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APPROXIMATE BIAS FOR FIRST-ORDER AUTOREGRESSIVE MODEL WITH UNIFORM INNOVATIONS. SMALL SAMPLE CASE

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Abstract

The first-order autoregressive model with uniform innovations is considered. The approximate bias of the maximum likelihood estimator (MLE) of the parameter is obtained. Also, a formula for the approximate bias is given when a single outlier occurs at a specified time with a known amplitude. Simulation procedures confirm that our formulas are suitable. A small sample case is considered only.

Keywords: autoregressive model, bias, outlier, uniform distribution.

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1. Introduction

Consider the following autoregressive model

$$Y_t = \rho Y_{t-1} + \varepsilon_t \tag{1}$$

where the ε_t's are *i.i.d.* and distributed according to a uniform distribution $U(0,1)$. Bell and Smith (1986) studied the estimating and testing problem on the parameter ρ for the model (1). Confidence intervals for ρ are obtained in Bickel and Doksum (1977) and in Choi (1980). We assume Y_0 distributed as $U(0, 1/(1-\rho))$ and observe the segment of observations

$$Y_1, Y_2, \ldots, Y_n, \qquad n \quad \text{fixed.} \tag{2}$$

The maximum likelihood estimator (MLE) for ρ is (Bell and Smith, 1986) $\hat{\rho}_Y = \min_{2 \leq t \leq n}(Y_t/Y_{t-1})$. Then

$$E(\hat{\rho}_Y - \rho) = E\left(\min_{2 \leq t \leq n}\left(\frac{\varepsilon_t}{Y_{t-1}}\right)\right). \tag{3}$$

Since the process is mean stationary with mean $m = \dfrac{1}{2(1-\rho)}$, we can use the method proposed by Anděl (1988) in exponential models with substituting m for Y_{t-1} in (3) and obtain

$$E(\hat{\rho}_Y - \rho) \simeq \frac{1}{m} E\left(\min_{2 \leq t \leq n} \varepsilon_t\right).$$

The aim of this paper is to approximate the bias of the maximum likelihood estimator of ρ in the model (1) and in the case of two kinds of contamination of this model. The first one is obtained when we observe the process

$$X_t = \rho X_{t-1} + \varepsilon_t + \Delta\ \delta_{t,k}, \qquad n \quad \text{fixed} \tag{4}$$

where

$$\delta_{t,k} = \begin{cases} 1 & \text{if} \quad t = k \\ 0 & \text{if} \quad t \neq k \end{cases}$$

instead of (1). The other is obtained when we observe the process

$$Z_t = Y_t \quad \forall t \neq k \quad \text{and} \quad Z_k = Y_k + \Delta \tag{5}$$

instead of (1), Δ being a known positive magnitude of the contamination of (1) which occurs at $t = k$ with $1 < k < n$.

The processes (4) and (5) are called an innovation outlier (IO) model and an additive outlier model respectively (Fox, 1972). An exhaustive simulation study confirms that our formulas hold.

2. Non contaminated model

Proposition. *When the model* (1) *is observed, the bias of the maximum likelihood estimator of* ρ *can be approximated by the expression*

$$B_0(\rho, n) \simeq \frac{2(1-\rho)}{n}. \tag{6}$$

Proof. Using Anděl's method, the bias of $\hat{\rho}_Y$ is approximated by

$$E(\hat{\rho}_Y - \rho) \simeq \frac{1}{m} E\left(\min_{2 \le t \le n} \varepsilon_t\right).$$

Since the random variable $S = \min_{2 \le t \le n} \varepsilon_t$ is distributed according to Beta$(1, n-1)$, we obtain

$$E(\hat{\rho}_Y - \rho) \simeq \frac{1}{nm} = \frac{2(1-\rho)}{n}.$$

■

Comment 1. For a given n, if $0 < \rho < 1$, the maximal bias is approximatively $2/n$.

Comment 2. The formula (6) can be used as a method of reduction of the bias. Indeed, the modified estimator $\tilde{\rho}_Y = (n\hat{\rho}_Y - 2)/(n-2)$ has a smaller bias than the MLE.

