Discussiones Mathematicae
Probability and Statistics 22 (2002) 37–51

EXACT DISTRIBUTION FOR THE GENERALIZED *F* TESTS

Miguel Fonseca, João Tiago Mexia

Department of Mathematics, Faculty of Science and Technology
New University of Lisbon
Monte da Caparica 2829–516 Caparica, Portugal
e-mail: fonsecamig@yahoo.com

and

Roman Zmyślony

Institute of Mathematics, University of Zielona Góra
Podgórna 50 65–246 Zielona Góra, Poland
e-mail: r.zmyslony@im.uz.zgora.pl

Abstract

Generalized F statistics are the quotients of convex combinations of central chi-squares divided by their degrees of freedom. Exact expressions are obtained for the distribution of these statistics when the degrees of freedom either in the numerator or in the denominator are even. An example is given to show how these expressions may be used to check the accuracy of Monte-Carlo methods in tabling these distributions. Moreover, when carrying out adaptative tests, these expressions enable us to estimate the p-values whenever they are available.

Keywords: exact distribution theory, hypothesis testing, generalized F distribution, adaptative test.

2000 Mathematics Subject Classification: 62E15, 62H10, 62H15, 62J10.

1. Introduction

When, under the tested hypothesis, the test statistic may be written as the quotient of convex combinations of independent central chi-squares divided by their degrees of freedom we have a *generalized F test*. Such tests were introduced by [3] and [4]. In the first of these papers, the nullity of variance components is tested. The statistics were the quotients of the positive part by the negative part of quadratic unbiased estimators. In the other paper, this technique was extended to mixed models containing also fixed parameters.

If the degrees of freedom in the numerator [denominator] are even we say that the first [second] evenness condition holds. We are going to obtain exact expressions for the distribution of generalized F test statistics when either of these conditions hold.

These expressions are useful in checking the precision of Monte-Carlo methods in tabling such distributions. Moreover, if adaptative tests, see [1], are carried out we can use the exact expressions of the distributions, if available, to estimate the p-values.

2. Exact expressions

Let $F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, \underset{\sim}{\mathbf{g}}_1^r, \underset{\sim}{\mathbf{g}}_2^s)$ be the distribution of

$$\mathcal{F} = \frac{\sum\limits_{j=1}^{r} c_{1j}\chi^2_{g_{1j}}}{\sum\limits_{j=1}^{s} c_{2j}\chi^2_{g_{2j}}} \tag{1}$$

when the chi-squares are independent. Clearly, the c_{1j} and g_{1j}, $j = 1, ...r$ [c_{2j} and g_{2j}, $j = 1, ...s$] will be the components of $\underset{\sim}{\mathbf{c}}_1^r$ and $\underset{\sim}{\mathbf{g}}_1^r$ [$\underset{\sim}{\mathbf{c}}_2^s$ and $\underset{\sim}{\mathbf{g}}_2^s$], respectively. The generalized F distributions will belong to this family, namely they correspond to the case in which $\underset{\sim}{\mathbf{c}}_1^r > \underset{\sim}{\mathbf{0}}^r$, $\underset{\sim}{\mathbf{c}}_2^s > \underset{\sim}{\mathbf{0}}^s$ and $\sum_{j=1}^{r} c_{1j}g_{1j} = \sum_{j=1}^{s} c_{2j}g_{2j}$.

We start by assuming that $\underset{\sim}{\mathbf{c}}_1^r > \underset{\sim}{\mathbf{0}}^r$, $\underset{\sim}{\mathbf{c}}_2^s > \underset{\sim}{\mathbf{0}}^s$ and that $\underset{\sim}{\mathbf{g}}_1^r = 2\underset{\sim}{\mathbf{m}}^r$. Our results extend directly to the case in which $\underset{\sim}{\mathbf{g}}_2^s = 2\underset{\sim}{\mathbf{m}}^s$, since

$$F\left(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, \underset{\sim}{\mathbf{g}}_1^r, 2\underset{\sim}{\mathbf{m}}^s\right) = 1 - F\left(u^{-1}|\underset{\sim}{\mathbf{c}}_2^s, \underset{\sim}{\mathbf{c}}_1^r, 2\underset{\sim}{\mathbf{m}}^s, \underset{\sim}{\mathbf{g}}_1^r\right). \tag{2}$$

Thus we will cover the cases of generalized F distributions whenever one of the evenness conditions holds.

