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BAND COPULAS AS SPECTRAL MEASURES FOR TWO-DIMENSIONAL STABLE RANDOM VECTORS

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Abstract

In this paper, we study basic properties of symmetric stable random vectors for which the spectral measure is a copula, i.e., a distribution having uniformly distributed marginals.

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1. Introduction

We say that a symmetric random variable is stable if there exists a positive constant A and index of stability $\alpha \in (0, 2]$ such that

$$\mathbf{E}e^{itX} = \exp\{-A|t|^{\alpha}\}, \qquad t \in \mathbb{R}.$$

A random vector $X = (X_1, \ldots, X_n)$ is symmetric α-stable if for every $\xi = (\xi_1, \ldots, \xi_n)$ the random variable $< \xi, X >= \sum_{k=1}^{n} \xi_k X_k$ is symmetric α-stable. The following, well known theorem was proven by Feldheim in 1937 and presented in P. Levy [5] in 1937 (first edition).

Theorem 1.1. *A random vector $X = (X_1, \dots, X_n)$ is symmetric α-stable if and only if there exists a finite measure ν on the unit sphere $S_{n-1} \subset I\!R^n$ such that*

$$\mathbf{E}e^{i<\xi,X>} = \exp\Big\{ - \underset{S_{n-1}}{\int \dots \int} | < \xi, x > |^\alpha \nu(dx) \Big\}.$$

The measure ν on S_{n-1} is uniquely determined and it is called the canonical spectral measure for the symmetric α-stable random vector X.

Remark 1. Usually the measure ν given in the previous theorem is simply called the spectral measure for the symmetric α-stable vector X. However we will also consider other representations for the characteristic functions of X, so in this paper a *canonical spectral measure* will always mean the measure concentrated on the unit sphere. The existence of many representations of the characteristic functions for the given symmetric α-stable vector X follows from the following theorem:

Theorem 1.2. *For every symmetric finite measure ν on $I\!R^n$ such that:*

$$\underset{I\!R^n}{\int \dots \int} \|x\|^\alpha \nu(dx) < \infty$$

the following function:

$$\varphi(\xi) \overset{def}{=} \exp\Big\{ - \underset{I\!R^n}{\int \dots \int} | < \xi, x > |^\alpha \nu(dx) \Big\} \tag{1}$$

is a characteristic function of a symmetric α-stable vector $X = (X_1, \dots, X_n)$. The measure ν given by equality (1) *we will call a spectral measure for the random vector X. This measure is not uniquely determined.*

Proof. We shall prove that for every fixed $\xi \in I\!R^n$ the function $\varphi(t\xi)$, as function of $t \in I\!R$, is a characteristic function of an SαS random variable, i.e., there exists $A > 0$ such that $\varphi(\xi t) = e^{-A|t|^\alpha}$. Indeed:

$$\begin{aligned} \varphi(\xi t) &= \exp\Big\{ - \underset{I\!R^n}{\int \dots \int} | < \xi t, x > |^\alpha \nu(dx) \Big\} \\ &= \exp\Big\{ -|t|^\alpha \underset{I\!R^n}{\int \dots \int} | < \xi, x > |^\alpha \nu(dx) \Big\}. \end{aligned}$$

It is enough to take

$$A = A(\xi) = \int \ldots \int_{\mathbb{R}^n} | < \xi, x > |^{\alpha} \nu(dx).$$

■

Remark 2. If the characteristic function of a symmetric α-stable random vector X is given by the formula (1) with the spectral measure ν, then the canonical spectral measure ν_0 for this vector we obtain substituting $u = x/\|x\|$, and $r = \|x\|$ and integrating with respect to r. Notice that if ν has an atom at zero, then this atom has no influence on the formula (1), thus we can always assume that $\nu(\{0\}) = 0$.

