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AUTOREGRESSIVE ERROR-PROCESSES, CUBIC SPLINES AND TRIDIAGONAL MATRICES

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Abstract

In the paper formulate for the inversion of some tridiagonal matrices are given. The results can be applied to the autoregressive processes.

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1. Introduction

The covariance-matrix of an autoregressive error-process is given by ($|\rho| < 1$)

$$\Omega = \sigma_u^2 \quad (\rho^{|i-j|};\ i,j = 1,\ldots,n), \tag{1.1}$$

where σ_u^2 is the variance of the error terms. The inverse Ω^{-1} of Ω is equal to

$$\Omega^{-1} = (\sigma_u^{-2})\frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho & & & & \\ -\rho^1 & 1+\rho^2 & -\rho & & 0 & \\ & & & \ddots & & \\ & 0 & & -\rho & 1+\rho^2 & -\rho \\ & & & & -\rho & 1 \end{pmatrix}. \tag{1.2}$$

Ω^{-1} is thus a tridiagonal matrix, i. e., if $\Omega^{-1} = (\omega_{ij})$ then $\omega_{ij} = 0$ if $|i-j| > 1$. Ω^{-1} is needed for the computation of the Aitken-estimators

$$\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y \tag{1.3}$$

in a linear regression-model $y = X\beta + u$.
In cubic equidistant spline interpolation the inversion of the matrix

$$A = \begin{pmatrix} 4\,1 & & & \\ 1\,4\,1 & & & 0 \\ & & \ddots & \\ 0 & & & 1\,4\,1 \\ & & & 1\,4 \end{pmatrix} \tag{1.4}$$

is required. This matrix can be represented in the form

$$A = \frac{a}{1-\rho^2} \begin{pmatrix} 1+\rho^2 & -\rho & & & & \\ -\rho & 1+\rho^2 & -\rho & & 0 & \\ & & & \ddots & & \\ & 0 & & -\rho & 1+\rho^2 & -\rho \\ & & & & -\rho & 1+\rho^2 \end{pmatrix} \tag{1.5}$$

for some $|\rho| < 1$, namely

$$\rho = -2 + \sqrt{3} = -0,2679491\ldots \tag{1.6}$$

Indeed, $\dfrac{a}{1-\rho^2}(1+\rho^2) = 4$ and $\dfrac{a}{1-\rho^2}(-\rho) = 1$ implies $(1+\rho^2)(-\rho)^{-1} = 4$ or $1+\rho^2 = -4\rho$. Since $(\rho^2+4\rho)+1 = (\rho+2)^2 - 3 = 0$ if $\rho + 2 = \pm\sqrt{3},\ \rho = -2+\sqrt{3}$ or $\rho = -2-\sqrt{3}$. Only the first ρ satisfies $|\rho| < 1$ and fulfills the desired requirement. Moreover,

$$a = \frac{1-\rho^2}{-\rho} = 2\sqrt{3}\,. \tag{1.7}$$

Therefore

$$A^{-1} = a^{-1}\,\Omega_0\,, \tag{1.8}$$

where Ω_0 only slightly differs from the covariance-matrix of an autoregressive error-process. The Törnquist-Egervary formula allows to compute this difference. It turns out that this computation leads to surprisingly simple results which can only be found in some scattered literature (see Graybill [3], p. 286; Nabben [4], p. 298).

2. AUTOREGRESSIVE ERROR-PROCESSES

In this section we follow Schönfeld [6], pp. 152–164. We consider the linear regression model

$$y_t = x_t'\beta + u_t, \quad t = 1, 2, \ldots, T, \tag{2.1}$$

where β is an $k \times 1$ parameter-vector to be estimated and x_t is a $k \times 1$ design-vector. Thus y_t is a random variable and the disturbance-(or error-)term u_t is assumed to follow an autoregressive process of the first order

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad |\rho| < 1 \tag{2.2}$$

for $t = 2, \ldots, T$, where $\varepsilon_t,\ t = 2, \ldots, T$ are uncorrelated random variables with mean zero and varicance σ^2. (2.2) represents an inhomogeneous (stochastic) difference equation which is solved by $u_t = c(t)\rho^t$. $c(t)$ must obey the equation

$$\rho^t(c(t) - c(t-1)) = \varepsilon_t, \quad t = 2, \ldots, T\,, \tag{2.3}$$

i. e., $c(t) = c(1) + \sum_{\tau=1}^{t}(c(\tau)) - c(\tau - 1)) = c(1) + \sum_{\tau=2}^{t} \rho^{-\tau}\varepsilon_\tau$ and finally

$$u_t = \rho^t c(1) + \sum_{\tau=2}^{t} \rho^{t-\tau}\varepsilon_\tau\,. \tag{2.4}$$

