Discussiones Mathematicae
Probability and Statistics 22 (2002) 73–104

ON THE BEHRENS-FISHER DISTRIBUTION AND ITS GENERALIZATION TO THE PAIRWISE COMPARISONS*

Viktor Witkovský

Institute of Measurement Science
Slovak Academy of Sciences
Dúbravská cesta 9
842 19 Bratislava, Slovak Republic
e-mail: Witkovsky@savba.sk

Abstract

Weerahandi (1995b) suggested a generalization of the Fisher's solution of the Behrens-Fisher problem to the problem of multiple comparisons with unequal variances by the method of generalized p-values. In this paper, we present a brief outline of the Fisher's solution and its generalization as well as the methods to calculate the p-values required for deriving the conservative joint confidence interval estimates for the pairwise mean differences, refered to as the generalized Scheffé intervals. Further, we present the corresponding tables with critical values for simultaneous comparisons of the mean differences of up to $k = 6$ normal populations with unequal variances based on independent random samples with very small sample sizes.

Keywords: Behrens-Fisher distribution, pairwise comparisons, unequal variances, generalized p-values.

2000 Mathematics Subject Classification: 62F04; 62E15.

*The research has been supported by the grant VEGA 1/7295/20 from the Scientific Grant Agency of the Slovak Republic.

1. Introduction

According to the *Encyclopedia of Statistical Sciences*, see Robinson (1982), the Behrens-Fisher problem is that of testing whether the means of two normal populations are the same, without making any assumption about the variances. An essentially equivalent problem is that of finding an interval estimate for the difference between the population means.

Several different solutions to the problem have been proposed in the literature, among the others see e.g. Fisher (1935), Welch (1947), Scheffé (1970), Lee and Gurland (1975), and Robinson (1982). Based on the result of Robinson (1976), a strong theoretical support was given to the original Fisher's solution who derived the result using his theory of statistical inference called fiducial probability which is known, in other cases, to lead to paradoxes. But this is not the only way in which the solution to the problem can be justified, see e.g. Barnard (1984), Meng (1994) and Weerahandi (1995b).

In this paper, we use the equivalent solution to the Behrens-Fisher problem based on the method of generalized p-values, originally suggested by Tsui and Weerahandi (1989). This method allows us to generalize the Fisher's solution to the case of multiple comparisons, see Weerahandi (1995a) and Weerahandi (1995b). In general, the solution requires $(k-1)$-dimensional numerical integration for the evaluation of the exact p-value, where k denotes the number of independent random samples from the normal populations. However, the exact p-values required for deriving the conservative joint confidence interval estimates for the pairwise mean differences, here refered to as the generalized Scheffé intervals, can be calculated by one-dimensional numerical integration. In a special case, when the required degrees of freedom are odd, we present an exact formula, which is a well defined finite linear combination of the cdf's of the Fisher-Snedecor's F-distributions.

Selected parts of the paper were presented at the following conferences: DATASTAT 01, Chata na Čiháku, Czech Republic, August 27–31, 2001, 24th European Meeting of Statisticians, Prague, Czech Republic, August 19–23, 2002 and XXVIII Konferencja Statystyka Matematyczna, Wisła, Poland, December 2–6, 2002.

2. The Fisher's solution

In ours notation, the Fisher's solution is based on the $d(\theta)$ statistic,

$$d(\theta) = \frac{(\bar{Y}_1 - \bar{Y}_2) - \theta}{\sqrt{S_1^2/n_1 + S_2^2/n_2}}, \tag{1}$$

where $\theta = \mu_1 - \mu_2$, and $\bar{Y}_i$ and S_i^2 denote the sample mean and the sample variance, respectively, of the random sample $Y_{i1}, \dots, Y_{in_i}$, with sample size $n_i \geq 2$, from the normal population with mean μ_i and variance σ_i^2, $i = 1, 2$. We denote by $\bar{y}_i$ the observed sample mean and by s_i^2 the observed sample variance. Then, given θ, we denote by $d_{obs}(\theta)$ the observed value of the $d(\theta)$ statistic. Fisher derived, using his fiducial argument, that given s_1^2, s_2^2, and the true value of θ, the distribution of $d(\theta)$ (or simply the distribution of d) is that of a linear combination of two independent Student's t random variables t_{f_1} and t_{f_2} with $f_1 = n_1 - 1$ and $f_2 = n_2 - 1$ degrees of freedom, i.e.

$$d \sim s_\varphi t_{f_1} - c_\varphi t_{f_2} \equiv s_\varphi t_{f_1} + c_\varphi t_{f_2}, \tag{2}$$

with

$$s_\varphi = \sqrt{\frac{(s_1^2/n_1)}{(s_1^2/n_1) + (s_2^2/n_2)}}, \quad c_\varphi = \sqrt{\frac{(s_2^2/n_2)}{(s_1^2/n_1) + (s_2^2/n_2)}}, \tag{3}$$

or equivalently, $s_\varphi = \sin(\varphi)$, $c_\varphi = \cos(\varphi)$, and $\varphi = \arctan\sqrt{(s_1^2/n_1)/(s_2^2/n_2)}$. The Fisher's test rejects the null hypothesis $H_0 : \theta = \theta_0$ (typically $\theta_0 = 0$) against the alternative $H_1 : \theta \neq \theta_0$ if

$$|d_{obs}(\theta_0)| > \gamma_{1-\frac{\alpha}{2}}, \tag{4}$$

or if the adequate Behrens-Fisher p-value $p(d_{obs}(\theta_0))$ is sufficiently small, that is if

$$\begin{aligned} p_0 = p(d_{obs}(\theta_0)) &= 2\left[1 - \Pr\{s_\varphi t_{f_1} + c_\varphi t_{f_2} \leq |d_{obs}(\theta_0)|\}\right] \\ &= 2\left[1 - \mathcal{F}^{(d)}_{[f_1,f_2,\varphi]}(|d_{obs}(\theta_0)|)\right] < \alpha, \end{aligned} \tag{5}$$

where α is the chosen nominal significance level of the test and $\mathcal{F}^{(d)}_{[f_1,f_2,\varphi]}$ is the cdf of the random variable d. By $\gamma_{1-\frac{\alpha}{2}}$ we denote the critical value (the upper cut-off point) of the distribution of d,

$$\gamma_{1-\frac{\alpha}{2}} = \mathcal{F}^{-1(d)}_{[f_1,f_2,\varphi]}\left(1-\frac{\alpha}{2}\right). \tag{6}$$

Because of symmetry of the distribution we have $\gamma_{\frac{\alpha}{2}} = -\gamma_{1-\frac{\alpha}{2}}$. Note that equivalently we also have

$$p_0 = 1 - \Pr\left\{(s_\varphi t_{f_1} + c_\varphi t_{f_2})^2 \leq d^2_{obs}(\theta_0)\right\} = 1 - \mathcal{F}^{(d^2)}_{[f_1,f_2,\varphi]}(d^2_{obs}(\theta_0)). \tag{7}$$

The $100 \times (1-\alpha)\%$ interval estimate of θ based on $d(\theta)$ is defined as a set $\Theta_{1-\alpha} = \{\theta_0 : p(d_{obs}(\theta_0)) \geq \alpha\}$ and is given as

$$(\bar{y}_1 - \bar{y}_2) \pm \gamma_{1-\frac{\alpha}{2}}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}. \tag{8}$$

The methods for the evaluation of the cdf $\mathcal{F}^{(d)}_{[f_1,f_2,\varphi]}$, needed for the evaluation of the p-value given by (5), have been discussed broadly in the literature. See e.g. the method suggested by Witkovský (2001a), which requires one-dimensional numerical integration, or the method suggested by Walker and Saw (1978) (the authors consider only the case when the degrees of freedom f_1 and f_2 are both odd). In the latter case, the cdf $\mathcal{F}^{(d)}_{[f_1,f_2,\varphi]}$ is expressed as a well defined finite linear combination of cdf's of Student's t-distributions. Other exact methods are discussed in subsequent sections.

The critical values $\gamma_{1-\frac{\alpha}{2}}$ have been tabulated for some combinations of α, f_1, f_2, and φ, see Fisher and Yates (1975). However, $\gamma_{1-\frac{\alpha}{2}}$ can be calculated by numerical optimization which typically requires a repeated usage of one of the above mentioned procedures.

3. Solution based on the generalized p-values

The concept of generalized p-values has been introduced by Tsui and Weerahandi (1989). Here is the brief outline of the method: Consider an observable random vector X such that its distribution depends on the vector parameter (θ, ϑ), where θ is the parameter of interest and ϑ is a vector of the other nuisance parameters. Further, consider the problem of testing one-sided hypothesis

(9) $$H_0 : \theta \le \theta_0, \quad \text{against} \quad H_1 : \theta > \theta_0,$$

where θ_0 is a prespecified value of θ. Let x be an observed value of X, then an observed significance level for hypothesis testing is defined on the basis of a data-based generalized extreme region, a subset of the sample space, with x on its boundary. In order to define such an extreme region, a stochastic ordering of the sample space according to the possible values of θ is required. This could be accomplished by means of generalized test variables.

A random variable $T(X, x, \theta_0, \vartheta)$ is said to be a generalized test variable if it has the following properties:

1. $T_{obs}(\theta_0) = T(x, x, \theta_0, \vartheta)$ does not depend on the unknown nuisance parameters.
2. If $\theta = \theta_0$ (θ is the true value of the parameter), the probability distribution of $T(X, x, \theta_0, \vartheta)$ is free of nuisance vector parameter ϑ.
3. For fixed x and ϑ, and for any given t, $\Pr\{T(X, x, \theta_0, \vartheta) \le t|\theta\}$ is a monotonic function of θ.

   If $\Pr\{T(X, x, \theta_0, \vartheta) \le t|\theta\}$ is a nondecreasing function of θ, then the test variable $T(X, x, \theta_0, \vartheta)$ is said to be stochastically increasing in θ. If $\Pr\{T(X, x, \theta_0, \vartheta) \le t|\theta\}$ is a nonincreasing function of θ, then the test variable $T(X, x, \theta_0, \vartheta)$ is said to be stochastically decreasing in θ.

If $T(X, x, \theta_0, \vartheta)$ is a stochastically increasing test variable, then the subset of the sample space $C_x(\theta_0) = \{x_* : T(x_*, x, \theta_0, \vartheta) \ge T_{obs}(\theta_0)\}$ is said to be a generalized extreme region for testing H_0 against H_1 and $p = \sup_{\theta \le \theta_0} \Pr\{X \in C_x(\theta_0)|\theta\}$ is said to be its generalized p-value for testing H_0. If $T(X, x, \theta_0, \vartheta)$ is stochastically increasing, then $p = \Pr\{T(X, x, \theta_0, \vartheta) \ge T_{obs}(\theta_0)|\theta = \theta_0\}$. Notice that if $T(X, x, \theta_0, \vartheta)$ is stochastically decreasing, then the p-value is $p = \Pr\{T(X, x, \theta_0, \vartheta) \le T_{obs}(\theta_0)|\theta = \theta_0\}$.