In order to illustrate that, we simulate n observations from the model (1) for a given n and ρ. The following Table 1 presents simulated values of the bias of the MLE and the modified estimator given above for $n = 10, 20$ and $\rho = 0.2, 0.4, 0.6$.

n	ρ	$\hat{\rho}_Y$	$\tilde{\rho}_Y$
10	0.2	0.1403	0.1209
	0.4	0.0895	0.0779
	0.6	0.0560	0.0484
20	0.2	0.0744	0.0581
	0.4	0.0467	0.0367
	0.6	0.0282	0.0223

Table 1. Simulated values of the bias of $\hat{\rho}_Y$ and $\tilde{\rho}_Y$, 100000 runs.

3. INNOVATION OUTLIER MODEL

When the IO model (4) is observed, the MLE estimator of ρ is

$$\hat{\rho}_X = min\{X_2/X_1, X_3/X_2, \ldots, X_n/X_{n-1}\}. \tag{7}$$

Since

$$X_t = Y_t \quad \forall t < k \quad \text{and} \quad X_t = Y_t + \rho^{t-k}\Delta \quad \text{for} \quad k \leq t \leq n$$

the bias of $\hat{\rho}_X$ can be approximated in the same way by

$$E(\hat{\rho}_X - \rho) \simeq$$

$$E(min\{\varepsilon_2/m, \varepsilon_3/m, \ldots, (\varepsilon_k + \Delta)/m, \varepsilon_{k+1}/(m+\Delta), \ldots, \varepsilon_n/(m + \rho^{n-k-1}\Delta)\}).$$

Let us introduce the notations : $\delta_0 = \dfrac{-m + \sqrt{m^2 + 4m}}{2}$ and

$W = \{\varepsilon_2/m, \varepsilon_3/m, \ldots, (\varepsilon_k + \Delta)/m, \varepsilon_{k+1}/(m+\Delta), \ldots, \varepsilon_n/(m+\rho^{n-k-1}\Delta)\}$.

Then, the probability distribution of W can be given in the two following cases:

Case 1. $0 \leq \Delta \leq \delta_0$

$$F_W(x) =$$

$$\begin{cases} 0 \quad \text{if} \quad x < 0 \\ 1-(1-mx)^{k-2} \displaystyle\prod_{j=0}^{n-k-1} \left(1-mx-\left(\frac{2m-1}{2m}\right)^j \Delta x\right) \quad \text{if} \quad 0 \leq x < \Delta/m \\ 1-(1-mx)^{k-2}(\Delta + 1-mx) \displaystyle\prod_{j=0}^{n-k-1} (1-mx-\left(\frac{2m-1}{2m}\right)^j \Delta x) \\ \qquad\qquad \text{if} \quad \Delta/m \leq x < 1/(m+\Delta) \\ 1 \quad \text{if} \quad x \geq 1/(m+\Delta) \end{cases}$$

Case 2. $\Delta > \delta_0$

$$F_W(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ 1-(1-mx)^{k-2} \prod_{j=0}^{n-k-1} \left(1-mx-\left(\frac{2m-1}{2m}\right)^j \Delta x\right) & \text{if} \quad 0<x< 1/(m+\Delta) \\ 1 & \text{if} \quad x \geq 1/(m+\Delta) \end{cases}$$

Since analytic treatments of the function $F_W(.)$ are rather complicated, we propose to give the expression of the approximate bias $B_{n,m}(\Delta)$ of estimator for $n = 3$.