To make the notation clear we put $a_i = c_{1i}$ and $g_i = g_{1i}$, $i = 1, ..., r$, as well as $a_{i+r} = c_{2i}$ and $g_{i+r} = g_{2i}$, $i = 1, ..., s$. Define also $X_i = a_i \chi^2_{g_i i}$, $i = 1, ..., r + s$.
We now carry out successive integrations. The density of X_i will be

$$\text{(3)} \qquad \begin{cases} f_i(x) = \dfrac{x^{m_i - 1}}{(m_i - 1)!(2a_i)^{m_i}} \, e^{-\frac{x}{2a_i}} & , x > 0, i = 1, ..., r \\ f_i(x) = \dfrac{x^{\frac{g_i}{2} - 1}}{\Gamma(\frac{g_i}{2})(2a_i)^{\frac{g_i}{2}}} \, e^{-\frac{x}{2a_i}} & , x > 0, i = r + 1, ..., r + s \end{cases}$$

Then

$$F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, 2\underset{\sim}{\mathbf{m}}^r, \underset{\sim}{\mathbf{g}}_2^s) = \mathbb{P}\left[\sum_{i=1}^{r} X_i \leq u \sum_{i=r+1}^{r+s} X_i\right]$$

$$\text{(4)} \qquad = \int_0^{+\infty} \cdots \int_0^{+\infty} \int_0^{u \sum_{i=r+1}^{r+s} x_i} \cdots$$

$$\cdots \int_0^{u \sum_{i=r+1}^{r+s} x_i - \sum_{i=2}^{r} x_i} \prod_{j=1}^{r+s} f_j(x_j) dx_j.$$

We now point out that, with b a non-negative integer, through a successive integration by parts we get

$$\text{(5)} \qquad \int_0^d x^b e^{-\frac{x}{a}} dx = a^{b+1} b! \left(1 - e^{-\frac{d}{a}} \sum_{j=0}^{b} \frac{d^j}{a^j j!}\right).$$

This expression can be rewritten as

$$\text{(6)} \qquad \int_0^d x^b e^{-\frac{x}{a}} dx = a^{b+1} b! \sum_{k=0}^{1} (-1)^k e^{-\frac{kd}{a}} \sum_{j=0}^{kb} \frac{d^j}{a^j j!}.$$

Using (6), with $y = u \sum_{i=r+1}^{r+s} x_i$ and $b_i = m_i - 1$, $i = 1, ..., r$, and in the first place, $a = 2a_1$ and $d = y - \sum_{i=2}^{r} x_i$, we get

$$\int_0^{y-\sum\limits_{i=2}^{r} x_i} f_1(x_1)\,dx_1 = \int_0^{y-\sum\limits_{i=2}^{r} x_i} \frac{x_1^{b_1} e^{-\frac{x_1}{a}}}{b_1!(2a_1)^{b_1+1}}\,dx_1$$

$$= \sum_{k_1=0}^{1} (-1)^{k_1} e^{-\frac{k_1}{2a_1}\left(y-\sum\limits_{i=2}^{r} x_i\right)} \sum_{j_1=0}^{k_1 b_1} \frac{\left(y - \sum\limits_{i=2}^{r} x_i\right)^{j_1}}{j_1!(2a_1)^{j_1}}$$

(7)

$$= \sum_{k_1=0}^{1} (-1)^{k_1} e^{-\frac{k_1}{2a_1}\left(y-\sum\limits_{i=2}^{r} x_i\right)} \sum_{j_1=0}^{k_1 b_1} \frac{1}{(2a_1)^{j_1}}$$

$$\sum_{\left(\sum\limits_{i=1}^{j_1} t_{1,i}=j_1\right)} (-1)^{j_1-t_{1,1}} \frac{y^{t_{1,1}}}{t_{1,1}!} \prod_{i=2}^{r} \frac{x_i^{t_{1,i}}}{t_{1,i}!}$$

where $\left(\sum\limits_{i=1}^{j_1} t_{1,i} = j_1\right)$ indicates summation for all sets of non negative integers $t_{1,1}, ..., t_{1,r}$ with sum j_1. Thus, we can apply again (6) to get

$$\int_0^{y-\sum\limits_{i=3}^{r} x_i} \int_0^{y-\sum\limits_{i=2}^{r} x_i} f_1(x_1) f_2(x_2)\,dx_1 dx_2 =$$