If the null hypothesis is right-sided, then the generalized p-value for testing H_0 is $p = \Pr\{T(X, x, \theta_0, \vartheta) \le T_{obs}(\theta_0)|\theta = \theta_0\}$, if $T(X, x, \theta_0, \vartheta)$ is stochastically increasing, and $p = \Pr\{T(X, x, \theta_0, \vartheta) \ge T_{obs}(\theta_0)|\theta = \theta_0\}$, if $T(X, x, \theta_0, \vartheta)$ is stochastically decreasing. Based on the above discussion the concept of generalized p-values can be easily generalized for testing the two-sided null hypothesis $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

The solution to the Berens-Fisher problem by the method of generalized p-values is based on the random vector $(\bar{Y}_1, \bar{Y}_2, S_1^2, S_2^2)$ which consists of a set of sufficient statistics for the parameters of the distribution. We notice that

$$\bar{Y}_1 \sim \mathcal{N}\left(\mu_1, \frac{\sigma_1^2}{n_1}\right), \quad \text{and} \quad \bar{Y}_2 \sim \mathcal{N}\left(\mu_2, \frac{\sigma_2^2}{n_2}\right), \tag{10}$$

$$\frac{f_1}{\sigma_1^2} S_1^2 \sim \chi^2_{f_1}, \quad \text{and} \quad \frac{f_2}{\sigma_2^2} S_2^2 \sim \chi^2_{f_2}, \tag{11}$$

are mutually independent random variables.

The hypothesis of interest is

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0. \tag{12}$$

In this testing problem the parameter of interest is θ, $\theta = \mu_1 - \mu_2$, and (σ_1^2, σ_2^2) is the vector of nuisance parameters.

For testing H_0 and interval estimation of θ we shall define a generalized test variable $D^2(\theta_0) = D^2((\bar{Y}_1, \bar{Y}_2, S_1^2, S_2^2), (\bar{y}_1, \bar{y}_2, s_1^2, s_2^2), \theta_0, (\sigma_1^2, \sigma_2^2))$

$$D^2(\theta_0) = \frac{(\bar{Y}_1 - \bar{Y}_2 - \theta_0)^2}{(\sigma_1^2/n_1 + \sigma_2^2/n_2)} \left(\frac{\sigma_1^2}{n_1}\frac{s_1^2}{S_1^2} + \frac{\sigma_2^2}{n_2}\frac{s_2^2}{S_2^2}\right). \tag{13}$$

Notice that $D^2_{obs}(\theta_0) = ((\bar{y}_1 - \bar{y}_2) - \theta_0)^2$ does not depend on the nuisance parameters.

If $\theta = \theta_0$, the distribution of $D^2(\theta_0)$ (or simply the distribution of D^2) is given as

$$D^2 \sim \chi_1^2 \left(\frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} + \frac{(s_2^2/n_2)}{\chi^2_{f_2}/f_2}\right), \tag{14}$$

where by χ^2_1, $\chi^2_{f_1}$ and $\chi^2_{f_2}$ we denote the independent random variables with chi-square distribution with 1, f_1 and f_2 degrees of freedom. Given $(\bar{y}_1, \bar{y}_2, s^2_1, s^2_2)$ and (σ^2_1, σ^2_2), $D^2(\theta_0)$ is stochastically increasing for $\theta > \theta_0$ and stochastically decreasing for $\theta < \theta_0$.

For given θ_0, the generalized p-value is defined as

$$p(D^2_{obs}(\theta_0)) = \Pr\left\{D^2 > D^2_{obs}(\theta_0)\right\}. \tag{15}$$

Hence, the significance test of the hypothesis H_0 is based on $p_0 = p(D^2_{obs}(\theta_0))$:

$$\begin{aligned} p_0 &= 1 - \Pr\left\{\chi^2_1\left(\frac{(s^2_1/n_1)}{\chi^2_{f_1}/f_1} + \frac{(s^2_2/n_2)}{\chi^2_{f_2}/f_2}\right) \leq ((\bar{y}_1 - \bar{y}_2) - \theta_0)^2\right\} \\ &= 1 - \Pr\left\{\chi^2_1\left(\frac{f_1 s^2_\varphi}{\chi^2_{f_1}} + \frac{f_2 c^2_\varphi}{\chi^2_{f_2}}\right) \leq d^2_{obs}(\theta_0)\right\} \qquad (16) \\ &= 1 - \mathcal{F}^{(F^1_{12})}_{[1,f_1,f_2,\varphi]}(d^2_{obs}(\theta_0)), \end{aligned}$$

where $\mathcal{F}^{(F^1_{12})}_{[1,f_1,f_2,\varphi]}$ is the cdf of the random variable

$$F^1_{12} \sim \chi^2_1\left(\frac{f_1 s^2_\varphi}{\chi^2_{f_1}} + \frac{f_2 c^2_\varphi}{\chi^2_{f_2}}\right). \tag{17}$$

We reject H_0 if the p-value p_0 is small (smaller than the nominal significance level α), or if

$$d^2_{obs}(\theta_0) > \mathcal{F}^{-1(F^1_{12})}_{[1,f_1,f_2,\varphi]}(1-\alpha). \tag{18}$$

Note that (16) is equivalent to

$$p_0 = 1 - \Pr\left\{-\frac{d^2_{obs}(\theta_0)}{\chi^2_1} + \frac{f_1 s^2_\varphi}{\chi^2_{f_1}} + \frac{f_2 c^2_\varphi}{\chi^2_{f_2}} \leq 0\right\}, \tag{19}$$

i.e., the p-value p_0 is a function of the cdf of a linear combination of independent inverted chi-square random variables. Exact values can be calculated by the method suggested by Witkovský (2001b), which requires one-dimensional numerical integration. A brief description of the method is given in Section 5.

We note that the p-value (16) is equal to the Behrens-Fisher p-value (5). In particular, $\mathcal{F}^{(d^2)}_{[f_1,f_2,\varphi]} = \mathcal{F}^{(F^1_{12})}_{[1,f_1,f_2,\varphi]}$. So, if we denote

$$\gamma^1_{12,1-\alpha} = \sqrt{\mathcal{F}^{-1}{}^{(F^1_{12})}_{[1,f_1,f_2,\varphi]}(1-\alpha)}, \tag{20}$$

then $\gamma^1_{12,1-\alpha} = \gamma_{1-\frac{\alpha}{2}}$, where $\gamma_{1-\frac{\alpha}{2}}$ is defined by (6), and the $100 \times (1-\alpha)\%$ interval estimate of θ based on the generalized p-values is given as

$$(\bar{y}_1 - \bar{y}_2) \pm \gamma^1_{12,1-\alpha}\sqrt{\frac{s^2_1}{n_1} + \frac{s^2_2}{n_2}}. \tag{21}$$

4. Generalization to the multiple comparisons

Consider now $k \geq 2$ independent random samples from normal populations, $Y_{i1}, \ldots, Y_{in_i}$, with sample sizes n_i and with possibly unequal means μ_i and variances σ^2_i, $i = 1, \ldots, k$. Let $\bar{Y}_i = (1/n_i)\sum_{j=1}^{n_i} Y_{ij}$ be the sample means and let $S^2_i = (1/f_i)\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_i)^2$ be the sample variances, $f_i = n_i - 1$. Then $\bar{Y}_i \sim \mathcal{N}(\mu_i, \sigma^2_i/n_i)$ and $f_i S^2_i/\sigma^2_i \sim \chi^2_{f_i}$ are mutually independent random variables. Further, let $\bar{y}_i$ be the observed value of $\bar{Y}_i$ and let s^2_i be the observed value of S^2_i, $i = 1, \ldots, k$. Hence, the observed value of the random vector $\bar{Y} = (\bar{Y}_1, \ldots, \bar{Y}_k)'$ is $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_k)'$, and the observed value of the random vector $S^2 = (S^2_1, \ldots, S^2_k)'$ is $s^2 = (s^2_1, \ldots, s^2_k)'$.

Let $\mu = (\mu_1, \ldots, \mu_k)'$, $\sigma^2 = (\sigma_1^2, \ldots, \sigma_k^2)'$, and let

$$R = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 1 & 0 & -1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \ldots & -1 \end{pmatrix}, \tag{22}$$

R being an $(\kappa \times k)$-matrix, $\kappa = k - 1$. We will denote $\theta = R\mu$, i.e. $\theta = (\theta_1, \ldots, \theta_\kappa)'$ with $\theta_i = \mu_1 - \mu_{i+1}$. Then, additionally assuming that $R\bar{y} - \theta_0 \neq 0$, the significance test of the null hypothesis on all contrasts $H_0 : \lambda'\theta = \lambda'\theta_0$ for all $\lambda \in \mathcal{R}^\kappa$ against the alternative $H_1 : \lambda'\theta \neq \lambda'\theta_0$ for some $\lambda \in \mathcal{R}^\kappa$, with $\theta_0 = (\theta_{01}, \ldots, \theta_{0\kappa})'$, (typically $\theta_0 = 0$), will be based on the generalized test variable $F^\kappa(\theta_0) = F^\kappa(\bar{Y}, S^2, \bar{y}, s^2, \theta_0, \sigma^2)$, where

$$F^\kappa(\theta_0) = \frac{(R\bar{Y} - \theta_0)'[RVR']^{-1}(R\bar{Y} - \theta_0)}{(R\bar{y} - \theta_0)'[RWR']^{-1}(R\bar{y} - \theta_0)}, \tag{23}$$

with

$$V = \operatorname{diag}\left(\frac{\sigma_1^2}{n_1}, \ldots, \frac{\sigma_k^2}{n_k}\right) \text{ and } W = \operatorname{diag}\left(\frac{\sigma_1^2}{n_1}\frac{s_1^2}{S_1^2}, \ldots, \frac{\sigma_k^2}{n_k}\frac{s_k^2}{S_k^2}\right). \tag{24}$$

Note that $F^\kappa_{obs}(\theta_0) = F^\kappa(\bar{y}, s^2, \bar{y}, s^2, \theta_0, \sigma^2) = 1$. Under H_0 we have $(R\bar{Y} - \theta_0) \sim \mathcal{N}(0, RVR)$, and hence, $(R\bar{Y} - \theta_0)'[RVR']^{-1}(R\bar{Y} - \theta_0) \sim \chi^2_\kappa$. On the other hand, the denominator of $F^\kappa(\theta_0)$ is stochastically independent of the nominator and $(R\bar{y} - \theta_0)'[RWR']^{-1}(R\bar{y} - \theta_0) \sim q(\bar{y}, s^2, \theta_0, \chi^2_{f_1}, \ldots, \chi^2_{f_k})$ with

(25)

$$RWR' \sim \begin{pmatrix} \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} + \frac{(s_2^2/n_2)}{\chi^2_{f_2}/f_2} & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} & \cdots & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} \\ \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} + \frac{(s_3^2/n_3)}{\chi^2_{f_3}/f_3} & \cdots & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} & \cdots & \frac{(s_1^2/n_1)}{\chi^2_{f_1}/f_1} + \frac{(s_k^2/n_k)}{\chi^2_{f_k}/f_k} \end{pmatrix}.$$

If $\theta = \theta_0$, the distribution of $F^\kappa(\theta_0)$ (or simply the distribution of F^κ) is given as

$$F^\kappa \sim \frac{\chi^2_\kappa}{q(\bar{y}, s^2, \theta_0, \chi^2_{f_1}, \dots, \chi^2_{f_k})}, \tag{26}$$

and does not depend on the vector of nuisance parameters σ^2. The test variable $F^\kappa(\theta_0)$ is stochastically increasing in

$$\eta^2 = (\theta - \theta_0)'[RVR']^{-1}(\theta - \theta_0) = (\mu - \mu_0)'R'[RVR']^{-1}R(\mu - \mu_0). \tag{27}$$

For given θ_0 the generalized p-value is defined as

$$p(F^\kappa_{obs}(\theta_0)) = \Pr\left\{F^\kappa > F^\kappa_{obs}(\theta_0)\right\}. \tag{28}$$

