SELECTIVE *F* TESTS FOR SUB-NORMAL MODELS

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Abstract

F tests that are specially powerful for selected alternatives are built for sub-normal models. In these models the observation vector is the sum of a vector that stands for what is measured with a normal error vector, both vectors being independent. The results now presented generalize the treatment given by Dias (1994) for normal fixed-effects models, and consider the testing of hypothesis on the ordering of mean values and components.

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1. Introduction

While *t* tests can be one-sided or two-sided, *F* tests for fixed-effects models generally do not select alternatives. To overcome this restrictions Dias (1994) introduced selective *F* tests. We now present an extension of his results to sub-normal models.

After discussing how to define the selected alternatives we proceed with the test observation and show how max-min tests may be obtained. These tests maximize the minimum power for selected alternatives.

2. Models and hypothesis

We will use superscripts to indicate the number of components of vectors. In sub-normal models the vector Y^n of observations is the sum of a vector Z^n that stands for what is measured with an error vector e^n. It is assumed that:

- Z^n and e^n are independent, we put $Z^n(i)e^n$;
- e^n is normal with null mean vector and variance-covariance matrix $\sigma^2 I_n$, we write $e^n \sim N(0^n, \sigma^2 I_n)$;
- Z^n belongs to the range space $\Omega = R(X)$ of a $n \times m$ matrix with $\mathrm{rank}(X) = m$.

Usual F tests have been derived for these models, see Mexia and Dias (2001), we now consider F tests specially powerful for selected alternatives.

Since $Z^n \in \Omega = R(X)$ with $\mathrm{rank}(X) = m$ we have $Z^n = XV^m$ and, with X^+ the Moore-Penrose inverse of X, $V^m = X^+ Z^n$. Moreover the orthogonal projection matrix on Ω will be

$$(2.1) \qquad\qquad Q(\Omega) = X(X^\top X)^{-1} X^\top$$

so that $X^\top Q(\Omega) = X^\top$ and, if v_Ω^n is the orthogonal projection of v^n on Ω

$$(2.2) \qquad\qquad X^\top v_\Omega^n = X^\top v^n.$$

If we had a fixed-effects model $Y^n \sim N(X\beta^m, \sigma^2 I_n)$ and $R(W^\top) \subseteq R(X^\top)$ we would have the hypothesis

$$(2.3) \qquad\qquad H_0 : W\beta^m = \psi_0^s.$$

To select alternatives to H_0, Dias (1994, pgs 21 to 24), used polar coordinates $(r, \theta_1, ..., \theta_{s-1})$. With θ_1^{s-1} the vector of central angles for $\psi_1^s - \psi_0^s$, the selected alternatives

$$(2.4) \qquad\qquad H_1(\psi_1^s) : W\beta^m = \psi_1^s,$$

were defined by $\|\psi_1^s - \psi_0^s\| \geq d$ and $\theta_1^{s-1} \in D$. Let $\mathcal{V}_1$ be the set of these ψ_1^s.

Returning to sub-normal models, we test

$$(2.5) \qquad H_0 : WV^m = \psi_0^s.$$

The selected alternatives satisfying the condition $WV^m \in \mathcal{V}_1$. For these alternatives the support of the distribution $G(u^s)$ of WV^m will be contained in $\mathcal{V}_1$.

3. TEST DERIVATION

Since $e^n \sim N(0^n, \sigma^2 I_n)$, e_Ω^n and $e_{\Omega^\perp}^n$ will be independent. Now $Z_\Omega^n = Z^n$ is independent from e^n, e_Ω^n and $e_{\Omega^\perp}^n$ so that $Y_\Omega^n = Z_\Omega^n + e_\Omega^n = Z^n + e_\Omega^n$ will be independent from $Y_{\Omega^\perp}^n = e_{\Omega^\perp}^n$ as well as from $S = \|Y_{\Omega^\perp}^n\|^2 = \|e_{\Omega^\perp}^n\|^2$ which will be the product by σ^2 of a central chi-square with $n - m$ degrees of freedom.

In what follows the statistic

$$(3.1) \qquad \widetilde{\psi}^s = W(X^\top X)^{-1} X^\top Y^n$$

will play a central part. We start by pointing that, since $\widetilde{\psi}^s = W(X^\top X)^{-1} X^\top Y_\Omega^n$, and $S = \|Y_{\Omega^\perp}^n\|^2$, $\widetilde{\psi}^s$ and S will be independent since Y_Ω^n and $Y_{\Omega^\perp}^n$ are independent. Moreover

$$
\begin{aligned}
\widetilde{\psi}^s &= W(X^\top X)^{-1} X^\top (Z^n + e^n) \\
(3.2) \qquad &= W(X^\top X)^{-1} X^\top (XV^m + e^n) \\
&= WV^m + W(X^\top X)^{-1} X^\top e^n.
\end{aligned}
$$