(8) $$\sum_{k_1=0}^{1} (-1)^{k_1} e^{-\frac{k_1}{2a_1}\left(y-\sum\limits_{y=3}^{r} x_i\right)} \sum_{j_1=0}^{k_1 b_1} \sum_{\left(\sum\limits_{i=1}^{r} t_{1,i}=j_1\right)} \frac{y^{t_{1,1}}}{t_{1,1}!} \prod_{i=3}^{r} \frac{x_i^{t_{1,i}}}{t_{1,i}!} \cdot \frac{1}{b_2! t_{1,2}!} \times$$

$$\int_0^{y-\sum\limits_{i=3}^{r} x_i} \frac{x_2^{b_2+t_{1,2}}}{(2a_2)^{b_2+1}} e^{-\left(\frac{1}{2a_2}-\frac{k_1}{2a_1}\right)}\,dx_2 = \sum_{k_1=0}^{1} \frac{(-1)^{k_1}}{(2a_2)^{b_2+1}} \sum_{j_1=0}^{k_1 b_1} \frac{1}{(2a_1)^{j_1}} \times$$

$$\sum_{\left(\sum_{i=1}^{r} t_{1,i}=j_1\right)} (-1)^{j_1-t_{1,1}} \left(\frac{1}{2a_2}-\frac{k_1}{2a_1}\right)^{-(b_2+t_{1,2}+1)} \frac{(b_2+t_{1,2})!}{b_2!t_{1,2}!}\times$$

$$\sum_{k_2=0}^{1}(-1)^{k_2} e^{-\left(\frac{k_1}{2a_1}+k_2\left(\frac{1}{2a_2}-\frac{k_1}{2a_2}\right)\right)\left(y-\sum_{i=3}^{r} x_i\right)} \sum_{j_2=0}^{k_2(b_2+t_{1,2})} \left(\frac{1}{2a_2}-\frac{k_1}{2a_1}\right)^{j_2}\times$$

$$\sum_{\left(\sum_{i=1}^{r-1} t_{2,1}=j_2\right)} (-1)^{j_2-t_{2,1}} \frac{y^{t_{1,1}+t_{2,1}}}{t_{1,1}!t_{2,1}!} \prod_{i=3}^{r} \frac{x_i^{t_{1,i}+t_{2,i-1}}}{t_{1,i}!t_{2,i-1}!}.$$

Now, with $b_j^+ = b_j + \sum_{u=1}^{j-1} t_{u,j+1-u}$, $j = 1, ..., w$, we will have $b_1^+ = b_1$ and $b_2^+ = b_2 + t_{1,2}$. Besides this, taking $\underset{\sim}{\mathbf{k}}^n = (k_1, ..., k_n)'$, we have $\underset{\sim}{\mathbf{k}}^2 = (k_1, k_2)'$. Let $d(\underset{\sim}{\mathbf{0}}^n) = 0$ and, for $\underset{\sim}{\mathbf{k}}^n \neq \underset{\sim}{\mathbf{0}}^n$, $d(\underset{\sim}{\mathbf{k}}^n) = \frac{1}{a_{w_n}}$ with w_n the largest index for non null components of $\underset{\sim}{\mathbf{k}}^n$, $n = 1, ..., r$. It is easy to check that $\frac{k_1}{2a_1} + k_2(\frac{1}{2a_2} - \frac{k_1}{2a_1}) = d(\underset{\sim}{\mathbf{k}}^2)$, and that $d(\underset{\sim}{\mathbf{k}}^n) + k_{n+1}\left(\frac{1}{2a_{n+1}} - d(\underset{\sim}{\mathbf{k}}^n)\right) = d(\underset{\sim}{\mathbf{k}}^{n+1})$.

We may now rewrite (8) as

$$\int_0^{y-\sum_{i=3}^{r} x_i}\int_0^{y-\sum_{i=2}^{r} x_i} f_1(x_1) f_2(x_2) dx_1 dx_2 =$$

(9)
$$\frac{1}{(2a_2)^{b_2+1}} \sum_{k_1=0}^{1}\sum_{j_1=0}^{k_1 b_1^+} \frac{1}{(2a_1)^{j_1}} \sum_{\left(\sum_{i=1}^{r} t_{1,i}=j_1\right)} \sum_{k_2=0}^{1}\sum_{j_2=0}^{k_2 b_2^+} \sum_{\left(\sum_{i=1}^{r} t_{2,i}=j_2\right)}$$

$$(-1)^{\sum_{i=1}^{2}(k_i+j_i-t_{i,1})}\frac{b_2^+!}{b_2!t_{1,2}!}\left(\frac{1}{2a_2}-\frac{k_1}{2a_1}\right)^{j_2-b_2^+-1}\frac{e^{-d(\mathbf{k}^2)y}y^{t_{1,1}+t_{2,1}}}{t_{1,1}!t_{2,1}!}\times$$

$$\prod_{i=3}^{r}\frac{e^{-d(\mathbf{k}^2)x_i}x_i^{t_{1,i}+t_{2,i-1}}}{t_{1,i}!t_{2,i-1}!}.$$