Hence, the significance test of the hypothesis H_0 is based on $p_0 = p(F^\kappa_{obs}(\theta_0))$:

$$p_0 = 1 - \Pr\left\{\frac{\chi^2_\kappa}{q(\bar{y}, s^2, \theta_0, \chi^2_{f_1}, \dots, \chi^2_{f_k})} \le 1\right\}, \tag{29}$$

and the $100 \times (1-\alpha)\%$ confidence set for θ based on the generalized p-values is defined as a set

$$\Theta^\kappa_{1-\alpha} = \left\{\theta_0 : p(F^\kappa_{obs}(\theta_0)) \ge \alpha\right\}. \tag{30}$$

From (29) we have

$$p_0 = 1 - \Pr\left\{\chi^2_\kappa \le q(\bar{y}, s^2, \theta_0, \chi^2_{f_1}, \dots, \chi^2_{f_k})\right\}$$

(31)
$$= 1 - \Pr\left\{\chi^2_\kappa \le (R\bar{y} - \theta_0)'[RWR']^{-1}(R\bar{y} - \theta_0)\right\}$$

$$\le 1 - \Pr\left\{\chi^2_\kappa \le \tfrac{(\lambda'(R\bar{y}-\theta_0))^2}{\lambda' RWR'\lambda}\right\} = p_0^\lambda,$$

for each fixed non-zero vector $\lambda = (\lambda_1, \dots, \lambda_\kappa)'$, $\lambda \in \mathcal{R}^\kappa$. In general,

(32)
$$p_0 \le p_0^{\lambda_*} = \inf_{\lambda \in \mathcal{R}^\kappa} p_0^\lambda,$$

note that the equality $p_0 = p_0^{\lambda_*}$ holds true if $\kappa = 1$. The p-value p_0 can be calculated numerically by a method which requires in general κ-dimensional numerical integration, see Weerahandi (1995b). However, for any $\lambda \in \mathcal{R}^\kappa$ we can calculate

(33)
$$p_0^\lambda = 1 - \Pr\left\{\chi^2_\kappa\left(\sum_{i=0}^{\kappa} \lambda_i^2 \frac{(s^2_{1+i}/n_{1+i})}{\chi^2_{f_{1+i}}/f_{1+i}}\right) \le \left(\sum_{i=1}^{\kappa} \lambda_i((\bar{y}_1 - \bar{y}_{1+i}) - \theta_{0i})\right)^2\right\},$$

where $\lambda_0^2 = \left(\sum_{i=1}^{\kappa} \lambda_i\right)^2$. Note that the p-value p_0^λ can be represented as a function of the cdf of a linear combination of independent inverted chi-square random variables and the exact values can be calculated by the method suggested by Witkovský (2001b).

Consider now the problem of pairwise multiple comparisons. In particular, let $\lambda^{(1j)} = (0, \dots, 1, \dots, 0)'$, with 1 on the position $(j-1)$, for $j = 2, \dots, k$, and let $\lambda^{(ij)} = (0, \dots, -1, \dots, 1, \dots, 0)'$, with -1 on the position $(i-1)$ and 1 on the position $(j-1)$, for $i = 2, \dots, k$ and $j = i+1, \dots, k$. Further, let $\theta^{(ij)} = \mu_i - \mu_j$, note that $\theta^{(ij)} = \theta_{j-1} - \theta_{i-1}$, seting $\theta_{i-1} = 0$ for $i = 1$. Then for $i = 1, \dots, k$ and $j = i+1, \dots, k$ we define $p_0^{(ij)} = p_0^{\lambda^{(ij)}}$, i.e.

$$p_0^{(ij)} = 1 - \Pr\left\{\chi^2_\kappa\left(\frac{(s_i^2/n_i)}{\chi^2_{f_i}/f_i} + \frac{(s_j^2/n_j)}{\chi^2_{f_j}/f_j}\right) \leq \left((\bar{y}_i - \bar{y}_j) - \theta_0^{(ij)}\right)^2\right\}$$

$$(34) \qquad = 1 - \Pr\left\{\chi^2_\kappa\left(\frac{f_i s^2_{\varphi_{ij}}}{\chi^2_{f_i}} + \frac{f_j c^2_{\varphi_{ij}}}{\chi^2_{f_j}}\right) \leq d^2_{ij,obs}\left(\theta_0^{(ij)}\right)\right\}$$

$$= 1 - \mathcal{F}^{(F^\kappa_{ij})}_{[\kappa, f_i, f_j, \varphi_{ij}]}\left(d^2_{ij,obs}\left(\theta_0^{(ij)}\right)\right),$$

where $\mathcal{F}^{(F^\kappa_{ij})}_{[\kappa, f_i, f_j, \varphi_{ij}]}$ is the cdf of the random variable

$$(35) \qquad F^\kappa_{ij} \sim \chi^2_\kappa\left(\frac{f_i s^2_{\varphi_{ij}}}{\chi^2_{f_i}} + \frac{f_j c^2_{\varphi_{ij}}}{\chi^2_{f_j}}\right),$$

and further,

$$(36) \qquad d_{ij,obs}\left(\theta_0^{(ij)}\right) = \frac{(\bar{y}_i - \bar{y}_j) - \theta_0^{(ij)}}{\sqrt{s_i^2/n_i + s_j^2/n_j}},$$

and

$$(37) \qquad s^2_{\varphi_{ij}} = \frac{(s_i^2/n_i)}{(s_i^2/n_i) + (s_j^2/n_j)}, \quad c^2_{\varphi_{ij}} = \frac{(s_j^2/n_j)}{(s_i^2/n_i) + (s_j^2/n_j)},$$

or equivalently, $s^2_{\varphi_{ij}} = \sin^2(\varphi_{ij})$, $c^2_{\varphi_{ij}} = \cos^2(\varphi_{ij})$, and $\varphi_{ij} = \arctan\sqrt{(s_i^2/n_i)/(s_j^2/n_j)}$. Note that the p-value $p_0^{(ij)}$ given by (34) is a generalization of the Behrens-Fisher p-value p_0 given by (5) and (16).

As a conservative approximation to the significance test of H_0 based on the generalized p-value p_0 given by (31) we suggest to reject the null hypothesis on pairwise mean differences $H_0^*: \theta^{(ij)} = \theta_0^{(ij)}$ for all $i = 1, \ldots, k$ and $j = i+1, \ldots, k$ against the alternative $H_1^*: \theta^{(ij)} \neq \theta_0^{(ij)}$ for some i and j, if

(38) $$p_0^* = \min_{i,j} p^{(ij)} < \alpha,$$

where α is the nominal significance level of the test, or if

(39) $$\left| d_{ij,obs}\left(\theta_0^{(ij)}\right) \right| > \gamma_{ij,1-\alpha}^{\kappa} = \sqrt{\mathcal{F}^{-1}{}^{(F_{ij}^{\kappa})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)},$$

for some $i = 1,\ldots,k$ and $j = i+1,\ldots,k$. Based on (32), note that $p_0 \leq p_0^{\lambda_*} \leq p_0^*$.

The conservative joint $100 \times (1-\alpha)\%$ confidence interval estimates for the pairwise mean differences $\theta^{(ij)} = \mu_i - \mu_j$, refered to as the generalized Scheffé intervals, are given as

(40) $$(\bar{y}_i - \bar{y}_j) \pm \gamma_{ij,1-\alpha}^{\kappa}\sqrt{\frac{s_i^2}{n_i} + \frac{s_j^2}{n_j}},$$

$i = 1,\ldots,k$, and $j = i+1,\ldots,k$.

The critical values

(41) $$\gamma_{ij,1-\alpha}^{\kappa} = \sqrt{\mathcal{F}^{-1}{}^{(F_{ij}^{\kappa})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)},$$

can be calculated numerically by solving the equation

(42) $$1-\alpha = \Pr\left\{ -\frac{\left(\gamma_{ij,1-\alpha}^{\kappa}\right)^2}{\chi_\kappa^2} + \frac{f_i s_{\varphi_{ij}}^2}{\chi_{f_i}^2} + \frac{f_j c_{\varphi_{ij}}^2}{\chi_{f_j}^2} \leq 0 \right\}.$$

The Tables A – E present the critical values $\gamma_{ij,1-\alpha}^{\kappa}$ for $\alpha = 0.05$ and $\kappa = 1,\ldots,5$, $f_i = 1,\ldots,10$, $f_j = f_i,\ldots,10$, $\varphi_{ij} = [0^\circ : 10^\circ : 90^\circ]$. Note that for comparing the random samples with $f_j < f_i$ we can use the critical values from the given Tables A – E after changing the ordering of the two samples.

Table A. Critical values $\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{\mathcal{F}^{-1}{}^{(F^{\kappa}_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)}$, for $\alpha = 0.05$ and $\kappa = 1$.

κ	f_i	f_j	φ_{ij}: 0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
1	1	1	12.706	14.720	16.286	17.357	17.901	17.969	17.901	17.357	16.286	14.720	12.706
1	1	2	4.3027	5.6297	6.9936	8.3262	9.5629	10.127	10.647	11.333	12.184	12.578	12.706
1	1	3	3.1824	4.3137	5.6518	7.1227	8.6010	9.3032	9.9628	11.112	11.981	12.522	12.706
1	1	4	2.7765	3.8227	5.1832	6.7704	8.3792	9.1362	9.8414	11.056	11.961	12.518	12.706
1	1	5	2.5706	3.5711	4.9639	6.6365	8.3118	9.0902	9.8107	11.043	11.956	12.517	12.706
1	1	6	2.4469	3.4195	4.8444	6.5766	8.2859	9.0732	9.7995	11.038	11.955	12.517	12.706
1	1	7	2.3646	3.3187	4.7725	6.5460	8.2736	9.0651	9.7941	11.036	11.954	12.517	12.706
1	1	8	2.3060	3.2470	4.7261	6.5287	8.2666	9.0604	9.7908	11.034	11.953	12.516	12.706
1	1	9	2.2622	3.1936	4.6945	6.5178	8.2622	9.0572	9.7885	11.033	11.953	12.516	12.706
1	1	10	2.2282	3.1522	4.6719	6.5106	8.2590	9.0550	9.7869	11.032	11.952	12.516	12.706
1	2	2	4.3027	4.3624	4.4680	4.5629	4.6169	4.6240	4.6169	4.5629	4.4680	4.3624	4.3027
1	2	3	3.1824	3.2767	3.4536	3.6452	3.8229	3.9025	3.9752	4.1002	4.1998	4.2722	4.3027
1	2	4	2.7765	2.8837	3.0848	3.3124	3.5414	3.6528	3.7610	3.9636	4.1365	4.2579	4.3027
1	2	5	2.5706	2.6842	2.8974	3.1447	3.4041	3.5345	3.6636	3.9086	4.1149	4.2537	4.3027
1	2	6	2.4469	2.5643	2.7847	3.0450	3.3251	3.4684	3.6111	3.8817	4.1054	4.2519	4.3027
1	2	7	2.3646	2.4844	2.7097	2.9794	3.2747	3.4273	3.5795	3.8666	4.1002	4.2509	4.3027
1	2	8	2.3060	2.4275	2.6563	2.9332	3.2402	3.3997	3.5588	3.8571	4.0969	4.2502	4.3027
1	2	9	2.2622	2.3849	2.6164	2.8990	3.2153	3.3802	3.5445	3.8507	4.0947	4.2498	4.3027
1	2	10	2.2282	2.3518	2.5855	2.8727	3.1965	3.3657	3.5340	3.8462	4.0931	4.2494	4.3027
1	3	3	3.1824	3.1848	3.2015	3.2254	3.2417	3.2440	3.2417	3.2254	3.2015	3.1848	3.1824
1	3	4	2.7765	2.7934	2.8450	2.9134	2.9811	3.0116	3.0394	3.0878	3.1300	3.1659	3.1824
1	3	5	2.5706	2.5957	2.6652	2.7564	2.8512	2.8972	2.9416	3.0257	3.1016	3.1597	3.1824
1	3	6	2.4469	2.4771	2.5575	2.6625	2.7746	2.8305	2.8857	2.9922	3.0875	3.1568	3.1824
1	3	7	2.3646	2.3983	2.4859	2.6004	2.7245	2.7874	2.8501	2.9719	3.0794	3.1551	3.1824
1	3	8	2.3060	2.3423	2.4350	2.5564	2.6894	2.7575	2.8258	2.9585	3.0743	3.1541	3.1824
1	3	9	2.2622	2.3003	2.3970	2.5236	2.6635	2.7357	2.8082	2.9491	3.0707	3.1533	3.1824
1	3	10	2.2282	2.2678	2.3675	2.4983	2.6437	2.7191	2.7950	2.9421	3.0681	3.1528	3.1824