Since the empty sum equals zero, $c(1) = u_1\rho^{-1}$ and thus finally

$$u_t = \rho^{t-1} u_1 + \sum_{\tau=2}^{t} \rho^{t-\tau} \varepsilon_\tau \, . \tag{2.5}$$

We assume that $E(u_1) = 0$ and u_1 is uncorrelated with the ε_t. From this it follows that

$$\begin{aligned} \operatorname{Var}(u_t) &= \sigma^2 \left\{ \rho^{2(t-1)} \frac{\operatorname{Var}(u_1)}{\sigma^2} + \sum_{\tau=2}^{t} \rho^{2(t-\tau)} \right\} \\ &= \rho^{2(t-1)} \operatorname{Var}(u_1) + \sigma^2 \frac{1-\rho^{2(t-1)}}{1-\rho^2} \, . \end{aligned} \tag{2.6}$$

If we make the assumption “Nature does not jump”, i. e. $\operatorname{Var}(u_t) = \operatorname{Var}(u_1)$, we get

$$\operatorname{Var}(u_1)(1-\rho^{2(t-1)}) = \frac{\sigma^2(1-\rho^{2(t-1)})}{1-\rho^2} \, , \tag{2.7}$$

i. e. $\operatorname{Var}(u_1) = \dfrac{\sigma^2}{1-\rho^2}$. This result can also be obtained in another way. If we assume that $u_t = \rho u_{t-1} + \varepsilon_t$, $t \leq T$, i. e., $t = \ldots - n$, $-(n-1), \ldots, 0, 1, \ldots, T$, then it follows by mathematical induction that

$$u_t = \sum_{\tau=0}^{\infty} \rho^\tau \varepsilon_{t-\tau} \, . \tag{2.8}$$

From this follows

$$\operatorname{Var}(u_t) = \operatorname{Var}(u_1) = \sum_{\tau=0}^{\infty} \rho^{2\tau} = \frac{\sigma^2}{1-\rho^2} \, , \tag{2.9}$$

since $|\rho| < 1$ has been assumed. An elementary computation also shows that for $s \in \mathbb{N}$:

$$\operatorname{Cov}(u_t, u_{t+s}) = \frac{\sigma^2}{1-\rho^2} \rho^s \, . \tag{2.10}$$

Thus

$$\Omega = \mathrm{Cov}\,(u_1,\dots,u_T)' = \frac{\sigma^2}{1-\rho^2}(\rho^{|i-j|};\ i,j=1,\dots,T) \tag{2.11}$$
$$= \sigma_u^2(\rho^{|i-j|};\ i,j=1,\dots,T).$$

The formula for Ω^{-1} can be proved by simple verification. There is, however, also a statistical approach for determining Ω^{-1}. Consider the matrix

$$B = \begin{pmatrix} 1 & & & & 0 \\ -\rho & 1 & & & \\ & -\rho & \ddots & & \\ & & & \ddots & \\ 0 & & & -\rho & 1 \end{pmatrix}. \tag{2.12}$$

Since $\det B = 1$, B is regular and $\mathrm{im}\, B = \mathrm{im}\, B\Omega = \mathbb{R}^T$. Consequently if $X' = (x_1,\dots,x_T)$ and we consider the linear regression model $y = X\beta + u$ the statistic By is a linearly sufficient statistic (Drygas, 1984) since $\mathrm{im}\, X \subseteq \mathrm{im}\,(\Omega + XX')B' = \mathrm{im}\,(\Omega + XX') = \mathrm{im}\,(\Omega) + \mathrm{im}\,(X)$. Now

$$By = (y_1, y_2 - \rho y_1, \dots, y_2 - \rho y_{\tau-1})' = (y_1, \tilde{y}_2, \dots, \tilde{y}_T)' \tag{2.13}$$

and

$$E(y_1) = E(x_1'\beta + u_1) = x_1'\beta,\ \mathrm{Var}\,(y_1) = \mathrm{Var}\,(u_1) = \frac{\sigma^2}{(1-\rho^2)}\,, \tag{2.14}$$

$$E(y_i - \rho y_{i-1}) = E(x_i - \rho x_{i-1} + \varepsilon_i) = x_i - \rho x_{i-1}, \tag{2.15}$$
$$\mathrm{Var}\,(y_i - \rho y_{i-1}) = \mathrm{Var}\,(\varepsilon_i) = \sigma^2\,.$$