Remark 3. Assume that $n = 2$ and assume that the canonical measure ν in formula (1) is absolutely continuous with the density function $f(x, y)$. Then we can write:

$$\int \ldots \int_{\mathbb{R}^n} | < \xi, x > |^{\alpha} \nu(dx) =$$

$$= \int_0^{2\pi} |\xi_1 \cos t + \xi_2 \sin t|^{\alpha} \int_0^{\infty} r^{\alpha+1} f(r \cos t, r \sin t) dr dt.$$

This means that the canonical spectral measure ν_0 for this random vector has the density given by:

$$g(u) = \int_0^{\infty} f(ru) r^{\alpha+1} dr, \qquad u \in S_1 \subset \mathbb{R}^2.$$

2. Copulaes

In general, by the term copula we understand a two dimensional (or n-dimensional) distribution with given marginals. The inversion method restricts the problem of constructing such distributions into constructing distributions on $[0,1]^2$ (or $[-1,1]^2$) having marginals uniform on the interval $[0,1]$ (or $[-1,1]$ respectively). Many types of copulas are well known in the literature. Recently there appeared a book written by Nelsen [6] which is entirely devoted to the theory of copulaes and a two dimensional distribution on $[0,1]^2$. In this paper, we will use copulaes from a very wide class constructed independently by T.S. Ferguson in [2] and J. Bojarski in [1]. The construction is follows:

Construction:
Let Z be a random variable with a density function $f(z)$, concentrated on an interval $[-2,2]$ such that $f(z)=f(-z)$. We define a two-dimensional density function $g(x,y)$ concentrated on $[-1,1]^2$ by the formula

$$g(x,y)=\begin{cases} f(x-y)+f(x+y-2) & \text{for} \quad x+y\geq 0,\\ \\ f(x-y)+f(x+y+2) & \text{for} \quad x+y\leq 0.\end{cases}$$

The density $g(x,y)$ has marginals uniform on the interval $[-1,1]$, thus it defines a two-dimensional copulae.

3. Copulae as a spectral measure for an SαS random vector

Let $x^{<p>}=|x|^p\mathrm{sign}(x)$. This notation is very useful in describing properties and moments of random variables with a infinite variance. In our considerations, we will use the following formulas:

$$\begin{aligned} \int (ax+b)^{<\alpha>}\,dx &= \frac{1}{a(\alpha+1)}|ax+b|^{\alpha+1}+C,\\ \int |ax+b|^{\alpha}\,dx &= \frac{1}{a(\alpha+1)}\,(ax+b)^{<\alpha+1>}+C. \end{aligned} \tag{2}$$

Theorem 3.1. *Assume that Z is a random variable with the density function $f(z)$ concentrated on $[-2,2]$. If the spectral measure ν of an $S\alpha S$ random vector (X,Y) has the density $g(x,y)$ given by formula (2), then the characteristic function of (X,Y) at the point (a,b) is given by* $\exp\{-c(a,b)^\alpha\}$ *where*

$$c(a,b)^\alpha =$$

$$= \frac{2(1+\alpha)^{-1}}{(a^2-b^2)}\mathbf{E}\left[b\,(b-a(1-|Z|))^{<\alpha+1>}+a\,(b(1-|Z|)-a)^{<\alpha+1>}\right].$$

The James correlation coefficient for the random vector (X, Y) is given by:

$$\rho_\alpha(X,Y) \stackrel{def}{=} \int \ldots \int x^{<\alpha-1>} y \nu(dx, dy)$$

$$= \frac{2}{\alpha(\alpha+1)} \mathbf{E}\left[(\alpha+1)(1-|Z|) - (1-|Z|)^{<\alpha+1>}\right].$$

Proof. The proof is only a matter of laborious calculations, and the integral formulas given at the begining of this section simplify these calculations slightly. The formula for ρ_α holds for every $\alpha \in (0, 2]$ as long as the right hand side makes sense. ■