For example, if $n = 3$ and $k = 2$, then

$$W = min\left\{\frac{\varepsilon_2 + \Delta}{m}, \frac{\varepsilon_3}{m+\Delta}\right\}.$$

The expression of $F_W(.)$ becomes:

Case 1. $0 \leq \Delta \leq \delta_0$

$$F_W(x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ (\Delta + m)x & \text{if} \quad 0 \leq x < \Delta/m \\ 1-(1+\Delta-mx)(1-mx-\Delta x) & \text{if} \quad \Delta/m \leq x < 1/(m+\Delta) \\ 1 & \text{if} \quad x \geq 1/(m+\Delta) \end{cases}$$

Case 2. $\Delta > \delta_0$

$$F_W(x) = \begin{cases} 0 & \text{if} \quad x \leq 0 \\ (m+\Delta)x & \text{if} \quad 0 \leq x < 1/(m+\Delta) \\ 1 & \text{if} \quad x \geq 1/(m+\Delta) \end{cases}$$

Using easy computations, we obtain the expression of the approximation of the bias of $\hat{\rho}_X$:

(8)

$$B_{3,m}(\Delta) = \begin{cases} \dfrac{(\Delta^6+3m\Delta^5+m(m-1)\Delta^4+m^2(m-6)\Delta^3+3m^2(1-m)\Delta^2+3m^2(1+m)\Delta+2m^3}{6m^2(m+\Delta)^2} & \text{if} \quad 0 \leq \Delta < \delta_0 \\ \dfrac{1}{2(m+\Delta)} & \text{if} \quad \delta \geq \delta_0 \end{cases}$$

One can remark that

(i) $B_{3,m}(0) = \frac{1}{3m} = \dfrac{2(1-\rho)}{3}$.

(ii) $\lim_{\Delta \longrightarrow +\infty} B_{3,\rho}(\Delta) = 0$.

For every n, we can use the expressions of $F_W(.)$ to obtain the approximate bias of the estimator with numerical integration.

To illustrate the given formulas, Table 2 presents approximated and simulated values of the bias for $\rho = 0.5$ and $n = 3, 5, 10$. The simulated bias is calculated using the expression of $\hat{\rho_X}$ (formula (7)).

	$n = 3$, $k = 2$		$n = 5$, $k = 3$		$n = 20$, $k = 18$	
Δ	Approx.	Simul.	Approx.	Simul.	Approx.	Simul.
0	0.333	0.353	0.200	0.218	0.050	0.049
0.1	0.357	0.372	0.211	0.213	0.051	0.051
0.2	0.365	0.373	0.214	0.233	0.051	0.051
0.3	0.362	0.366	0.211	0.230	0.051	0.050
0.4	0.349	0.356	0.205	0.222	0.050	0.050
0.5	0.332	0.343	0.197	0.214	0.050	0.049
0.6	0.312	0.331	0.189	0.205	0.050	0.049
0.7	0.294	0.318	0.181	0.197	0.049	0.049
0.8	0.277	0.306	0.174	0.189	0.049	0.048
0.9	0.263	0.293	0.167	0.181	0.048	0.048
1.0	0.250	0.281	0.161	0.174	0.048	0.048
5	0.083	0.087	0.063	0.065	0.035	0.035
10	0.045	0.046	0.036	0.036	0.025	0.025
20	0.023	0.024	0.019	0.019	0.015	0.015
50	0.009	0.009	0.008	0.008	0.007	0.007

Table 2. Approximated and simulated values of the bias. $\rho = 0.5$, 100000 runs.

One can notice that the simulated and approximated values seems to be close especially when Δ and n big enough. This allows us to say that the approximation is valid. Also, we remark that the bias grows until a maximal value, which lies in $[0, \delta_0]$ and decreases to zero when Δ tends to infinity. For example, when $n = 5$ and $k = 3$, the maximal value is 0.211.

Comment 3. Notice that the quantity

$$min\{\varepsilon_2/m, \varepsilon_3/m, \ldots, (\varepsilon_k + \Delta)/m, \varepsilon_{k+1}/(m + \Delta), \ldots, \varepsilon_n/(m + \rho^{n-k-1}\Delta)\}$$

tends to zero when Δ leads to infinity. This allows us to say that the bias tends to zero when Δ grows to infinity.

4. Additive outlier model

Now, assume that we observe the segment

$$Z_1, \dots, Z_n, \quad n \quad \text{fixed}$$

of the process defined by the model (5).