If we, moreover, replace y_1 by

$$\tilde{y}_1 = (1-\rho^2)^{1/2} y_1\,, \tag{2.16}$$

then $\mathrm{Cov}\,(\tilde{y}_1, \tilde{y}_2, \dots, \tilde{y}_t)' = \sigma^2 I_T$. Therefore there exist two equivalent possibilities to compute the Gauss-Markov estimator (GME) in the regression model $y = X\beta + u$. Either we can use the Aitken-formula

$$\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y \tag{2.17}$$

or we can use the formula

$$\hat{\beta} = (\tilde{X}'\tilde{X})^{-1}\tilde{X}'\tilde{y}, \tag{2.18}$$

where $E\tilde{y} = \tilde{X}\beta$, $\tilde{X}' = (x_1(1-\rho^2)^{1/2},\ x_i - \rho x_{i-1},\ i = 2,\dots,T)$. We specialize to the case $k = 1$ and let $X = e_i$, the i-th unit-vector. Then

$$\hat{\beta} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y = \frac{\sum_{j=1}^{T}(\Omega^{-1})_{i,j}\, y_j}{(\Omega^{-1})_{i,i}}\,. \tag{2.19}$$

On the other hand for $i = 1$

$$\tilde{X} = ((1-\rho^2)^{1/2}, -\rho, 0, \dots, 0)'\,. \tag{2.20}$$

$$\tilde{X} = (0,\dots,0,\underset{(i)}{1},-\rho,0,\dots,0)' \tag{2.21}$$

for $i = 2,\dots,T-1$ and finally for $i = N$

$$\tilde{X} = (0,\dots,0,1)'\,. \tag{2.22}$$

Thus

$$\tilde{X}'\tilde{X} = 1 (i=1), \quad \tilde{X}'\tilde{X} = 1+\rho^2 (2 \le i \le T-1), \quad \tilde{X}'\tilde{X} = 1 (i = T)\,, \tag{2.23}$$

$$(\tilde{X}'X)^{-1} = 1,\ i = 1, T, \quad (\tilde{X}'\tilde{X})^{-1} = \frac{1}{1+\rho^2}\ (2 \le i \le T-1)\,. \tag{2.24}$$

For $i = 1$ we get

$$y_1 + \frac{(\Omega^{-1})_{1,2}\, y_2}{(\Omega^{-1})_{1,1}} = (1-\rho^2)y_1 - \rho(y_2 - \rho y_1) = y_1 - \rho y_2\,. \tag{2.25}$$

This implies that the first line of (Ω^{-1}) is equal to $(\Omega^{-1})_{1,1}(1,-\rho,0,\dots,0)$. Multiplying this with the first column of Ω yields $(\Omega^{-1})_{1,1}(1-\rho^2)=1$, i. e., $(\Omega^{-1})_{1,1}=\dfrac{1}{1-\rho^2}$. For $i=T$ we get

$$\sum_{j=1}^{T}(\Omega^{-1})_{T,j}\,y_j=(\Omega^{-1})_{i,T}(y_T-\rho y_{T-1})\,, \tag{2.26}$$

i. e., the last line of Ω^{-1} is proportional to $(0,\dots,0,-\rho,1)$ which again yields $(\Omega^{-1})_{T,T}=(1-\rho^2)^{-1}$. For $2\le i\le T-1$

$$\begin{aligned}\sum_{j=1}^{T}(\Omega^{-1})_{i,j}\,y_j&=\frac{(\Omega^{-1})_{i,i}}{1+\rho^2}(y_i-\rho y_{i-1}-\rho(y_{i+1}-\rho y_i))\\&=\frac{(\Omega^{-1})_{i,i}}{1+\rho^2}[(1+\rho^2)y_i-\rho y_{i-1}-\rho y_{i+1}]\,.\end{aligned} \tag{2.27}$$

Thus the i-th line of Ω^{-1} is proportional to $(0,\dots,-\rho,\underset{(i)}{1+\rho^2},-\rho,0,\dots,0)$.
The constant $(\Omega^{-1})_{i,i}$ must be found from the equation

$$\begin{aligned}\frac{(\Omega^{-1})_{i,i}}{1+\rho^2}((1+\rho^2)-2\rho^2)=\frac{(\Omega^{-1})_{i,i}}{1+\rho^2}(1-\rho^2)&=1,\\(\Omega^{-1})_{i,i}&=\frac{1+\rho^2}{1-\rho^2}\end{aligned} \tag{2.28}$$

and the i-th line of Ω^{-1} is equal to

$$\frac{1}{1-\rho^2}\,(0,\dots,-\rho,\underset{(i)}{1+\rho^2},-\rho,\dots,0)\,. \tag{2.29}$$