Let us establish

Lemma 1. *With*

$$L_{r,m}(y,x_{m+1},...,x_r)=\int_0^{y-\sum_{i=m+1}^{r}x_i}\cdots\int_0^{y-\sum_{i=2}^{r}x_i}\prod_{v=1}^{m}f_v(x_v)\prod_{v=1}^{m}dx_v$$

we have

$$L_{r,m}(y,x_m+1,...,x_r)=\int_0^{y-\sum_{i=m+1}^{r}x_i}\cdots\int_0^{y-\sum_{i=2}^{r}x_i}\prod_{v=1}^{m}f_v(x_v)\prod_{v=1}^{m}dx_v=$$

$$\frac{1}{\prod_{v=2}^{m}(2a_v)^{b_v+1}}\sum_{k_1=0}^{1}\sum_{j_1=0}^{k_1b_1^+}\sum_{\left(\sum_{i=1}^{r}t_{1,i}=j_1\right)}\cdots\sum_{k_m=0}^{1}\sum_{j_m}^{k_mb_m^+}\sum_{\left(\sum_{i=1}^{r+1-m}t_{m,i}=j_m\right)}$$

$$(-1)^{\sum_{i=1}^{m}(k_i+j_i-t_{i,1})}\frac{1}{(2a_1)^{j_1}}\prod_{v=2}^{m}\left(\frac{b_v^+!}{b_v!\prod_{i=1}^{v-1}t_{i,v+1-i}!}\left(\frac{1}{2a_v}-d(\underset{\sim}{\mathbf{k}}^{v-1})\right)^{j_v-b_v^+-1}\right)$$

$$\frac{e^{-d(\mathbf{k}^m)y}y^{\sum_{i=1}^{m}t_{i,1}}}{\prod_{i=1}^{m}t_{i,1}!}\prod_{i=m+1}^{r}\frac{e^{d(\mathbf{k}^m)x_i}x_i^{\sum_{v=1}^{m}t_{v,i+1-v}}}{\prod_{v=1}^{m}t_{v,i+1-v}!}.$$

Proof. We use induction in m. If $m = 2$ the thesis holds since, then, we have (9). Let us assume that it holds for $m \leq w < r$, since $b^{+}_{w+1} = b_w + \sum_{v=1}^{w} t_{v,w+1-v}$, we get using (6)

$$\int_0^{y-\sum_{i=w+2}^{r} x_i} e^{d(\underset{\sim}{\mathbf{k}}^w)x_{w+1}} \frac{x_{w+1}^{\sum_{v=1}^{w} t_{v,w+2-v}}}{\prod_{v=1}^{w} t_{v,w+2-v}!} f_{w+1}(x_{w+1})\, dx_{w+1} =$$

$$\frac{1}{b_{w+1}! \prod_{v=1}^{w} t_{v,w+2-v}!} \cdot \frac{1}{(2a_{w+1})^{b_{w+1}+1}} \int_0^{y-\sum_{i=w+2}^{r} x_i} x_{w+1}^{b_{w+1}+\sum_{v=1}^{w}(t_{v,w+2-v})}$$

$$e^{-\left(\frac{1}{2a_{w+1}} - d(\underset{\sim}{\mathbf{k}}^w)\right)x_{w+1}}\, dx_{w+1} = \frac{b^{+}_{w+1}!}{b_{w+1}! \prod_{v=1}^{w} t_{v,w+2-v}!} \cdot \frac{1}{(2a_{w+1})^{b_{w+1}+1}}$$

$$\left(\frac{1}{2a_{w+1}} - d(\underset{\sim}{\mathbf{k}}^w)\right)^{-b^{+}_{w+1}-1} \sum_{k_{w+1}=0}^{1} \sum_{j_{w+1}=0}^{k_{w+1}b^{+}_{w+1}} e^{-k_{w+1}\left(\frac{1}{2a_{w+1}} - d(\underset{\sim}{\mathbf{k}}^w)\right)\left(y-\sum_{i=w+2}^{r} x_i\right)}$$

$$\frac{(-1)^{k_{w+1}}}{j_{w+1}!} \left(\frac{1}{2a_{w+1}} - d(\underset{\sim}{\mathbf{k}}^w)\right)^{j_{w+1}} \left(y - \sum_{i=w+2}^{r} x_i\right)^{j_{w+1}} =$$