κ	f_i	f_j	φ_{ij}										
			0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
1	4	4	2.7765	2.7728	2.7726	2.7793	2.7856	2.7866	2.7856	2.7793	2.7726	2.7728	2.7765
1	4	5	2.5706	2.5743	2.5931	2.6247	2.6591	2.6752	2.6900	2.7167	2.7421	2.7655	2.7765
1	4	6	2.4469	2.4553	2.4858	2.5324	2.5842	2.6097	2.6345	2.6819	2.7263	2.7620	2.7765
1	4	7	2.3646	2.3763	2.4146	2.4714	2.5350	2.5670	2.5987	2.6603	2.7170	2.7600	2.7765
1	4	8	2.3060	2.3201	2.3641	2.4281	2.5003	2.5372	2.5739	2.6457	2.7109	2.7587	2.7765
1	4	9	2.2622	2.2781	2.3263	2.3959	2.4747	2.5152	2.5558	2.6353	2.7066	2.7578	2.7765
1	4	10	2.2282	2.2455	2.2970	2.3709	2.4550	2.4984	2.5421	2.6276	2.7035	2.7571	2.7765
1	5	5	2.5706	2.5664	2.5618	2.5624	2.5647	2.5652	2.5647	2.5624	2.5618	2.5664	2.5706
1	5	6	2.4469	2.4471	2.4542	2.4705	2.4906	2.5004	2.5097	2.5273	2.5453	2.5626	2.5706
1	5	7	2.3646	2.3679	2.3829	2.4097	2.4418	2.4580	2.4740	2.5052	2.5353	2.5603	2.5706
1	5	8	2.3060	2.3115	2.3322	2.3666	2.4073	2.4283	2.4492	2.4902	2.5288	2.5589	2.5706
1	5	9	2.2622	2.2694	2.2944	2.3345	2.3818	2.4064	2.4310	2.4794	2.5242	2.5578	2.5706
1	5	10	2.2282	2.2367	2.2651	2.3096	2.3622	2.3896	2.4171	2.4713	2.5208	2.5571	2.5706
1	6	6	2.4469	2.4431	2.4373	2.4353	2.4358	2.4359	2.4358	2.4353	2.4373	2.4431	2.4469
1	6	7	2.3646	2.3637	2.3658	2.3745	2.3871	2.3937	2.4002	2.4130	2.4270	2.4407	2.4469
1	6	8	2.3060	2.3072	2.3149	2.3314	2.3528	2.3641	2.3753	2.3978	2.4202	2.4391	2.4469
1	6	9	2.2622	2.2650	2.2770	2.2993	2.3274	2.3422	2.3571	2.3868	2.4154	2.4380	2.4469
1	6	10	2.2282	2.2322	2.2477	2.2745	2.3078	2.3254	2.3432	2.3786	2.4118	2.4372	2.4469
1	7	7	2.3646	2.3612	2.3553	2.3522	2.3516	2.3516	2.3516	2.3522	2.3553	2.3612	2.3646
1	7	8	2.3060	2.3046	2.3043	2.3090	2.3173	2.3220	2.3268	2.3368	2.3483	2.3596	2.3646
1	7	9	2.2622	2.2623	2.2663	2.2769	2.2919	2.3001	2.3085	2.3257	2.3433	2.3584	2.3646
1	7	10	2.2282	2.2296	2.2369	2.2520	2.2723	2.2833	2.2946	2.3174	2.3396	2.3575	2.3646
1	8	8	2.3060	2.3029	2.2972	2.2936	2.2925	2.2924	2.2925	2.2936	2.2972	2.3029	2.3060
1	8	9	2.2622	2.2606	2.2591	2.2614	2.2671	2.2705	2.2742	2.2824	2.2922	2.3017	2.3060
1	8	10	2.2282	2.2278	2.2296	2.2365	2.2475	2.2537	2.2602	2.2740	2.2884	2.3009	2.3060
1	9	9	2.2622	2.2594	2.2540	2.2502	2.2488	2.2486	2.2488	2.2502	2.2540	2.2594	2.2622
1	9	10	2.2282	2.2266	2.2245	2.2253	2.2292	2.2318	2.2348	2.2417	2.2502	2.2585	2.2622
1	10	10	2.2282	2.2257	2.2206	2.2168	2.2152	2.2150	2.2152	2.2168	2.2206	2.2257	2.2282

Table B. Critical values $\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{\mathcal{F}^{-1}{}^{(F^{\kappa}_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)}$, for $\alpha = 0.05$ and $\kappa = 2$.

			φ_{ij}										
κ	f_i	f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
2	1	1	19.975	23.140	25.602	27.286	28.141	28.249	28.141	27.286	25.602	23.140	19.975
2	1	2	6.1644	8.2322	10.418	12.598	14.649	15.593	16.466	17.963	19.072	19.750	19.975
2	1	3	4.3708	6.1259	8.3331	10.810	13.282	14.444	15.529	17.404	18.810	19.680	19.975
2	1	4	3.7267	5.3500	7.6604	10.366	13.029	14.261	15.399	17.347	18.790	19.676	19.975
2	1	5	3.4018	4.9579	7.3802	10.229	12.968	14.220	15.373	17.336	18.786	19.675	19.975
2	1	6	3.2073	4.7253	7.2469	10.179	12.947	14.206	15.363	17.332	18.784	19.675	19.975
2	1	7	3.0781	4.5732	7.1773	10.156	12.938	14.200	15.359	17.329	18.783	19.675	19.975
2	1	8	2.9863	4.4670	7.1378	10.144	12.932	14.196	15.356	17.328	18.783	19.675	19.975
2	1	9	2.9177	4.3892	7.1136	10.136	12.929	14.193	15.354	17.327	18.782	19.674	19.975
2	1	10	2.8646	4.3301	7.0979	10.131	12.926	14.191	15.352	17.326	18.782	19.674	19.975
2	2	2	6.1644	6.2310	6.3540	6.4676	6.5332	6.5419	6.5332	6.4676	6.3540	6.2310	6.1644
2	2	3	4.3708	4.4911	4.7326	5.0103	5.2854	5.4161	5.5409	5.7697	5.9643	6.1070	6.1644
2	2	4	3.7267	3.8677	4.1509	4.4915	4.8569	5.0431	5.2285	5.5833	5.8851	6.0909	6.1644
2	2	5	3.4018	3.5533	3.8575	4.2342	4.6555	4.8758	5.0968	5.5167	5.8620	6.0868	6.1644
2	2	6	3.2073	3.3650	3.6821	4.0836	4.5444	4.7877	5.0313	5.4875	5.8526	6.0850	6.1644
2	2	7	3.0781	3.2399	3.5659	3.9860	4.4765	4.7361	4.9948	5.4724	5.8477	6.0840	6.1644
2	2	8	2.9863	3.1509	3.4835	3.9183	4.4319	4.7035	4.9724	5.4635	5.8447	6.0834	6.1644
2	2	9	2.9177	3.0845	3.4221	3.8690	4.4009	4.6815	4.9577	5.4576	5.8427	6.0829	6.1644
2	2	10	2.8646	3.0329	3.3748	3.8316	4.3785	4.6659	4.9474	5.4535	5.8412	6.0826	6.1644
2	3	3	4.3708	4.3606	4.3561	4.3647	4.3741	4.3755	4.3741	4.3647	4.3561	4.3606	4.3708
2	3	4	3.7267	3.7383	3.7907	3.8746	3.9705	4.0190	4.0673	4.1636	4.2580	4.3373	4.3708
2	3	5	3.4018	3.4260	3.5085	3.6311	3.7729	3.8475	3.9237	4.0778	4.2226	4.3305	4.3708
2	3	6	3.2073	3.2397	3.3406	3.4870	3.6582	3.7498	3.8440	4.0343	4.2065	4.3275	4.3708
2	3	7	3.0781	3.1163	3.2296	3.3922	3.5843	3.6879	3.7949	4.0092	4.1977	4.3259	4.3708
2	3	8	2.9863	3.0287	3.1510	3.3255	3.5332	3.6459	3.7623	3.9934	4.1923	4.3248	4.3708
2	3	9	2.9177	2.9633	3.0923	3.2761	3.4960	3.6158	3.7394	3.9826	4.1886	4.3240	4.3708
2	3	10	2.8646	2.9127	3.0470	3.2381	3.4678	3.5933	3.7225	3.9750	4.1860	4.3235	4.3708

			φ_{ij}										
κ	f_i	f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
2	4	4	3.7267	3.7118	3.6883	3.6755	3.6720	3.6718	3.6720	3.6755	3.6883	3.7118	3.7267
2	4	5	3.4018	3.3979	3.4053	3.4346	3.4783	3.5031	3.5292	3.5860	3.6479	3.7033	3.7267
2	4	6	3.2073	3.2106	3.2375	3.2924	3.3651	3.4055	3.4480	3.5380	3.6283	3.6994	3.7267
2	4	7	3.0781	3.0866	3.1267	3.1989	3.2915	3.3428	3.3967	3.5092	3.6172	3.6973	3.7267
2	4	8	2.9863	2.9987	3.0484	3.1330	3.2402	3.2995	3.3617	3.4904	3.6102	3.6959	3.7267
2	4	9	2.9177	2.9331	2.9901	3.0841	3.2025	3.2680	3.3366	3.4773	3.6055	3.6949	3.7267
2	4	10	2.8646	2.8823	2.9450	3.0465	3.1737	3.2442	3.3178	3.4678	3.6020	3.6942	3.7267
2	5	5	3.4018	3.3883	3.3627	3.3444	3.3368	3.3359	3.3368	3.3444	3.3627	3.3883	3.4018
2	5	6	3.2073	3.2004	3.1938	3.2021	3.2242	3.2389	3.2555	3.2949	3.3413	3.3839	3.4018
2	5	7	3.0781	3.0759	3.0825	3.1087	3.1508	3.1761	3.2036	3.2646	3.3289	3.3814	3.4018
2	5	8	2.9863	2.9876	3.0039	3.0429	3.0995	3.1325	3.1680	3.2445	3.3210	3.3798	3.4018
2	5	9	2.9177	2.9218	2.9453	2.9941	3.0618	3.1007	3.1422	3.2303	3.3155	3.3786	3.4018
2	5	10	2.8646	2.8708	2.9001	2.9565	3.0329	3.0764	3.1228	3.2199	3.3116	3.3778	3.4018
2	6	6	3.2073	3.1956	3.1714	3.1520	3.1430	3.1419	3.1430	3.1520	3.1714	3.1956	3.2073
2	6	7	3.0781	3.0709	3.0595	3.0583	3.0697	3.0792	3.0909	3.1210	3.1583	3.1928	3.2073
2	6	8	2.9863	2.9823	2.9804	2.9923	3.0184	3.0355	3.0550	3.1002	3.1498	3.1910	3.2073
2	6	9	2.9177	2.9164	2.9216	2.9434	2.9805	3.0035	3.0289	3.0854	3.1438	3.1898	3.2073
2	6	10	2.8646	2.8653	2.8762	2.9057	2.9515	2.9791	3.0091	3.0745	3.1395	3.1889	3.2073
2	7	7	3.0781	3.0679	3.0459	3.0269	3.0175	3.0164	3.0175	3.0269	3.0459	3.0679	3.0781
2	7	8	2.9863	2.9793	2.9664	2.9607	2.9661	2.9726	2.9814	3.0057	3.0370	3.0660	3.0781
2	7	9	2.9177	2.9132	2.9074	2.9116	2.9282	2.9405	2.9551	2.9906	3.0308	3.0647	3.0781
2	7	10	2.8646	2.8620	2.8618	2.8738	2.8992	2.9160	2.9352	2.9793	3.0262	3.0637	3.0781
2	8	8	2.9863	2.9773	2.9573	2.9392	2.9299	2.9288	2.9299	2.9392	2.9573	2.9773	2.9863
2	8	9	2.9177	2.9111	2.8981	2.8900	2.8919	2.8966	2.9035	2.9239	2.9509	2.9759	2.9863
2	8	10	2.8646	2.8599	2.8523	2.8520	2.8628	2.8720	2.8834	2.9124	2.9461	2.9749	2.9863
2	9	9	2.9177	2.9097	2.8915	2.8745	2.8654	2.8643	2.8654	2.8745	2.8915	2.9097	2.9177
2	9	10	2.8646	2.8584	2.8456	2.8364	2.8362	2.8396	2.8453	2.8629	2.8866	2.9086	2.9177
2	10	10	2.8646	2.8573	2.8407	2.8247	2.8160	2.8149	2.8160	2.8247	2.8407	2.8573	2.8646