3. CUBIC EQUIDISTANT SPLINES

Consider the interval $[0,1]$ and the points

$$x=0,\quad x_k=\frac{k}{n},\quad k=1,2,\dots,n-1,\quad x_n=1\,. \tag{3.1}$$

$f(x),\ x \in [0,1]$ is called a cubic spline if

$$(3.2)\qquad f(x) = \sum_{j=0}^{3} a_j^{(k)}(x - x_k)^j, \quad x_b \leq x \leq x_{k+1}, \quad k = 0, 1, 2, \ldots, n-1$$

and

$$(3.3)\qquad f(x_k) = y_k, \quad k = 0, \ldots, n,$$

where the $y_k = g(x_k)$ is a given function. $f(x)$ is considered as an interpolation of $g(x)$. Moreover, it is required that $f(x)$ is twice continuously differentiable. Let $f''(x_k) = M_k$. The M_k are called moments. Then $M_k,\ k = 1, \ldots, n-1$, obeys the equation

$$(3.4)\qquad \begin{pmatrix} 4\,1 & & & & \\ 1\,4\,1 & & & 0 & \\ & 1\,4\,1 & & & \\ & & \ddots & & \\ 0 & & & 1\,4\,1 & \\ & & & & 1\,4 \end{pmatrix} \begin{pmatrix} M_1 \\ \vdots \\ \vdots \\ \vdots \\ M_{n-1} \end{pmatrix} = \begin{pmatrix} V_1 \\ \vdots \\ \vdots \\ \vdots \\ V_{n-1} \end{pmatrix},$$

where $V_1, \ldots, V_{n-1}$ are linear functions of y_i (see Schwarz [7], p. 125, Stoer [8], p. 81, Törnig/Spellucci [9], p. 77). The matrix

$$(3.5)\qquad A = \begin{pmatrix} 4\,1 & & & 0 \\ 1\,4\,1 & & & \\ & & \ddots & \\ & & & 1\,4\,1 \\ 0 & & & 1\,4 \end{pmatrix}$$

is a tridiagonal matrix. Usually the equation system $Am = v$ is solved by representing A as a product of two bidiagonal matrices (see Schwarz [7]). For example

$$(3.6)\qquad \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ 0 & \frac{4}{15} & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 0 & \frac{15}{4} & 1 \\ 0 & 0 & \frac{56}{15} \end{pmatrix}$$

and therefore

$$
\begin{aligned}
\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}^{-1} &= \begin{pmatrix} \frac{1}{4} & -\frac{1}{15} & \frac{1}{56} \\ 0 & \frac{4}{15} & -\frac{1}{14} \\ 0 & 0 & \frac{15}{56} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & 1 & 0 \\ \frac{1}{15} & -\frac{4}{15} & 1 \end{pmatrix} \\
&= \frac{1}{56} \begin{pmatrix} 15 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 15 \end{pmatrix}.
\end{aligned}
\tag{3.7}
$$

However, A is very similar to Ω^{-1} and therefore there might be an explicit formula for A^{-1} which may perhaps also be convenient from the computational point of view. As shown in the introduction

$$
A = \frac{1-\rho^2}{(-\rho)} \begin{pmatrix} \frac{1+\rho^2}{1-\rho^2} & \frac{-\rho}{(1-\rho^2)} & & & 0 \\ \frac{-\rho}{(1-\rho^2)} & \frac{1+\rho^2}{1-\rho^2} & \frac{-\rho}{1-\rho^2} & & \\ 0 & & & \ddots & \\ & & & \frac{-\rho}{1-\rho^2} & \frac{1+\rho^2}{1-\rho^2} \end{pmatrix},
\tag{3.8}
$$

where $\rho = -2 + \sqrt{3}$. Thus

$$
A = \frac{(1-\rho^2)}{-\rho} \left\{ \Omega^{-1} + \frac{\rho^2}{1-\rho^2} \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdots\cdots 0 \\ 0 \cdots\cdots 1 \end{pmatrix} \right\}.
\tag{3.9}
$$

If we apply the well-known Törnquist-Egervary formula

$$
(B + CD)^{-1} = B^{-1} - B^{-1}C(I + DB^{-1}C)^{-1}DB^{-1}
\tag{3.10}
$$

we can find an explicit formula for A^{-1}. This is done in the next section.

4. The inversion of some tridiagonal matrices

We consider matrices of the form

$$A = \begin{pmatrix} \beta & 1 & & & & 0 \\ 1 & \beta & 1 & & & \\ & & & \ddots & & \\ & & & 1 & \beta & 1 \\ 0 & & & & 1 & \beta \end{pmatrix}, \tag{4.1}$$

where $\beta \in \mathbb{C}$ and $\rho = -\frac{\beta}{2} + \frac{1}{2}\sqrt{\beta^2 - 4}$. If $\beta \in \mathbb{R}$ and $\beta^2 > 4$, then $|\rho| < 1$. If $\beta^2 < 4$, then $\rho \in \mathbb{C}$ and $|\rho| = 1$. Special attention will be paid to the cases $\rho^2 = 1$ and $\rho^{2(n+1)} = 1$ because in these cases the derived formulae may not be valid. The case $\beta = 4$ is needed in spline interpolation. We denote the matrix A in (4.1) by $A_n(\beta)$.