The MLE estimator of ρ is

$$\hat{\rho}_Z = min\{Z_2/Z_1, Z_3/Z_2, \dots, Z_n/Z_{n-1}\}. \tag{9}$$

Since

$$Z_t = Y_t \quad \forall t \neq k \quad \text{and} \quad Z_k = Y_k + \Delta$$

we can, as in Section 3 approximate the bias of the estimator $\hat{\rho}_Z$ by

$$E(\hat{\rho}_Z - \rho) \simeq$$

$$\frac{1}{m} E\left(min \left\{ \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{k-1}, (\varepsilon_k + \Delta), \frac{(\varepsilon_{k+1} - (\frac{2m-1}{2m})\Delta)m}{(m+\Delta)}, \varepsilon_{k+2}, \dots, \varepsilon_n \right\} \right).$$

The distribution function of the random variable

$$T = min \left\{ \varepsilon_2, \varepsilon_3, \dots, \varepsilon_{k-1}, (\varepsilon_k + \Delta), \frac{(\varepsilon_{k+1} - (\frac{2m-1}{2m})\Delta)m}{(m+\Delta)}, \varepsilon_{k+2}, \dots, \varepsilon_n \right\}$$

can be given in the three following cases:

Case 1. $0 \le \Delta < 1/2$

$$F_T(x) = \begin{cases} 0 & \text{if } x < \frac{-(2m-1)\Delta}{2(m+\Delta)} \\ \frac{m+\Delta}{m}x + \frac{(2m-1)\Delta}{2m} & \text{if } \frac{-(2m-1)\Delta}{2(m+\Delta)} \le x \le 0 \\ 1-(1-x)^{n-3}\left(1-\frac{m+\Delta}{m}x-\frac{(2m-1)\Delta}{2m}\right) & \text{if } 0 \le x < \Delta \\ 1-(1-x+\Delta)(1-x)^{n-3}\left(1-\frac{m+\Delta}{m}x-\frac{(2m-1)\Delta}{2m}\right) & \text{if } \Delta \le x \le \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \\ 1 & \text{if } x \ge \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \end{cases}$$

Case 2. $1/2 \le \Delta < \frac{2(m+\Delta)}{2m-1}$

$$F_T(x) = \begin{cases} 0 & \text{if } x < \frac{-(2m-1)\Delta}{2(m+\Delta)} \\ \frac{m+\Delta}{m}x + \frac{(2m-1)\Delta}{2m} & \text{if } \frac{-(2m-1)\Delta}{2(m+\Delta)} \le x \le 0 \\ 1-(1-x)^{n-3}\left(1-\frac{m+\Delta}{m}x-\frac{(2m-1)\Delta}{2m}\right) & \text{if } 0 \le x < \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \\ 1 & \text{if } x \ge \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \end{cases}$$

Case 3. $\Delta \ge \frac{2m}{2m-1}$

$$F_T(x) = \begin{cases} 0 & \text{if } \ x < \frac{-(2m-1)\Delta}{2(m+\Delta)} \\ \frac{m+\Delta}{m}x + \frac{(2m-1)\Delta}{2m} & \text{if } \ \frac{-(2m-1)\Delta}{2(m+\Delta)} \le x \le \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \\ 1 & \text{if } \ x \ge \frac{2m-(2m-1)\Delta}{2(m+\Delta)} \end{cases}$$

For every n, the expression of the approximate bias is

(10)

$$B_{n,m}(\Delta) =$$

$$\begin{cases} \frac{1}{2nm^2 I_n}\left\{\frac{(\Delta(2m-1))^n}{(2\Delta+2m)^{n-1}}\Phi^{(1)}_{m,n,\Delta} - n\Phi^{(2)}_{m,n,\Delta} - (1-\Delta)^{n-3}\Phi^{(3)}_{m,n,\Delta}\right\} & \text{if } 0 \le \Delta < 1/2 \\ \frac{1}{2m^2}\left\{-\frac{(\Delta(2m-1))^2}{4(m+d)} + \frac{1}{I_n}\left(\frac{(\Delta(2m+1))^{n-1}}{(2\Delta+2m)^{n-2}} - \Phi^{(2)}_{m,n,\Delta}\right)\right\} & \text{if } 1/2 \le \Delta < \frac{2m}{2m-1} \\ \frac{m-(2m-1)\Delta}{2m(m+\Delta)} & \text{if } \Delta \ge \frac{2m}{2m-1} \end{cases}$$