Table C. Critical values $\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{\mathcal{F}^{-1}{}^{(F^{\kappa}_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)}$, for $\alpha = 0.05$ and $\kappa = 3$.

	φ_{ij}										
κ f_i f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
3 1 1	25.439	29.470	32.605	34.750	35.839	35.976	35.839	34.750	32.605	29.470	25.439
3 1 2	7.5824	10.209	13.015	15.831	18.493	19.720	20.857	22.807	24.256	25.143	25.439
3 1 3	5.2754	7.5037	10.375	13.611	16.823	18.327	19.728	22.142	23.946	25.061	25.439
3 1 4	4.4468	6.5118	9.5594	13.103	16.544	18.127	19.587	22.080	23.925	25.057	25.439
3 1 5	4.0285	6.0148	9.2423	12.961	16.482	18.086	19.560	22.069	23.921	25.056	25.439
3 1 6	3.7777	5.7235	9.1037	12.912	16.463	18.073	19.551	22.065	23.919	25.056	25.439
3 1 7	3.6112	5.5360	9.0369	12.891	16.453	18.066	19.547	22.062	23.918	25.056	25.439
3 1 8	3.4926	5.4073	9.0014	12.880	16.448	18.062	19.544	22.061	23.918	25.055	25.439
3 1 9	3.4041	5.3149	8.9807	12.873	16.444	18.060	19.542	22.060	23.917	25.055	25.439
3 1 10	3.3354	5.2461	8.9675	12.869	16.442	18.057	19.540	22.059	23.917	25.055	25.439
3 2 2	7.5824	7.6553	7.7925	7.9207	7.9953	8.0051	7.9953	7.9207	7.7925	7.6553	7.5824
3 2 3	5.2754	5.4157	5.7065	6.0504	6.4008	6.5714	6.7366	7.0453	7.3114	7.5056	7.5824
3 2 4	4.4468	4.6134	4.9593	5.3878	5.8606	6.1057	6.3515	6.8229	7.2207	7.4879	7.5824
3 2 5	4.0284	4.2084	4.5826	5.0613	5.6120	5.9034	6.1962	6.7487	7.1961	7.4837	7.5824
3 2 6	3.7777	3.9657	4.3575	4.8718	5.4785	5.8011	6.1229	6.7181	7.1866	7.4819	7.5824
3 2 7	3.6112	3.8044	4.2086	4.7503	5.3995	5.7437	6.0839	6.7028	7.1817	7.4809	7.5824
3 2 8	3.4927	3.6896	4.1032	4.6670	5.3492	5.7087	6.0610	6.6940	7.1787	7.4803	7.5824
3 2 9	3.4041	3.6038	4.0248	4.6071	5.3153	5.6859	6.0463	6.6883	7.1766	7.4798	7.5824
3 2 10	3.3354	3.5372	3.9643	4.5622	5.2915	5.6702	6.0362	6.6842	7.1751	7.4794	7.5824
3 3 3	5.2754	5.2560	5.2351	5.2312	5.2345	5.2352	5.2345	5.2312	5.2351	5.2560	5.2754
3 3 4	4.4468	4.4542	4.5064	4.6015	4.7191	4.7820	4.8469	4.9817	5.1175	5.2296	5.2754
3 3 5	4.0284	4.0517	4.1432	4.2896	4.4685	4.5662	4.6681	4.8789	5.0773	5.2223	5.2754
3 3 6	3.7777	3.8113	3.9272	4.1054	4.3240	4.4447	4.5709	4.8286	5.0599	5.2192	5.2754
3 3 7	3.6112	3.6521	3.7845	3.9847	4.2317	4.3689	4.5123	4.8006	5.0507	5.2175	5.2754
3 3 8	3.4927	3.5391	3.6834	3.8998	4.1685	4.3182	4.4742	4.7835	5.0451	5.2164	5.2754
3 3 9	3.4041	3.4547	3.6082	3.8371	4.1229	4.2823	4.4479	4.7721	5.0414	5.2156	5.2754
3 3 10	3.3354	3.3894	3.5500	3.7890	4.0887	4.2560	4.4289	4.7641	5.0387	5.2150	5.2754

κ	f_i	f_j	φ_{ij} 0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
3	4	4	4.4468	4.4234	4.3813	4.3518	4.3397	4.3384	4.3397	4.3518	4.3813	4.4234	4.4468
3	4	5	4.0284	4.0185	4.0160	4.0423	4.0928	4.1245	4.1599	4.2418	4.3338	4.4140	4.4468
3	4	6	3.7777	3.7768	3.7996	3.8600	3.9491	4.0016	4.0588	4.1842	4.3116	4.4099	4.4468
3	4	7	3.6112	3.6167	3.6570	3.7405	3.8561	3.9232	3.9955	4.1504	4.2994	4.4077	4.4468
3	4	8	3.4927	3.5032	3.5562	3.6563	3.7915	3.8695	3.9529	4.1288	4.2919	4.4062	4.4468
3	4	9	3.4041	3.4184	3.4812	3.5940	3.7443	3.8307	3.9227	4.1140	4.2869	4.4052	4.4468
3	4	10	3.3354	3.3528	3.4233	3.5461	3.7085	3.8015	3.9004	4.1034	4.2832	4.4045	4.4468
3	5	5	4.0284	4.0077	3.9650	3.9303	3.9137	3.9118	3.9137	3.9303	3.9650	4.0077	4.0284
3	5	6	3.7777	3.7652	3.7469	3.7474	3.7704	3.7890	3.8118	3.8699	3.9402	4.0030	4.0284
3	5	7	3.6112	3.6045	3.6033	3.6277	3.6774	3.7100	3.7472	3.8335	3.9262	4.0003	4.0284
3	5	8	3.4927	3.4904	3.5019	3.5435	3.6126	3.6555	3.7032	3.8097	3.9174	3.9986	4.0284
3	5	9	3.4041	3.4053	3.4264	3.4811	3.5650	3.6158	3.6715	3.7932	3.9114	3.9974	4.0284
3	5	10	3.3354	3.3394	3.3682	3.4331	3.5287	3.5857	3.6479	3.7811	3.9071	3.9965	4.0284
3	6	6	3.7777	3.7599	3.7205	3.6858	3.6682	3.6661	3.6682	3.6858	3.7205	3.7599	3.7777
3	6	7	3.6112	3.5988	3.5759	3.5654	3.5750	3.5868	3.6031	3.6481	3.7053	3.7569	3.7777
3	6	8	3.4927	3.4844	3.4738	3.4808	3.5100	3.5319	3.5584	3.6231	3.6957	3.7550	3.7777
3	6	9	3.4041	3.3991	3.3979	3.4182	3.4622	3.4919	3.5261	3.6056	3.6891	3.7536	3.7777
3	6	10	3.3354	3.3331	3.3394	3.3700	3.4256	3.4614	3.5018	3.5927	3.6843	3.7527	3.7777
3	7	7	3.6112	3.5956	3.5600	3.5270	3.5095	3.5074	3.5095	3.5270	3.5600	3.5956	3.6112
3	7	8	3.4927	3.4811	3.4573	3.4419	3.4442	3.4522	3.4645	3.5013	3.5498	3.5935	3.6112
3	7	9	3.4041	3.3956	3.3811	3.3789	3.3962	3.4119	3.4318	3.4832	3.5427	3.5921	3.6112
3	7	10	3.3354	3.3294	3.3223	3.3306	3.3596	3.3812	3.4072	3.4698	3.5376	3.5910	3.6112
3	8	8	3.4927	3.4789	3.4467	3.4158	3.3990	3.3969	3.3990	3.4158	3.4467	3.4789	3.4927
3	8	9	3.4041	3.3933	3.3701	3.3526	3.3508	3.3564	3.3661	3.3973	3.4393	3.4773	3.4927
3	8	10	3.3354	3.3271	3.3111	3.3039	3.3139	3.3255	3.3412	3.3835	3.4340	3.4762	3.4927
3	9	9	3.4041	3.3917	3.3625	3.3338	3.3177	3.3157	3.3177	3.3338	3.3625	3.3917	3.4041
3	9	10	3.3354	3.3254	3.3033	3.2849	3.2807	3.2847	3.2926	3.3198	3.3570	3.3906	3.4041
3	10	10	3.3354	3.3243	3.2976	3.2708	3.2555	3.2536	3.2555	3.2708	3.2976	3.3243	3.3354

Table D. Critical values $\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{\mathcal{F}^{-1\,(F^{\kappa}_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)}$, for $\alpha = 0.05$ and $\kappa = 4$.