Theorem 4.1. *Let $A = A_n(\beta)$. Then*

$$A_n(\beta) = \frac{1-\rho^2}{(-\rho)} \left\{ \Omega^{-1} + \frac{\rho^2}{1-\rho^2}(e_1 e_n)(e_1, e_n)' \right\} \tag{4.2}$$

and

$$(A_n(\beta))^{-1} = (b_{ij}), \tag{4.3}$$

$$b_{ij} = \frac{-\rho^{i-j+1}(1-\rho^{2j})(1-\rho^{2(n-i+1)})}{(1-\rho^2) \quad (1-\rho^{2(n+1)})} \tag{4.4}$$

if $i \geq j$ and $\rho^2 \neq 1$, $\rho^{2(n+1)} \neq 1$. If $j \geq i$ then $b_{ij} = b_{ji}$ as above. This result is correct for $n \geq 2$.

Proof. According to (3.10) we have to compute

$$(I_2 + C\,\Omega\,D)^{-1} \tag{4.5}$$

where $C = \dfrac{\rho^2}{1-\rho^2} D',\ D = (e_1, e_n)',\ e_i$ the i-th unit-vector. We get

$$(I_2 + C\,\Omega\,D) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\rho^2}{1-\rho^2} \begin{pmatrix} 1 & 0 \\ \vdots & 0 \\ 0 & 1 \end{pmatrix} \Omega \begin{pmatrix} 1 \cdots 0 \\ 0 \cdots 1 \end{pmatrix}$$

$$(4.6) \qquad = I_2 + \frac{\rho^2}{1-\rho^2} \begin{pmatrix} 1 & \rho^{n-1} \\ \rho^{n-1} & 1 \end{pmatrix}$$

$$= \frac{1}{1-\rho^2} \begin{pmatrix} 1 & \rho^{n+1} \\ \rho^{n+1} & 1 \end{pmatrix},$$

$$(4.7) \qquad \left(I_2 + \frac{\rho^2}{1-\rho^2} D\,\Omega\,D \right)^{-1} = \frac{1-\rho^2}{(1-\rho^{2(n+1)})} \begin{pmatrix} 1 & -\rho^{n+1} \\ -\rho^{n+1} & 1 \end{pmatrix}.$$

From $\Omega \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho^{n-1} \\ \vdots & \vdots \\ \rho^{n-1} & 1 \end{pmatrix}$ it follows that

$$\frac{\rho^2}{1-\rho^2} \Omega\,D' \left(I_2 + \frac{\rho^2}{1-\rho^2} D'\,\Omega\,D \right)^{-1} D\,\Omega =$$

$$(4.8) \qquad = \frac{\rho^2}{1-\rho^{2(n+1)}} \begin{pmatrix} 1-\rho^{2n} & \rho^{n-1}-\rho^{n+1} \\ \vdots & \vdots \\ \rho^{i-1}-\rho^{2n-(i-1)} & \rho^{n-i}-\rho^{n+i} \\ \vdots & \vdots \\ \rho^{n-1}-\rho^{n+1} & \rho^{n}-\rho^{2n} \end{pmatrix} \begin{pmatrix} 1 \cdots \rho^{n-1} \\ \rho^{n-1} \cdots 1 \end{pmatrix}.$$

We have compute the inner product of the i-th row of the left hand matrix with the j-th column of the right hand matrix. This yields

$$(4.9)\quad \begin{aligned}&(\rho^{i-1}-\rho^{2n-(i-1)})\rho^{j-1}+(\rho^{n-i}-\rho^{n+i})\rho^{n-j}=\\ &=\rho^{i+j-2}-\rho^{2n-i+j}+\rho^{2n-i-j}-\rho^{2n+i-j}\,.\end{aligned}$$

Thus

$$(4.10)\quad \begin{aligned}&A^{-1}=\\ &=\frac{(-\rho)}{1-\rho^2}\left\{\rho^{|i-j|}-\frac{\rho^2}{1-\rho^{2(n+1)}}\left[\rho^{i+j-2}+\rho^{2n-(i+j)}-\rho^{2n-i+j}-\rho^{2n+i-j}\right]\right\}.\end{aligned}$$

We now consider the case $i \geq j$ – no restriction in view of symmetry. We get

$$\begin{aligned}b_{ij}&=\frac{(-\rho)}{1-\rho^2}\,\frac{1}{1-\rho^{2(n+1)}}\{\rho^{i-j}(1-\rho^{2(n+1)})\}-\rho^{i+j}-\rho^{2(n+1)-(i+j)}\\ &\qquad+\rho^{2(n+1)-i+j}+\rho^{2(n+1)-j+i}\\ &=\frac{(-\rho)}{1-\rho^2}\,\frac{1}{1-\rho^{2(n+1)}}\{\rho^{i-j}-\rho^{i+j}-\rho^{2(n+1)-(i+j)}-\rho^{2(n+1)-i+j}\}\end{aligned}$$

$$(4.11)\quad \begin{aligned}&=\frac{-\rho}{(1-\rho^2)}\,\frac{1}{(1-\rho^{2(n+1)})}(\rho^{i-j}-\rho^{i+j})\\ &=\frac{-\rho}{1-\rho^2}\,\frac{\rho^{i-j}(1-\rho^{2j})(1-\rho^{2(n-i+1)})}{1-\rho^{2(n+1)}}\\ &=-\,\frac{\rho^{i-j+1}(1-\rho^{2j})(1-\rho^{2(n-i+1)})}{(1-\rho^2)(1-\rho^{2(n+1)})}\,.\end{aligned}$$

