{N}\boldsymbol{N}']\boldsymbol{c}\sigma^2.$$

Here the off-diagonal elements of the matrix $\boldsymbol{N}\boldsymbol{N}'$, the so-called “concurrences”, are $\lambda_1 = 1$ and $\lambda_2 = 0$ (see also Caliński and Kageyama [5], p. 266). Hence, for an elementary contrast, the variance (3.3) is

$$\mathrm{Var}(\widehat{\tau_i - \tau_{i'}}) = \frac{2(k+1)}{kr}\sigma^2 = \frac{3}{5}\sigma^2 = 0.6\sigma^2, \quad \text{if} \quad \lambda_{ii'} = \lambda_1 = 1,$$

or

$$\mathrm{Var}(\widehat{\tau_i - \tau_{i'}}) = \frac{2[r+k(r-1)]}{kr(r-1)}\sigma^2 = \frac{19}{30}\sigma^2 = 0.6333\sigma^2, \quad \text{if} \quad \lambda_{ii'} = \lambda_2 = 0.$$

Evidently, these two variances do not differ much. In case of applying the corresponding resolvable BIB design (No. 16* in Table 9.1 given in Caliński and Kageyama [5]), the variance would be equal to $0.4\sigma^2$. However, most of the official variety testing experiments, not only in Poland, are carried out with the number of replications 2, 3, or 4 (see Patterson and Silvey [10], p. 225). Many of the affine resolvable proper block designs meet this requirement. Moreover, being $(\rho_0; \rho_1; 0)$-EB designs, they offer desirable optimality properties. In fact, for a fixed number of treatmants, v, and a fixed number of replications, r, the affine resolvable proper block designs are A-, D- and E-optimal in the class of all $(\rho_0; \rho_1, ..., \rho_{m-1}; 0)$-EB designs with constant $\mathrm{rank}(\boldsymbol{M}_0) \equiv \mathrm{rank}(\boldsymbol{N}) - 1$, and constant k and b (see Caliński and Kageyama [5], Section 7.1; also Bailey *et al.* [1], Section 3).

In a similar way, the information provided by $\{\mu_\beta, \rho_\beta, \boldsymbol{L}_\beta\}$, $\beta = 0, 1, ..., m$, can be used in the intra-block analysis of any other block design. Thus, if this information is available in advance for the design chosen by the researcher when planning an experiment, then it can easily be used in examining the statistical properties of the design and later in performing the analysis of the experimental data. Otherwise, the researcher has to evaluate the quantities $\{\varepsilon_\beta = 1 - \mu_\beta\}$, $\{\rho_\beta\}$ and $\{\boldsymbol{L}_\beta\}$ related to the design, before deciding on its use. In any case, the knowledge of these quantities allows the researcher to use the design for an experiment in such a way that best corresponds to the experimental problem. In particular, it helps the researcher to implement the design so that the contrasts considered as the most important can be estimated with the highest efficiency in the stratum of the smallest variance, which usually the intra-block stratum is.

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