								φ_{ij}					
κ	f_i	f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
4	1	1	29.972	34.722	38.416	40.943	42.226	42.387	42.226	40.943	38.416	34.722	29.972
4	1	2	8.7742	11.864	15.185	18.528	21.694	23.155	24.509	26.832	28.560	29.619	29.972
4	1	3	6.0389	8.6613	12.086	15.948	19.771	21.356	23.217	26.075	28.209	29.526	29.972
4	1	4	5.0550	7.4895	11.152	15.384	19.466	21.339	23.065	26.009	28.186	29.522	29.972
4	1	5	4.5573	6.9055	10.805	15.236	19.403	21.297	23.037	25.997	28.182	29.521	29.972
4	1	6	4.2585	6.5665	10.661	15.187	19.383	21.283	23.028	25.993	28.180	29.521	29.972
4	1	7	4.0597	6.3511	10.595	15.167	19.373	21.277	23.023	25.990	28.179	29.520	29.972
4	1	8	3.9181	6.2057	10.561	15.156	19.368	21.272	23.020	25.989	28.179	29.520	29.972
4	1	9	3.8121	6.1031	10.542	15.149	19.364	21.270	23.018	25.988	28.178	29.520	29.972
4	1	10	3.7299	6.0282	10.530	15.144	19.361	21.267	23.016	25.987	28.178	29.520	29.972
4	2	2	8.7742	8.8530	9.0028	9.1439	9.2263	9.2372	9.2263	9.1439	9.0028	8.8530	8.7742
4	2	3	6.0389	6.1963	6.5288	6.9283	7.3422	7.5461	7.7451	8.1205	8.4456	8.6817	8.7742
4	2	4	5.0550	5.2431	5.6418	6.1448	6.7085	7.0033	7.2998	7.8682	8.3449	8.6625	8.7742
4	2	5	4.5573	4.7612	5.1943	5.7599	6.4208	6.7724	7.1253	7.7875	8.3188	8.6582	8.7742
4	2	6	4.2585	4.4718	4.9268	5.5377	6.2695	6.6589	7.0458	7.7555	8.3091	8.6563	8.7742
4	2	7	4.0597	4.2793	4.7499	5.3965	6.1819	6.5971	7.0049	7.7399	8.3041	8.6553	8.7742
4	2	8	3.9181	4.1420	4.6246	5.3006	6.1276	6.5605	6.9814	7.7310	8.3010	8.6546	8.7742
4	2	9	3.8121	4.0393	4.5316	5.2323	6.0918	6.5372	6.9667	7.7252	8.2988	8.6541	8.7742
4	2	10	3.7299	3.9596	4.4599	5.1817	6.0671	6.5214	6.9567	7.7211	8.2973	8.6537	8.7742
4	3	3	6.0389	6.0122	5.9779	5.9634	5.9615	5.9616	5.9615	5.9634	5.9779	6.0122	6.0389
4	3	4	5.0550	5.0590	5.1110	5.2154	5.3514	5.4266	5.5056	5.6735	5.8440	5.9831	6.0389
4	3	5	4.5573	4.5796	4.6785	4.8450	5.0557	5.1732	5.2974	5.5565	5.7999	5.9753	6.0389
4	3	6	4.2585	4.2930	4.4212	4.6265	4.8859	5.0318	5.1857	5.5008	5.7813	5.9721	6.0389
4	3	7	4.0597	4.1029	4.2511	4.4833	4.7781	4.9445	5.1194	5.4706	5.7718	5.9703	6.0389
4	3	8	3.9181	3.9678	4.1306	4.3828	4.7047	4.8867	5.0769	5.4525	5.7660	5.9692	6.0389
4	3	9	3.8121	3.8669	4.0407	4.3087	4.6522	4.8463	5.0481	5.4406	5.7622	5.9684	6.0389
4	3	10	3.7299	3.7887	3.9713	4.2519	4.6132	4.8169	5.0276	5.4324	5.7594	5.9678	6.0389

			φ_{ij}										
κ	f_i	f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
4	4	4	5.0550	5.0246	4.9668	4.9230	4.9032	4.9009	4.9032	4.9230	4.9668	5.0246	5.0550
4	4	5	4.5573	4.5422	4.5312	4.5544	4.6106	4.6483	4.6919	4.7956	4.9135	5.0145	5.0550
4	4	6	4.2585	4.2539	4.2729	4.3376	4.4407	4.5038	4.5737	4.7301	4.8892	5.0102	5.0550
4	4	7	4.0597	4.0627	4.1027	4.1954	4.3310	4.4119	4.5004	4.6922	4.8762	5.0079	5.0550
4	4	8	3.9181	3.9268	3.9823	4.0954	4.2550	4.3493	4.4515	4.6684	4.8683	5.0064	5.0550
4	4	9	3.8121	3.8254	3.8927	4.0214	4.1997	4.3043	4.4171	4.6524	4.8630	5.0053	5.0550
4	4	10	3.7299	3.7469	3.8235	3.9645	4.1578	4.2708	4.3919	4.6410	4.8592	5.0045	5.0550
4	5	5	4.5573	4.5306	4.4732	4.4239	4.3994	4.3964	4.3994	4.4239	4.4732	4.5306	4.5573
4	5	6	4.2585	4.2411	4.2126	4.2059	4.2294	4.2514	4.2796	4.3543	4.4456	4.5255	4.5573
4	5	7	4.0597	4.0491	4.0409	4.0633	4.1194	4.1584	4.2041	4.3129	4.4303	4.5227	4.5573
4	5	8	3.9181	3.9127	3.9196	3.9630	4.0428	4.0944	4.1530	4.2861	4.4208	4.5209	4.5573
4	5	9	3.8121	3.8108	3.8294	3.8888	3.9867	4.0480	4.1164	4.2677	4.4145	4.5196	4.5573
4	5	10	3.7299	3.7319	3.7598	3.8317	3.9440	4.0130	4.0892	4.2544	4.4099	4.5187	4.5573
4	6	6	4.2585	4.2354	4.1828	4.1344	4.1091	4.1059	4.1091	4.1344	4.1828	4.2354	4.2585
4	6	7	4.0597	4.0430	4.0099	3.9908	3.9986	4.0124	4.0326	4.0911	4.1660	4.2323	4.2585
4	6	8	3.9181	3.9062	3.8877	3.8899	3.9215	3.9478	3.9804	4.0627	4.1555	4.2302	4.2585
4	6	9	3.8121	3.8040	3.7969	3.8153	3.8650	3.9007	3.9428	4.0429	4.1484	4.2288	4.2585
4	6	10	3.7299	3.7249	3.7268	3.7579	3.8219	3.8650	3.9147	4.0285	4.1432	4.2278	4.2585
4	7	7	4.0597	4.0395	3.9921	3.9464	3.9216	3.9185	3.9216	3.9464	3.9921	4.0395	4.0597
4	7	8	3.9181	3.9025	3.8691	3.8448	3.8442	3.8534	3.8688	3.9170	3.9808	4.0373	4.0597
4	7	9	3.8121	3.8002	3.7778	3.7697	3.7873	3.8059	3.8306	3.8963	3.9731	4.0358	4.0597
4	7	10	3.7299	3.7209	3.7073	3.7119	3.7439	3.7699	3.8019	3.8812	3.9675	4.0347	4.0597
4	8	8	3.9181	3.9002	3.8573	3.8147	3.7910	3.7880	3.7910	3.8147	3.8573	3.9002	3.9181
4	8	9	3.8121	3.7978	3.7656	3.7392	3.7338	3.7402	3.7524	3.7935	3.8492	3.8986	3.9181
4	8	10	3.7299	3.7184	3.6947	3.6811	3.6902	3.7039	3.7234	3.7779	3.8433	3.8974	3.9181
4	9	9	3.8121	3.7961	3.7571	3.7176	3.6950	3.6921	3.6950	3.7176	3.7571	3.7961	3.8121
4	9	10	3.7299	3.7166	3.6861	3.6592	3.6511	3.6556	3.6657	3.7016	3.7510	3.7948	3.8121
4	10	10	3.7299	3.7153	3.6797	3.6430	3.6216	3.6189	3.6216	3.6430	3.6797	3.7153	3.7299

Table E. Critical values $\gamma^{\kappa}_{ij,1-\alpha} = \sqrt{\mathcal{F}^{-1\,(F^{\kappa}_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}(1-\alpha)}$, for $\alpha = 0.05$ and $\kappa = 5$.

			φ_{ij}										
κ	f_i	f_j	0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
5	1	1	33.924	39.299	43.480	46.341	47.793	47.975	47.793	46.341	43.480	39.299	33.924
5	1	2	9.8225	13.317	17.085	20.887	24.490	26.154	27.696	30.344	32.313	33.520	33.924
5	1	3	6.7132	9.6797	13.585	17.992	22.344	24.373	26.260	29.504	31.924	33.418	33.924
5	1	4	5.5929	8.3507	12.549	17.378	22.016	24.140	26.098	29.434	31.901	33.414	33.924
5	1	5	5.0251	7.6909	12.175	17.222	21.950	24.096	26.069	29.422	31.896	33.413	33.924
5	1	6	4.6837	7.3109	12.026	17.173	21.929	24.083	26.059	29.417	31.894	33.412	33.924
5	1	7	4.4562	7.0719	11.960	17.152	21.920	24.076	26.054	29.414	31.893	33.412	33.924
5	1	8	4.2939	6.9128	11.926	17.141	21.914	24.071	26.051	29.413	31.892	33.412	33.924
5	1	9	4.1723	6.8024	11.907	17.134	21.910	24.068	26.048	29.412	31.892	33.412	33.924
5	1	10	4.0779	6.7232	11.895	17.129	21.907	24.066	26.047	29.411	31.892	33.412	33.924
5	2	2	9.8225	9.9068	10.068	10.221	10.310	10.322	10.310	10.221	10.068	9.9068	9.8225
5	2	3	6.7132	6.8858	7.2551	7.7036	8.1731	8.4062	8.6348	9.0680	9.4441	9.7165	9.8225
5	2	4	5.5929	5.8001	6.2453	6.8137	7.4575	7.7958	8.1367	8.7895	9.3345	9.6959	9.8225
5	2	5	5.0251	5.2500	5.7353	6.3775	7.1358	7.5402	7.9455	8.7029	9.3071	9.6913	9.8225
5	2	6	4.6837	4.9193	5.4301	6.1269	6.9691	7.4171	7.8606	8.6696	9.2970	9.6894	9.8225
5	2	7	4.4562	4.6988	5.2282	5.9685	6.8744	7.3516	7.8180	8.6536	9.2918	9.6883	9.8225
5	2	8	4.2939	4.5415	5.0854	5.8618	6.8167	7.3136	7.7940	8.6445	9.2886	9.6876	9.8225
5	2	9	4.1723	4.4237	4.9793	5.7864	6.7794	7.2897	7.7790	8.6386	9.2864	9.6871	9.8225
5	2	10	4.0779	4.3321	4.8976	5.7311	6.7540	7.2737	7.7689	8.6344	9.2848	9.6867	9.8225
5	3	3	6.7132	6.6802	6.6345	6.6108	6.6045	6.6040	6.6045	6.6108	6.6345	6.6802	6.7132
5	3	4	5.5929	5.5941	5.6459	5.7585	5.9109	5.9969	6.0885	6.2854	6.4863	6.6487	6.7132
5	3	5	5.0251	5.0467	5.1521	5.3364	5.5752	5.7104	5.8543	6.1562	6.4387	6.6406	6.7132
5	3	6	4.6837	4.7190	4.8580	5.0872	5.3831	5.5514	5.7300	6.0958	6.4192	6.6372	6.7132
5	3	7	4.4562	4.5013	4.6634	4.9240	5.2615	5.4541	5.6571	6.0637	6.4093	6.6354	6.7132
5	3	8	4.2939	4.3464	4.5254	4.8095	5.1793	5.3902	5.6110	6.0447	6.4033	6.6342	6.7132
5	3	9	4.1723	4.2306	4.4225	4.7252	5.1208	5.3461	5.5802	6.0324	6.3994	6.6333	6.7132
5	3	10	4.0779	4.1408	4.3428	4.6607	5.0776	5.3142	5.5584	6.0239	6.3965	6.6327	6.7132