∎

The theorem is not valid for $n = 1$, but if $n = 1$ and hence $i = j = 1$ then

$$
\begin{aligned}
& -\frac{\rho^{i-j+1}(1-\rho^{2j})(1-\rho^{2(n-i+1)})}{(1-\rho^2)(1-\rho^{2(n+1)})} = \\
= \quad & -\rho\frac{(1-\rho^2)}{1-\rho^4} = \frac{-\rho}{1+\rho^2} = \frac{-\rho}{-\beta\rho} = \frac{1}{\beta}\,.
\end{aligned} \tag{4.12}
$$

Thus the formula is also correct for $n = 1$. We prove the theorem again by additionally slightly generalizing it. If

$$
A = \begin{pmatrix} \beta & \alpha & & & 0 \\ \gamma & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \alpha \\ 0 & & \ddots & \gamma & \beta \end{pmatrix} = A_{u(\alpha,\beta,\gamma)} \tag{4.13}
$$

and $\alpha, \gamma \neq 0$ (The case $\alpha = 0$ or $\gamma = 0$ leads to a bidiagonal matrix easily invertible.), then

$$
A_n(\alpha,\beta,\gamma) = \sqrt{\alpha\gamma}A_n\left(\sqrt{\frac{\alpha}{\gamma}}, \frac{\beta}{\sqrt{\alpha\gamma}}, \sqrt{\frac{\gamma}{\alpha}}\right) \tag{4.14}
$$

and

$$
(A_n(\alpha,\beta,\gamma))^{-1} = (\alpha\gamma)^{-\frac{1}{2}}A_n^{-1}\left(\sqrt{\frac{\alpha}{\gamma}}, \frac{\beta}{\sqrt{\alpha\gamma}}, \sqrt{\frac{\gamma}{\alpha}}\right). \tag{4.15}
$$

Thus it is no restriction to assume that $\alpha\gamma = 1$.

Theorem 4.2. *Let $A = A_n(\alpha,\beta,\gamma)$, where $\alpha\gamma = 1$. Then $A^{-1}(\alpha,\beta,\gamma) = (b_{ij})$ and*

$$
b_{ij} = \frac{-\alpha^{i-j}\rho^{i-j+1}(1-\rho^{2j})(1-\rho^{2(n-i+1)})}{(1-\rho^2)(1-\rho^{2(n+1)})},\ i \geq j, \tag{4.16}
$$

where $\rho^2 + \beta\rho + 1 = 0$. If $i \leq j$, then

$$
b_{ij} = \frac{-\gamma^{j-i}\rho^{j-i+1}(1-\rho^{2j})(1-\rho^{2(n-j+1)})}{(1-\rho^2)(1-\rho^{2(n+1)})}\,. \tag{4.17}
$$

Proof. Only the case $i \geq j$ must be considered, since the case $j \leq i$ follows by transposition. The proof for $i \geq j$ is done by mathematical induction. For $n = 1$

$$b_{11} = -\rho \frac{(1-\rho^2)}{(1-\rho^4)} = \frac{-\rho}{1+\rho^2} = \frac{-\rho}{-\beta\rho} = +\frac{1}{\beta}\,, \tag{4.18}$$

if $\beta \neq 0$, i. e., $A_1(\alpha, \beta, \gamma)$ is invertible. We now assume that the formula is correct for n and we use the formula (see Rao [5], p. 33)

$$\begin{aligned} &\begin{pmatrix} A_n & \vdots & \alpha\, e_n \\ \cdots\cdots & \cdots & \cdots \\ \gamma\, e_n' & \vdots & \beta \end{pmatrix}^{-1} = \\ &= \begin{pmatrix} A_n^{-1} + A_n^{-1}\alpha\, e_n E_n^{-1}\gamma\, e_n' A_n^{-1} & \vdots & -\alpha A_n^{-1} e_n E_n^{-1} \\ \cdots\cdots\cdots & \cdots & \cdots\cdots \\ -E_n^{-1}\gamma\, e_n' A_n^{-1} & \vdots & E_n^{-1} \end{pmatrix}, \end{aligned} \tag{4.19}$$

where $E_n = \beta - \alpha\gamma\, e_n' A_n^{-1} e_n = \beta - e_n' A_n^{-1} e_n$ (Schur-complement). By assumption

$$e_n' A_n^{-1} e_n = \frac{-\rho}{1-\rho^2}\,\frac{(1-\rho^{2n})(1-\rho^2)}{(1-\rho^{2(n+1)})} = \frac{-\rho(1-\rho^{2n})}{(1-\rho^{2(n+1)})}\,. \tag{4.20}$$