Examples. In the following three examples we want to ilustrate the dependence between the distribution $f(x)$, the distribution of spectral measure $g(x, y)$, the shape of level curves of the characteristic function of the corresponding $S\alpha S$ vector for different α's. For each example we give also

$$h(\alpha) = \frac{\rho_\alpha(X,Y)}{\rho_\alpha(X,X)}$$

describing the dependence between α and the James correlation function. In the definition of the function $h(\alpha)$ we shall explain something more. Since $g(x, y)$ is a copula density function, then it has identical marginals, and from the Hölder inequality we obtain

$$\left|\int \ldots \int x^{<\alpha-1>} y \nu(dx, dy)\right|$$

$$\leq \left(\int \ldots \int |x|^\alpha \nu(dx, dy)\right)^{\frac{\alpha-1}{\alpha}} \left(\int \ldots \int |y|^\alpha \nu(dx, dy)\right)^{\frac{1}{\alpha}}$$

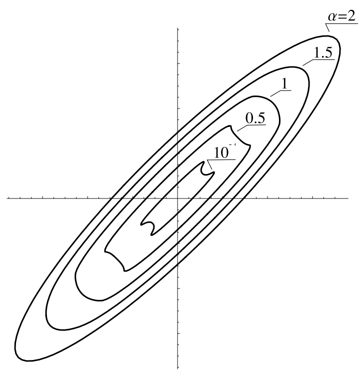
$$= \int \ldots \int |x|^\alpha \nu(dx, dy).$$

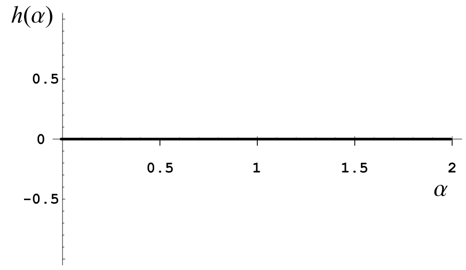
This means that

$$|h(\alpha)| \leq 1,$$

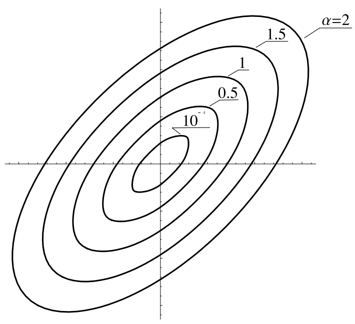
thus the function $h(\alpha)$ can play the same role for the $S\alpha S$ random vector as the correlation coefficient for the second order random vector. In our examples the function $h(\alpha)$ makes sense on the whole interval $(0, 2]$.

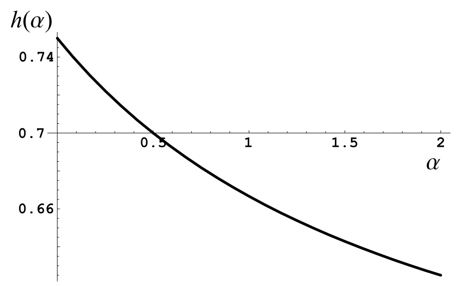
Example 1. Notice that the shape of level curves for the characteristic function suggests positive dependence coefficients, while in fact we have here $h(\alpha) = 0$ for every $\alpha \in (0, 2]$.



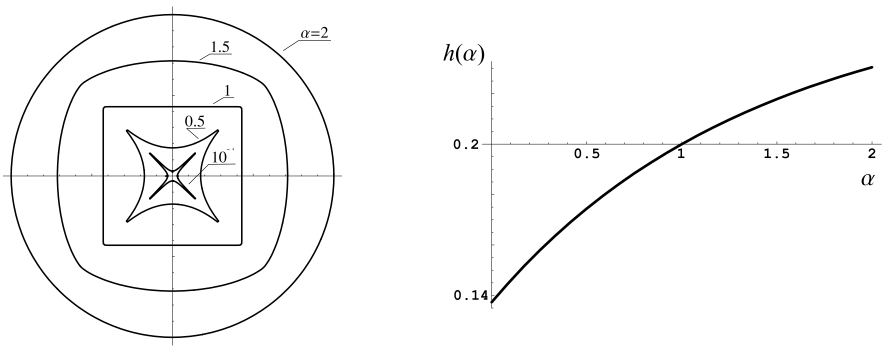


Example 2.





Example 3.



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