κ	f_i	f_j	φ_{ij} 0°	10°	20°	30°	40°	45°	50°	60°	70°	80°	90°
5	4	4	5.5929	5.5565	5.4850	5.4284	5.4017	5.3986	5.4017	5.4284	5.4850	5.5565	5.5929
5	4	5	5.0251	5.0057	4.9869	5.0075	5.0685	5.1116	5.1624	5.2858	5.4266	5.5458	5.5929
5	4	6	4.6837	4.6759	4.6914	4.7597	4.8752	4.9478	5.0292	5.2133	5.4005	5.5413	5.5929
5	4	7	4.4562	4.4568	4.4965	4.5973	4.7506	4.8440	4.9471	5.1720	5.3868	5.5388	5.5929
5	4	8	4.2939	4.3010	4.3585	4.4830	4.6645	4.7735	4.8927	5.1464	5.3785	5.5372	5.5929
5	4	9	4.1723	4.1846	4.2557	4.3985	4.6019	4.7232	4.8547	5.1293	5.3730	5.5361	5.5929
5	4	10	4.0779	4.0943	4.1763	4.3335	4.5546	4.6857	4.8270	5.1173	5.3691	5.5353	5.5929
5	5	5	5.0251	4.9932	4.9229	4.8605	4.8287	4.8248	4.8287	4.8605	4.9229	4.9932	5.0251
5	5	6	4.6837	4.6621	4.6243	4.6111	4.6350	4.6599	4.6931	4.7828	4.8928	4.9878	5.0251
5	5	7	4.4562	4.4421	4.4275	4.4479	4.5096	4.5544	4.6079	4.7369	4.8764	4.9849	5.0251
5	5	8	4.2939	4.2856	4.2884	4.3331	4.4224	4.4819	4.5504	4.7076	4.8664	4.9830	5.0251
5	5	9	4.1723	4.1687	4.1848	4.2482	4.3587	4.4295	4.5095	4.6876	4.8597	4.9817	5.0251
5	5	10	4.0779	4.0780	4.1049	4.1829	4.3102	4.3901	4.4791	4.6733	4.8549	4.9807	5.0251
5	6	6	4.6837	4.6560	4.5917	4.5308	4.4983	4.4943	4.4983	4.5308	4.5917	4.6560	4.6837
5	6	7	4.4562	4.4356	4.3933	4.3663	4.3722	4.3879	4.4118	4.4825	4.5735	4.6527	4.6837
5	6	8	4.2939	4.2787	4.2530	4.2506	4.2845	4.3145	4.3528	4.4511	4.5622	4.6506	4.6837
5	6	9	4.1723	4.1614	4.1486	4.1652	4.2201	4.2611	4.3105	4.4293	4.5547	4.6492	4.6837
5	6	10	4.0779	4.0704	4.0681	4.0994	4.1710	4.2208	4.2789	4.4136	4.5492	4.6481	4.6837
5	7	7	4.4562	4.4320	4.3738	4.3165	4.2849	4.2809	4.2849	4.3165	4.3738	4.4320	4.4562
5	7	8	4.2939	4.2748	4.2326	4.2000	4.1965	4.2069	4.2251	4.2838	4.3616	4.4296	4.4562
5	7	9	4.1723	4.1573	4.1276	4.1139	4.1317	4.1529	4.1820	4.2609	4.3534	4.4281	4.4562
5	7	10	4.0779	4.0662	4.0466	4.0477	4.0822	4.1120	4.1497	4.2442	4.3475	4.4269	4.4562
5	8	8	4.2939	4.2724	4.2198	4.1664	4.1362	4.1323	4.1362	4.1664	4.2198	4.2724	4.2939
5	8	9	4.1723	4.1547	4.1143	4.0797	4.0709	4.0780	4.0926	4.1428	4.2110	4.2706	4.2939
5	8	10	4.0779	4.0635	4.0328	4.0130	4.0211	4.0367	4.0597	4.1255	4.2047	4.2694	4.2939
5	9	9	4.1723	4.1529	4.1051	4.0556	4.0269	4.0232	4.0269	4.0556	4.1051	4.1529	4.1723
5	9	10	4.0779	4.0616	4.0234	3.9885	3.9768	3.9817	3.9937	4.0378	4.0985	4.1516	4.1723
5	10	10	4.0779	4.0602	4.0165	3.9704	3.9433	3.9398	3.9433	3.9704	4.0165	4.0602	4.0779

5. Numerical evaluation of the p-values

As mentioned earlier, the p-values p_0 given by (19), p_0^λ given by (33), and $p_0^{(ij)}$ given by (34), are functions of the cdf of a linear combination of independent inverted chi-square random variables. Moreover, for deriving the critical values (20) and (41) by solving (42) we need a repeated evaluation of the cdf of a linear combination of independent inverted chi-square random variables. For that we can use a method suggested by Witkovský (2001b), who derived the characteristic function of the inverted gamma distribution and suggested to calculate the exact distribution of a linear combination of independent inverted gamma variables by using the inversion formula of Gil-Pelaez (1951) which leads to one-dimensional numerical integration.

Let $X \sim G(\alpha, \beta)$ be a gamma random variable with the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$. Then the inverted gamma variable $Y = 1/X$, $Y \sim IG(\alpha, \beta)$, has its probability density function $f^{(Y)}_{[\alpha,\beta]}(y)$ defined for $y \geq 0$ by

$$f^{(Y)}_{[\alpha,\beta]}(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \left(\frac{1}{y}\right)^{\alpha+1} \exp\left(-\frac{1}{\beta y}\right), \tag{43}$$

and the characteristic function of Y is

$$\phi^{(Y)}_{[\alpha,\beta]}(u) = E\left(e^{iuY}\right) = \frac{2(-iu\beta)^{\frac{1}{2}\alpha} K_\alpha\left(\frac{2}{\beta}(-iu\beta)^{\frac{1}{2}}\right)}{\beta^\alpha \Gamma(\alpha)}, \tag{44}$$

where $K_\alpha(z)$ is the modified Bessel function of second kind, see Abramowitz and Stegun (1965).

Note that, for $\alpha = \nu/2$, $\nu = 1, 2, \ldots$ and $\beta = 2$, the random variable $X \sim G(\alpha, \beta)$ has the chi-square distribution with ν degrees of freedom, and hence the random variable $Y \sim IG(\alpha, \beta)$ has the inverted chi-square distribution with ν degrees of freedom.

Let $Y_{[\alpha_1,\beta_1]}, \ldots, Y_{[\alpha_m,\beta_m]}$ be a set of independent inverted gamma variables, where $Y_{[\alpha_j,\beta_j]} \sim IG(\alpha_j, \beta_j)$, with $\alpha_j > 0$ and $\beta_j > 0$, $j = 1, \ldots, m$. Let us define a general linear combination of such variables, say

$$L = \sum_{j=1}^{m} l_j Y_{[\alpha_j,\beta_j]} \tag{45}$$

with real coefficients l_j. If $\phi_j(u)$ denotes the characteristic function of the random variable $Y_{[\alpha_j,\beta_j]}$, then $\phi^{(L)}_{[\{l_j,\alpha_j,\beta_j\}_m]}(u)$, the characteristic function of L, is

$$\phi^{(L)}_{[\{l_j,\alpha_j,\beta_j\}_m]}(u) = \phi_1(l_1 u)\cdots\phi_m(l_m u). \tag{46}$$

The exact value of the distribution function $\mathcal{F}^{(L)}_{[\{l_j,\alpha_j,\beta_j\}_m]}(x) = \Pr(L \leq x)$ can be evaluated by using the inversion formula of Gil-Pelaez (1951) which leads to one-dimensional numerical integration, namely

$$\mathcal{F}^{(L)}_{[\{l_j,\alpha_j,\beta_j\}_m]}(x) = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left(\frac{e^{-iux}\phi^{(L)}_{[\{l_j,\alpha_j,\beta_j\}_m]}(u)}{u}\right) du. \tag{47}$$

Formula (47) is readily applicable to the numerical evaluation of the p-value p_0^λ given by (33) using a finite range of integration $0 \leq u \leq U$, $U < \infty$. In general, a complex-valued functions should be numerically evaluated. The degree of approximation depends on the error of truncation and the error of integration method. The method is quite general without any restrictions on the number of the variables, the values of the coefficients and the involved degrees of freedom. The numerical implementation is easy, provided an efficient algorithm for numerical evaluation of the modified Bessel function of second kind of a complex argument is available, see e.g. Amos (1986).

6. The exact distribution

Following the approach suggested by Walker and Saw (1978) and as a direct consequence of the result by Witkovský (2001c), under the assumption that f_i and f_j are odd, the distribution function $\mathcal{F}^{(F^\kappa_{ij})}_{[\kappa,f_i,f_j,\varphi_{ij}]}$ of the random variable F^κ_{ij}, given by (35), can be expressed as a well defined finite linear combination of the Fisher-Snedecor's F-distributions. In particular, let

$$Z \sim \chi^2_\kappa\left(\frac{a_1}{\chi^2_{2m_1-1}} + \frac{a_2}{\chi^2_{2m_2-1}}\right), \tag{48}$$

where $a_1 > 0$, $a_2 > 0$, and $m_1, m_2 \in \{1, 2, \dots\}$. Notice that $Z \sim F_{ij}^{\kappa}$ for $f_i = 2m_1 - 1$, $f_j = 2m_2 - 1$, $a_1 = f_i s_{\varphi_{ij}}^2$, and $a_2 = f_j c_{\varphi_{ij}}^2$. Then

$$\Pr(Z \leq z) = \mathcal{F}^{(Z)}_{[\kappa, m_1, m_2, a_1, a_2]}(z) = \sum_{j=1}^{m} \xi_j \mathcal{F}^{(F)}_{[\kappa, 2j-1]}\left(\frac{(2j-1)z}{\kappa\omega^2}\right), \tag{49}$$

where $m = 1 + \sum_{j=1}^{2}(m_j - 1)$, $\omega = \sum_{j=1}^{2} \sqrt{a_j}$, and $\mathcal{F}^{(F)}_{[\kappa, 2j-1]}$ denote the cdf's of the Fisher-Snedecor's F-distributions with κ and $2j - 1$ degrees of freedom. The vector of coefficients $\xi = (\xi_1, \dots, \xi_m)'$ is given by the following parade of definitions.

Let Q be an $(m \times m)$ lower-triangular matrix with its elements given by the following reccurence relation: First, set $Q_{1,1} = 1$, and $Q_{i,1} = Q_{i,2} = 1$ for $i = 2, \dots, m$, and note that $Q_{i,j} = 0$ for $j > i$. Then

$$Q_{i,j} = Q_{i-1,j} + \frac{Q_{i-2,j-2}}{(2i-3)(2i-5)}, \tag{50}$$

for $i = 3, \dots, m$, and $j = 3, \dots, i$. Further, we define $A_1 = \sqrt{a_1}/\omega$ and $A_2 = \sqrt{a_2}/\omega$, and

$$\begin{aligned} v_1 &= \operatorname{diag}\left(A_1^0, A_1^1, \dots, A_1^{m-1}\right) Q'_{m_1} \\ v_2 &= \operatorname{diag}\left(A_2^0, A_2^1, \dots, A_2^{m-1}\right) Q'_{m_2} \\ \zeta &= \operatorname{conv}(v_1, v_2), \end{aligned} \tag{51}$$

where Q_{m_1} and Q_{m_2} denote the m_1-th and m_2-th rows of the matrix Q, respectively. By $\operatorname{conv}(v_1, v_2)$ we denote the m-dimensional vector of polynomial convolution coefficients. For example, for $m = 4$ we have $\operatorname{conv}((1, 1, 0, 0)', (1, 1, 1/3, 0)') = (1, 2, 4/3, 1/3)'$. Then the vector of coefficients ξ is given as

$$\xi = \left(Q^{-1}\right)' \zeta. \tag{52}$$

Note that the result (49) can be easily extended for the distribution function of the random variable $Z \sim \chi^2_\kappa \sum_{j=1}^{k} a_j/\chi^2_{f_j}$ with odd degrees of freedom $f_j = 2m_j - 1$, $j = 1, \dots, k$.

7. Barnard's approximate formula

Based on another equivalent form of the exact Behrens-Fisher p-value, Barnard (1984) derived a simple approximate procedure to evaluate the p-value p_0 defined by (5). As the author mentioned, he checked the approximate p-values against the Fisher-Yates tables (see Table A in this paper) for moderate degrees of freedom and medium unbalancedness *"...and the maximum error found was less than one per cent of the true p-value"*. In this section, we present a natural generalization of the Barnard's approximate formula to evaluate the p-value $p_0^{(ij)}$ given by (34) for an arbitrary κ.