Thus, when $WV^m = \psi^s$, we have

$$(3.3) \qquad \widetilde{\psi}^s = \psi^s + W(X^\top X)^{-1} X^\top e^n \sim N(\psi^s, \sigma^2 C)$$

with

$$(3.4) \qquad C = W(X^\top X)^{-1} W^\top$$

and, see Dias (1994, pg 14), the (conditional) joint density of

$$(3.5) \qquad \Im = \frac{n - m}{s} \frac{(\widetilde{\psi}^s - \psi_0^s)^\top C^{-1} (\widetilde{\psi}^s - \psi_0^s)}{S}$$

and $\widetilde{\Theta}^{s-1} = \theta^{s-1}(\widetilde{\psi}^s - \psi_0^s)$, will be

$$f(z, \theta^{s-1}|C, \lambda^s, n-m) = \frac{e^{-\frac{1}{2}\lambda^\top C^{-1}\lambda}\sqrt{\frac{s}{n-m}\frac{h^2(\theta^{s-1})}{k(\theta^{s-1})z}}}{(2\pi)^{s/2}\Gamma(\frac{n-m}{2})\sqrt{\det(C)}}$$

(3.6)
$$\times \sum_{j=0}^{+\infty} \frac{2^{j/2}\Gamma(\frac{n-m+s+j}{2})a^j(\theta^{s-1})(\frac{s}{n-m}\frac{z}{k(\theta^{s-1})})^{\frac{j+s-1}{2}}}{j!(1+\frac{s}{n-m}z)^{\frac{n-m+s+j}{2}}}$$

where

(3.7)
$$\begin{cases} \lambda^s = \frac{1}{\sigma}(\psi^s - \psi_0^s) \\ k(\theta^{s-1}) = \ell^s(\theta^{s-1})^\top C^{-1}\ell^s(\theta^{s-1}) \\ a(\theta^{s-1}) = (\lambda^s)^\top C^{-1}\ell^s(\theta^{s-1}) \\ h(\theta^{s-1}) = \cos\theta_1^{s-2}, ..., \cos\theta_{s-2} \end{cases}.$$

The components of $\ell^s(\theta^{s-1})$ being

(3.8)
$$\begin{cases} \ell_1(\theta^{s-1}) = \cos\theta_1...\cos\theta_{s-1} \\ \ell_2(\theta^{s-1}) = \cos\theta_1...\cos\theta_{s-2}\sin\theta_{s-1} \\ \vdots \\ \ell_i(\theta^{s-1}) = \cos\theta_1...\cos\theta_{s-i}\sin\theta_{s-i+1} \\ \vdots \\ \ell_s(\theta^{s-1}) = \sin\theta_1 \end{cases}.$$

When H_0 holds, $\lambda^s = 0^s$ and the joint (unconditional) density is

$$f_0(z, \theta^{s-1}|C, 0^s, n-m) = \frac{\sqrt{\frac{s}{n-m}\frac{h^2(\theta^{s-1})}{k(\theta^{s-1})z}}}{(2\pi)^{s/2}\sqrt{\det(C)}}\frac{\Gamma(\frac{n-m+s}{2})(\frac{s}{n-m}\frac{z}{k(\theta^{s-1})})^{\frac{s-1}{2}}}{\Gamma(\frac{n-m}{2})(1+\frac{s}{n-m}z)^{\frac{n-m+s}{2}}}$$

(3.9)
$$= f(z|s, n-m)f^0(\theta^{s-1})$$

where $f(z|s, n-m)$ is the central F density with s and $n-m$ degrees of freedom, and

(3.10)
$$f^0(\theta^{s-1}) = \frac{\Gamma(\frac{s}{2})}{(2\pi)^{s/2}k(\theta^{s-1})^{s/2}}\frac{h(\theta^{s-1})}{\sqrt{\det(C)}}.$$

In writing these densities we did not indicate that, since we are using polar coordinates, $z > 0$, $-\frac{\pi}{2} \le \theta_j \le \frac{\pi}{2}$, $j = 1, ..., s - 2$, and $0 \le \theta_{s-1} \le 2\pi$. In what follows we will represent by $\overline{f}(z, \theta^{s-1} | C, \lambda^s, n - m)$ the joint density of

$$(3.11) \qquad \mathcal{T} = \frac{(\widetilde{\psi}^s - \psi_0^s)^\top C^{-1} (\widetilde{\psi}^s - \psi_0^s)}{S}$$

and $\widetilde{\Theta}^{s-1}$. Since $\mathcal{T} = \frac{s}{n-m} \Im$ we have

$$(3.12) \qquad \overline{f}(z, \theta^{s-1} | C, \lambda^s, n - m) = \frac{n - m}{s} f(\frac{n - m}{s} z, \theta^{s-1} | C, \lambda^s, n - m).$$

When we use the pair $(\mathcal{T}, \widetilde{\Theta}^{s-1})$ instead of the pair $(\Im, \widetilde{\Theta}^{s-1})$ the critical region (k, D) is replaced by $(\frac{s}{n-m} k, D)$.