Thus

$$\begin{aligned} E_n &= \frac{\beta(1-\rho^{2(n+1)} + \rho(1-\rho^{2n}))}{(1-\rho^{2(n+1)})} \\ &= \frac{-\rho^{2n+1}(1+\beta\rho) + \beta + \rho}{(1-\rho^{2(n+1)})} = \frac{\rho^{(2n+1)}\rho^2 + \beta + \rho}{(1-\rho^{2(n+1)})} \\ &= \frac{\rho^{2n+3} - \rho^{-1}}{(1-\rho^{2(n+1)})} = \frac{-\rho^{-1}(1-\rho^{2(n+2)})}{(1-\rho^{2(n+1)})} \end{aligned} \tag{4.21}$$

since $\rho^2 = -(1+\beta\rho),\ (\beta+\rho)\rho = \rho^2 + \beta\rho = -1$ and finally

$$E_n^{-1} = -\frac{\rho(1-\rho^{2(n+1)})}{1-\rho^{2(n+2)}}\,. \tag{4.22}$$

This finishes the induction-proof in the case of the $(n+1,\, n+1)$th element of $A_{n+1}(\alpha,\beta,\gamma)$. Since

$$(A_n^{-1}e_n)_j = -\,\frac{\alpha^{n-j}\rho^{n-j+1}(1-\rho^{2j})}{(1-\rho^{2(n+1)})}$$

it follows that

$$\begin{aligned} &-\alpha(A_n^{-1}e_n)_j E_n^{-1} = (A_{n+1}^{-1})_{j,n+1} = \\ &= \tfrac{-\alpha^{(n+1)-j}\rho^{(n+1-j+1)}(1-\rho^{2j})}{(1-\rho^{2(n+2)})}. \end{aligned} \tag{4.23}$$

This is the desired formula with n replaced by $n+1$. Similarly follows from

$$(e_n' A_n^{-1})_i \;=\; \frac{-\gamma^{n-i}\rho^{n-i+1}(1-\rho^{2i})}{(1-\rho^{2(n+1)})} \tag{4.24}$$

that

$$-\gamma(e_n' A_n^{-1})_i \;=\; \frac{-\gamma^{n+1-i}\rho^{n+1-i+1}(1-\rho^{2i})}{(1-\rho^{2(n+1)})}\,, \tag{4.25}$$

i.e., the formula with n replaced by $n+1$. Finally for $i \geq j$ we compute

$$\begin{aligned} C = &\frac{-\alpha^{i-j}\rho^{i-j+1}(1-\rho^{2j})(1-\rho^{2(n-i+2)})}{(1-\rho^{2(n+2)})(1-\rho^{2})} \\ &-\frac{-\rho^{i-j}\rho^{i-j+1}(1-\rho^{2(n-i+1)})(1-\rho^{2j})}{(1-\rho^{2(n+1)})(1-\rho^{2})}\,. \end{aligned} \tag{4.26}$$

The first term is asserted to be $(A_{n+1}^{-1})_{i,j})$, while the second term is $(A_n^{-1})_{i,j}$. We have to show that

$$C = (A_n^{-1})_{i,j}(A_n^{-1})_{i,n}E_n^{-1}$$

$$(4.27) \qquad = \frac{-\alpha^{n-j}\rho^{n-j+1}(1-\rho^{2j})(1-\rho^2)\gamma^{n-i}(1-\rho^{2i})(1-\rho^2)\rho^{n-i+1}\rho}{(1-\rho^2)(1-\rho^{2(n+1)})(1-\rho^{2(n+2)})(1-\rho^2)}$$

$$= \frac{(\alpha^{i-j}\rho^{i-j+1})\rho^{2(n-i+1)}(1-\rho^{2i})(1-\rho^{2j})(1-\rho^2)}{(1-\rho^2)(1-\rho^{2(n+1)})(1-\rho^{2(n+2)})}\,.$$

By shortening common factors we have to show that

$$(4.28) \qquad \begin{aligned} D &= (1-\rho^{2(n-i+2)})(1-\rho^{2(n+1)}) - (1-\rho^{2(n-i+1)})(1-\rho^{2(n+2)}) \\ &= \rho^{2(n-i+1)}(1-\rho^{2i})(1-\rho^2)\,. \end{aligned}$$

A simple algebraic manipulation shows that this indeed true. A similar argument holds for $i \leq j$. ■