For that, let $\varrho^{(ij)} = \sigma_j^2/\sigma_i^2$, and further we define

$$t^2_{ij,obs}\left(\theta_0^{(ij)}, \varrho^{(ij)}\right) = \frac{\left((\bar{y}_i - \bar{y}_j) - \theta_0^{(ij)}\right)^2}{\left(1/n_i + \varrho^{(ij)}/n_j\right)} \left(\frac{f_i + f_j}{f_i s_i^2 + f_j s_j^2/\varrho^{(ij)}}\right). \tag{53}$$

For given κ and $\varrho^{(ij)}$, we set

$$p\left(t^2_{ij,obs}\left(\theta_0^{(ij)}, \varrho^{(ij)}\right)\right) = 1 - \mathcal{F}^{(F)}_{[\kappa, f_1+f_2]}\left(\frac{1}{\kappa}\, t^2_{ij,obs}\left(\theta_0^{(ij)}, \varrho^{(ij)}\right)\right), \tag{54}$$

where $\mathcal{F}^{(F)}_{[\kappa, f_1+f_2]}$ denotes the cdf of the Fisher-Snedecor's F-distribution with κ and $f_1 + f_2$ degrees of freedom. Then, the approximation of the p-value (34) is given by

$$p_0^{(ij)} \approx \frac{1}{6}\sum_{l=1}^{3} h_l p\left(t^2_{ij,obs}\left(\theta_0^{(ij)}, \varrho_l^{(ij)}\right)\right), \tag{55}$$

with coefficients $h_1 = 1$, $h_2 = 4$, $h_3 = 1$, and

$$\varrho_l^{(ij)} = \frac{(s_2^2/s_1^2)}{\mathcal{F}^{-1}{}^{(F)}_{[f_2,f_1]}(u_l)}, \tag{56}$$

with $u_1 = \Phi(-\sqrt{3}) = 0.041632$, $u_2 = \Phi(0) = 0.05$, and $u_3 = \Phi(\sqrt{3}) = 1 - 0.041632$, Φ denotes the cdf of standard normal distribution and $\mathcal{F}^{-1}{}^{(F)}_{[f_2,f_1]}$ denotes the inverse of the cdf of the Fisher-Snedecor's F-distribution with f_2 and f_1 degrees of freedom.

8. An illustrative example: DNA analysis of patients with hematological malignacies

Bačová *et al.* (1998) used single cell gel electrophoresis to evaluate the level of DNA damage measured in per cent of tail DNA in peripheral blood, bone marrow, and lymphatic node cells of patients with acute lymphoblastic leukemia ALL, (ALL of T-cell subtype is denoted by TALL, ALL of early B-cell subtype is denoted by BALL), acute myeloid leukemia AML, chronic lymphocytic leukemia CLL, chronic myeloid leukemia CML and non-Hodgkin's lymphoma NHL. The level of DNA damage is to be compared with the level of basal DNA damage in control group CONTR, represented by healthy donors.

Table 1. The mean basal DNA damage of patients with hematological malignacies.

Diagnosis	n_i	$\bar{y}_i$	s_i^2
CONTR	11	6.7194	3.7092
AML	25	11.6970	40.4761
BALL	8	8.2017	7.8417
CLL	7	5.4709	0.9178
CML	7	16.5356	33.8495
NHL	11	15.2078	64.0564
TALL	12	10.7675	21.3017

The mean basal DNA damage increased in order

$$\text{CLL} < \text{BALL} < \text{TALL} < \text{AML} < \text{NHL} < \text{CML},$$

what correlated with survival prediction of patients with a particular hematological disease. A large heterogeneity was found in the level of basal DNA damage among patients with AML and NHL.

Table 2. Analysis of the DNA damage data. By $*$ we denoted the p-values less than 0.1, by $**$ we denoted the p-values less than 0.05.

		Individual comparisons			Multiple comparisons		
Contrast	$\bar{y}_i - \bar{y}_j$	p-value	95% CI		p-value	95% CI	
CONTR-AML	-4.9776	**0.0016	-7.8955	-2.0596	*0.0935	-10.4225	0.4674
CONTR-BALL	-1.4822	0.2433	-4.1410	1.1766	0.9476	-6.7265	3.7621
CONTR-CLL	1.2485	0.1074	-0.3122	2.8093	0.7990	-1.7766	4.2737
CONTR-CML	-9.8162	**0.0045	-15.3350	-4.2973	*0.0922	-21.1516	1.5193
CONTR-NHL	-8.4884	**0.0062	-14.0082	-2.9685	0.1663	-19.3035	2.3267
CONTR-TALL	-4.0481	**0.0166	-7.2406	-0.8555	0.3474	-10.1992	2.1031
AML-BALL	3.4954	*0.0501	-0.0025	6.9932	0.6371	-3.1253	10.1160
AML-CLL	6.2261	**0.0001	3.4599	8.9923	**0.0103	1.0441	11.4081
AML-CML	-4.8386	*0.0982	-10.7875	1.1103	0.7503	-16.7696	7.0923
AML-NHL	-3.5108	0.2277	-9.4697	2.4480	0.9461	-14.9971	7.9754
AML-TALL	0.9295	0.6292	-2.9869	4.8459	0.9997	-6.4290	8.2880
BALL-CLL	2.7307	**0.0355	0.2368	5.2247	0.4598	-2.2671	7.7286
BALL-CML	-8.3340	**0.0116	-14.1723	-2.4957	0.2110	-20.1458	3.4779
BALL-NHL	-7.0062	**0.0226	-12.8482	-1.1642	0.4000	-18.3539	4.3416
BALL-TALL	-2.5659	0.1630	-6.2975	1.1658	0.8936	-9.7417	4.6100
CLL-CML	11.0647	**0.0024	-16.5103	-5.6191	*0.0535	-22.3012	0.1718
CLL-NHL	-9.7369	**0.0025	-15.1810	-4.2929	*0.0829	-20.4394	0.9656
CLL-TALL	-5.2966	**0.0026	-8.3531	-2.2401	*0.0946	-11.2340	0.6408
CML-NHL	1.3278	0.7088	-6.2335	8.8890	0.9999	-13.4038	16.0594
CML-TALL	5.7681	*0.0605	-0.3187	11.8549	0.6143	-6.3927	17.9289
NHL-TALL	4.4403	0.1407	-1.6539	10.5346	0.8651	-7.2980	16.1787

Heparinized blood, bone marrow samples and lymphatic nodes of 70 patients with hematological malignancies and of 11 healthy donors, respectively, were obtained[†] from the National Cancer Institute and from the Department of Pediatric Oncology of University Children's Hospital, Bratislava, Slovak Republic. For illustration purposes, we present a simplified analysis. For each patient we calculated the mean level of DNA damage. Based on such data we calculated the sample means and the sample variances for each type of diagnosis, including the control group of healthy donors. The data in an agregated form are presented in Table 1. In Table 2 we present the results of individual as well as multiple comparisons. The individual comparisons lead to the problem of comparing equality of the mean basal DNA damage for each two diagnosis separately, in this case we set $\kappa = 1$. The multiple comparisons give us an answer to the question if the method based on measuring basal DNA damage could be a potential diagnostic tool for the patients with hematological maligancies, in this case we set $\kappa = 6$. The answers to the above questions are readily available from the results presented in Table 2. However, in this paper, we leave this discussion without any further comment.

9. Concluding remarks

Although sveral different procedures for pairwise multiple comparisons in the unequal variance case were suggested in the statistical literature, see e.g. Dunnet (1980), we believe that it would be beneficial for statisticians to have another well defined method leading to the joint conservative confidence intervals for the mean differences, refered to as the generalized Scheffé intervals, presented in such extent in one place with the exact as well as approximate methods for evaluation of the required p-values, followed by an example of application.

By construction, we can expect longer confidence intervals than those produced by the other methods based on approximately similar tests. However, following the theoretical support given to the original Fisher's solution of the Behrens-Fisher problem, see Robinson (1976) and Barnard (1984), we hope that also the generalized solution to the pairwise multiple comparisons persists the following two desirable properties: (i) Confidence regions cover the true value with probability always larger than the nominal confidence level, and (ii) There are no negatively biased relevant selections. For more details see Robinson (1976).

[†]The data were kindly provided by Gabriela Bačová of the Cancer Research Institute, Slovak Academy of Sciences, Bratislava, Slovak Republic.

Dedication

The paper is dedicated to the memory of my father, Ján Witkovský, who passed away after a long fight with cancer.

At the same time, I would like to express my last thank you to Prof. Stanisław Gnot, a wonderful man, and one of the top statisticians, who unfortunately also lost his heroic fight with cancer.

References

[1] D.E. Amos, *A portable package for bessel functions of a complex argument and nonnegative order*, ACM Transactions on Mathematical Software **12** (1986), 265–273.

[2] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York 1965.

[3] G. Bačová, A. Gábelová, O. Babušíková and D. Slameňová, *The single cell gel electrophoresis: A potential tool for DNA analysis of the patients with hematological maligancies*, Neoplasma **45** (1998), 349–359.

[4] G.A. Barnard, *Comparing the means of two independent samples*, Applied Statistics **33** (1984), 266–271.

[5] R.A. Fisher, *The fiducial argument in statistical inference*, Annals of Eugenics **6** (1935), 391–398.

[6] R.A. Fisher and F. Yates, *Statistical Tables*, Longman, London 1975.

[7] C.W. Dunnett, *Pairwise multiple comparisons in the unequal variance case*, Journal of the American Statistical Association **75** (1980), 796–800.

[8] J. Gil-Pelaez, *Note on the inversion theorem*, Biometrika **38** (1951), 481–482.

[9] A.F.S. Lee and J. Gurland, *Size and power of tests of equality of two normal populations with unequal variances*, Journal of the American Statistical Association **70** (1975), 933–941.

[10] X.-L. Meng, *Posterior predictive p-values*, Annals of Statistics **22** (1994), 1142–1160.

[11] G.K. Robinson, *Properties of Student t and of the Behrens-Fisher solution to the two means problem*, Annals of Statistics **4** (1976), 963–971.

[12] G.K. Robinson, *Behrens-Fisher problem*, Encyclopedia of Statistical Sciences, 9 volumes plus Supplement, Wiley, New York, (1982), 205–209.

[13] H. Scheffé, *Practical solutions of the Behrens-Fisher problem*, Journal of the American Statistical Association **65** (1970), 1501–1508.

[14] K.W. Tsui, and S. Weerahandi, *Generalized p values in significance testing of hypotheses in the presence of nuisance parameters*, Journal of the American Statistical Association **84** (1989), 602–607.

[15] G.A. Walker and J.G. Saw, *The distribution of linear combinations of t variables*, Journal of the American Statistical Association **73** (1978), 876–878.

[16] S. Weerahandi, *ANOVA under unequal error variances*, Biometrics **51** (1995a), 589–599.

[17] S. Weerahandi, *Exact Statistical Methods for Data Analysis*, Springer-Verlag, New York 1995b.

[18] B.L. Welch, *The generalization of 'Student's' problem when several different population variances are involved*, Biometrika **34** (1947), 28–35.

[19] V. Witkovský, *On the exact computation of the density and of the quantiles of linear combinations of t and F random variables*, Journal of Statistical Planning and Inference **94** (2001a), 1–13.

[20] V. Witkovský, *Computing the distribution of a linear combination of inverted gamma variables*, Kybernetika **37** (2001b), 79–90.

[21] V. Witkovský, *Exact distribution of positive linear combinations of inverted chi-square random variables with odd degrees of freedom,* Submitted to Statistics & Probability Letters 2001c.

Received 5 December 2002