with

$$I_n = (n-1)(n-2),$$

$$\Phi^{(1)}_{m,n,\Delta} = 2n\Delta + 4m(n-1) + n - 2,$$

$$\Phi^{(2)}_{m,n,\Delta} = (2m(n-1) - n + 3)\Delta + 2m(2-n),$$

$$\Phi^{(3)}_{m,n,\Delta} = 2m(n-2) + \Delta(m(24n^2 - 8n - 12) + n - 4)$$
$$+\Delta^2(m(-10n-12) + 12 - 4n)$$
$$+\Delta^3(4m(1-n) - 21n - 12) - \Delta^4(2n-4).$$

Remarks.

(i) $B_{n,m}(0) = 2(1-\rho)/n$.

(ii) $\lim_{\Delta \longrightarrow +\infty} B_{n,m}(\Delta) = -\rho$, $\forall n$.

(iii) $B_{n,m}(\Delta) = B_{n',m}(\Delta)$, $\forall n, n'$, $\forall \Delta \ge \frac{2m}{2m-1}$.

Application.
Table 3 presents the approximated values of the MLE's bias using formula (10) and simulated results obtained with formula (9) for $n = 3, 5, 20$ and $\rho = 0.5$. To see if our formula is suitable, the results are compared.

	$n = 3$, $k = 2$		$n = 5$, $k = 3$		$n = 20$, $k = 15$	
Δ	Approx.	Simul.	Approx.	Simul.	Approx.	Simul.
0	0.333	0.329	0.200	0.179	0.050	0.046
0.1	0.328	0.325	0.200	0.180	0.049	0.044
0.2	0.304	0.301	0.190	0.169	0.046	0.038
0.3	0.268	0.265	0.173	0.151	0.043	0.031
0.4	0.227	0.221	0.153	0.128	0.040	0.022
0.5	0.166	0.175	0.111	0.104	0.016	0.012
0.6	0.124	0.131	0.086	0.079	0.006	0.002
0.7	0.088	0.092	0.059	0.054	0.004	-0.008
0.8	0.055	0.057	0.035	0.031	-0.016	-0.020
0.9	0.026	0.027	0.012	0.008	-0.028	-0.032
1.0	0.000	-0.001	-0.009	-0.012	-0.040	-0.044
1.5	-0.100	-0.102	-0.100	-0.102	-0.104	-0.107
2	-0.166	-0.169	-0.166	-0.168	-0.166	-0.169
5	-0.333	-0.334	-0.333	-0.334	-0.333	-0.334
10	-0.409	-0.409	-0.409	-0.409	-0.409	-0.409
20	-0.452	-0.452	-0.452	-0.452	-0.452	-0.452
50	-0.480	-0.480	-0.480	-0.480	-0.480	-0.480
100	-0.490	-0.490	-0.490	-0.490	-0.490	-0.490

Table 3. Approximated and simulated values of the bias. $\rho = 0.5$, 100000 runs.

As in Section 3, notice that the approximated and the simulated values of the bias seems to be close especially when Δ and n big enough.

In all simulation results, we remarked that the position k has no effect on the value of the bias in the two kinds of contamination. Perhaps, one can say that the reason is that a change in the position k of the contaminant has no influence on the distribution of the random variables W and T.

5. Conclusions

In this work, we obtained the approximate bias of the MLE estimator of the AR(1) parameter. This has been used to reduce the bias of this estimator. Again, when an outlier occurs at a specified time with a known amplitude, a method of approximation of this bias is obtained with an explicit formula in the case of additive outlier contamination. Also, this work shows that the AO has a significant and IO has little effect on the estimated parameter.

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