A still simpler representation of A^{-1} is possible. Since $\rho^2 = -(\beta\rho+1)$, $\rho^n = a_n + b_n\rho$ for some a_n, $b_n \in \mathbb{C}$. Now $\rho^{n+1} = a_n\rho + b_n\rho^2 = a_{n+1} + b_{n+1}\rho = a_n\rho - b_n(\beta\rho+1) = (a_n - \beta b_n)\rho - b_n$. Thus a_{n+1} can be chosen equal to $-b_n$, while b_{n+1} can be chosen equal to $a_n - \beta b_n = -(b_{n-1} + \beta b_n)$. We get therefore the difference-equation

$$(4.29) \qquad b_{n+1} + \beta b_n + b_{n-1} \;=\; 0\,.$$

Obiously, $b_0 = 0$, $a_0 = 1$, $b_1 = 1$, $a_1 = -b_0 = 0$. Before formulating the next theorem we note that $\rho \neq 0$.

Theorem 4.3. $b_n = \dfrac{(1-\rho^{2n})\rho^{-(n-1)}}{1-\rho^2}\,, \quad n = 0, 1, 2, \ldots$

Proof. This formula is correct for $n = 0, 1$ and if it is correct for $n-1$ and n, then

$$
\begin{aligned}
b_{n+1} &= -(\beta b_n + b_{n-1}) \\
&= \frac{-1}{1-\rho^2}(\beta(1-\rho^{2n})\rho^{-(n-1)} + (1-\rho^{2(n-1)})\rho^{-(n-2)}) \\
&= \frac{-\rho^{-(n-1)}}{1-\rho^2}\,((1-\rho^{2n})\beta + (1-\rho^{2(n-1)})\rho) \\
&= \frac{-\rho^{-(n-2)}}{1-\rho^2}\,((\beta+\rho) - \rho^{2n-1}(\beta\rho+1)) \qquad (4.30)\\
&= \frac{\rho^{-(n-1)}}{1-\rho^2}(\rho^{-1} - \rho^{2n+1}) \\
&= \frac{\rho^{-(n-1)}\rho^{-1}}{1-\rho^2}(1-\rho^{2(n+1)}) \\
&= \frac{\rho^{-n}}{1-\rho^2}(1-\rho^{2(n+1)})\,.
\end{aligned}
$$

■

Corollary 4.4. *$A^{-1} = (b_{ij})$, where*

$$
\begin{aligned}
b_{ij} &= -\alpha^{i-j}\frac{b_j b_{n-i+1}}{b_{n+1}}, \quad i \geq j, \\
b_{ij} &= -\gamma^{j-i}\frac{b_i b_{n-j+1}}{b_{n+1}}, \quad j \geq i.
\end{aligned} \tag{4.31}
$$

Proof. Since $b_j = \dfrac{(1-\rho^{2j})\rho^{-(j-1)}}{1-\rho^2}$,

$$
b_{n-i+1} \;=\; \frac{(1-\rho^{2(n-i+1)})}{1-\rho^2}\;\rho^{-(n-i)}\,, \tag{4.32}
$$

and finally

$$
b_{n+1} = \frac{1-\rho^{2(n+1)}}{1-\rho^2}\rho^{-n},
$$

the Corollary follows immediately from Theorem 4.1. ■

The formulae given by Corollyry 4.4 are even simpler then the result of Theorem 4.1 and Theorem 4.2. However, the b_k may be very large numbers which can cause inaccuracies in a numerical result. The adventage of Theorem 4.1 and Theorem 4.2 lies in the fact that only small numbers of $[-1,1]$ must be multiplied.

Example 4.5. Let again $n = 3$ and $\beta = 4,\ \alpha = \gamma = 1$. Then $b_0 = 0,\ b_1 = 1,\ b_2 = -4,\ b_3 = 15,\ b_4 = -56$ and

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}^{-1} = \frac{-1}{b_4} \begin{pmatrix} b_3 & b_2 & b_1 \\ b_2 & b_2^2 & b_2 \\ b_1 & b_2 & b_3 \end{pmatrix} = \frac{1}{56} \begin{pmatrix} 15 & -4 & 1 \\ -4 & 16 & -4 \\ 1 & -4 & 15 \end{pmatrix}.$$

Remark 4.6. We did not yet discuss the case $\rho^2 = 1$ or $\rho^{2(n+1)} = 1$. If $\rho^2 = 1$, than $\rho = +1$ and $\rho = -1$, respectively, while $b_n = n$ and $b_n = (-1)^n n$, respectively. It turns out that the formulae of Theorem 4.1 and 4.2 are still correct in the sense that we pass to the limit $\rho^2 \to 1$ (Drygas [2]).

In a subsequent paper it will be shown that if $\rho^2 \neq 0$ then $b_n = 0$ is equivalent to $\rho^{2(n+1)} = 1$. It is not hard to prove that $\det(A_n(\alpha, \beta, \gamma)) = (-1)^n b_{n+1}$. Therefore the formulae of Theorems 4.1 and 4.2 apply in all cases when A^{-1} exists.

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