According to (3.9), when H_0 holds, $\Im(i)\widetilde{\Theta}^{s-1} = \theta^{s-1}(\widetilde{\psi}^s - \psi_0^s)$. Thus, the critical region is $]k, +\infty[\times D$ this is H_0 is rejected if and only if $\Im > k$ and $\widetilde{\Theta}^{s-1} \in D$, the test level will be

$$(3.13) \qquad level(k, D) = (1 - F(k | s, n - m)) \int_D \ldots \int f^0(\theta^{s-1}) d\theta_1 \ldots d\theta_{s-1}.$$

4. Max-min tests

From the last expression we see that, for a given test level, there is more than one pair (k, D). To choose a convenient pair Dias (1994, pg 53) introduced max-min tests. These tests maximize the minimum power for selected alternatives. In fixed-effects models these correspond to the $\psi_1^s \in \mathcal{V}_1$. If the minimum power is attained for $\psi_{c,1}^s$ the corresponding alternative will be critical. When we replace β^m by V^m a random vector that lies in Ω we obtain a sub-normal model associated to the fixed-effects model.

We now establish

Proposition 1. *When we use a pair (k, D) the test level is the same for associated models and if the test is max-min with one or more critical alternatives for the fixed-effects model, it is also max-min for the sub-normal model with the same minimum power for privileged alternatives.*

Proof. The first point of the thesis follows from expression (3.9) in the proceeding section since, when H_0 holds, the joint density of $\Im$ and $\widetilde{\Theta}^{s-1}$ is the same for associated models. Moreover, with $pow(\psi_1^s)$ the power of

the test in the fixed-effects model when $H_1(\psi_1^s)$ holds, the power for the sub-normal model will be

$$Pow(G) = \int_{\mathcal{V}_1} ... \int pow(u^s)dG(u^s).$$

Now, for every $u^s \in \mathcal{V}_1$, we have $pow(u^s) \geq pow(\psi_{c,1}^s)$, with $H_1(\psi_{c,1}^s)$ a critical alternative, so that to complete the proof we have only to point out that $G(.)$ may be degenerate having all it's probability concentrated at $\psi_{c,1}^s$.
∎

Thus the construction of max-min tests may be carried out in fixed-effects models and then, when there are critical alternatives, transferred to sub-normal models. Dias (1994, pgs 53 to 60) studied in detail the case in which $C = I_2$ and

$$\mathcal{V}_1 = \{\|\psi_1^2\| \geq d; \psi_1^2 \geq 0^2\}$$

which is of interest in fertilization studies.

If there are critical alternatives and the corresponding power of the test, for fixed- effects models, exceeds the test level the test is selectively unbiased. This is the power for selected alternatives exceeds the test level. According to proposition 1 this property also holds for the associated sub-normal model.

5.   Hypothesis on orderings

We start with normal models assuming $n = Jr$ and $\mu^n = \eta^J \otimes 1^r$, in order to test

(5.1) $H_0 : \eta_1 = ... = \eta_J$

while privileging the alternatives for which

(5.2) $\eta_{j_\ell} < \eta_{j'_\ell}, \quad \ell = 1, ..., L.$

where $j_1, ..., j_L$ and $j'_1, ..., j'_L$ are sequences of numbers (with possible replications) from the set $\{1, ..., J\}$.

For instance if we wanted to privilege the alternatives in which $\eta_J = min\{\eta_1, ..., \eta_J\}$ we would have

$$\eta_J < \eta_j, \quad j = 1, ..., J-1.$$

This could be of interest if the $\eta_j, \quad j = 1, ..., J$ were average yields for cultivars, the first $J-1$ of these having been obtained through plant breeding while the last one was a local cultivar.

In this situation we have J treatments and r replications. Let Y^J be the vector of the treatment mean values multiplied by $\sqrt{r}$ and S be the sum of sums of squares of residuals. Then $Y^J \sim N(\sqrt{r}\eta^J, \sigma^2 I_J)$ is independent from $S \sim \sigma^2 \chi_g^2$, with $g = J(r-1)$ degrees of freedom.

It would be now straightforward to apply the general theory. But practical difficulties occur, when L and n are not small, in computing the integrals

$$\int_D \cdots \int f^0(\theta^{s-1})d\theta_1, ..., d\theta_{s-1}$$

which is necessary to control the first type error. Fortunately a combinatorial solution for the problem exists. We will have $\Theta^{s-1} \in D$ if and only if

$$Y_{j_\ell} < Y_{j'_\ell}, \quad \ell = 1, ..., L.$$

Now when H_1 holds the $Y_1, ..., Y_J$ are i.i.d so that all possible orderings will have probability $\frac{1}{J!}$.

If there are m orderings satisfying the stated conditions we will have

$$\int_D \cdots \int f^0(\theta^{s-1})d\theta_1, ..., d\theta_{s-1} = \frac{m}{J!}.$$

For instance in the case of the example we gave this integral would be equal to $\frac{1}{J}$.

This result extends directly to sub-normal models since, as we saw, these inherit the test level of the associated normal models.

Thus we can control easily the first type errors both in normal and in sub-normal model, while testing for orderings.

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