$$\frac{b^{+}_{w+1}!}{b_{w+1}! \prod_{v=1}^{w} t_{v,w+2-v}!} \cdot \frac{1}{(2a_{w+1})^{b_{w+1}+1}} \sum_{k_{w+1}=0}^{1} \sum_{j_{w+1}=0}^{k_{w+1}b^{+}_{w+1}}$$

$$\sum_{\left(\sum_{i=1}^{r+1-(w+1)} t_{w+1,i}=j_{w+1}\right)} (-1)^{k_{w+1}+j_{w+1}-t_{w+1,1}} \left(\frac{1}{2a_{w+1}} - d(\underset{\sim}{\mathbf{k}}^w)\right)^{j_{w+1}-b^{+}_{w+1}-1} \times$$

$$e^{-k_{w+1}\left(\frac{1}{2a_{w+1}}-d(\mathbf{k}^w)\right)y}\frac{y^{t_{w+1,1}}}{t_{w+1,1}!}\prod_{i=w+2}^{r}\frac{e^{k_{w+1}\left(\frac{1}{2a_{w+1}}-d(\mathbf{k}^w)\right)x_i}x_i^{t_{w+1,(i+1)-(w+1)}}}{t_{w+1,(i+1)-(w+1)}!}.$$

Since $d(\underset{\sim}{\mathbf{k}}^w)+k_{w+1}\left(\frac{1}{2a_{w+1}}-d(\underset{\sim}{\mathbf{k}}^w)\right)=d(\underset{\sim}{\mathbf{k}}^{w+1})$, we will have

$$L_{w+1.r}(y,x_{w+2},...,x_r)=\int_0^{y-\sum\limits_{w+2}^{r}x_i}L_{w.r}(y,x_{w+1},...,x_r)f_{w+1}(x_{w+1})\,dx_{w+1}$$

$$=\frac{1}{\prod\limits_{v=2}^{w}(2a_v)^{b_v+1}}\sum_{k_1=0}^{1}\sum_{j_1=0}^{k_1b_1^+}\sum_{\left(\sum\limits_{i=1}^{r}t_{1,i}=j_1\right)}\cdots\sum_{k_w=0}^{1}\sum_{j_w=0}^{k_wb_w^+}\sum_{\left(\sum\limits_{i=1}^{r+1-w}t_{w,i}=j_w\right)}$$

$$\frac{(-1)^{\sum\limits_{i=1}^{w}(k_i+j_i-t_{i,1})}}{(2a_1)^{j_1}}\prod_{v=2}^{w}\left(\frac{b_v^+!}{b_v!\prod\limits_{i=1}^{v-1}t_{i,v+1-1}!}\left(\frac{1}{2a_v}-d(\underset{\sim}{\mathbf{k}}^{v-1})\right)^{j_v-b_v^+-1}\right)$$

$$\frac{e^{-d(\mathbf{k}^w)y}y^{\sum\limits_{i=1}^{w}t_{i,1}}}{\prod\limits_{i=1}^{w}t_{i,1}!}\prod_{i=w+2}^{r}\frac{e^{-d(\mathbf{k}^w)x_i}x_i^{\sum\limits_{v=1}^{w}t_{v,i+1-v}}}{\prod\limits_{v=1}^{w}t_{v,i+1-v}!}$$

$$\int_0^{y-\sum\limits_{i=w+2}^{r}x_i}\frac{e^{d(\mathbf{k}^w)x_{w+1}}x_{w+1}^{\sum\limits_{v=1}^{w}t_{v,w+2-v}}}{\prod\limits_{v=1}^{w}t_{v,w+2-v}!}f_{w+1}(x_{w+1})\,dx_{w+1}$$

$$=\frac{1}{\prod\limits_{v=2}^{w+1}(2a_v)^{b_v+1}}\sum_{k_1=0}^{1}\sum_{j_1=0}^{k_1b_1^+}\sum_{\left(\sum\limits_{i=1}^{r}t_{1,i}=j_1\right)}\cdots\sum_{k_w=0}^{1}\sum_{j_w=0}^{k_wb_w^+}\sum_{\left(\sum\limits_{i=1}^{r+1-w}t_{w,i}=j_w\right)}$$

$$\sum_{k_{w+1}=0}^{1}\sum_{j_{w+1}=0}^{k_{w+1}b_{w+1}^{+}}\sum_{\left(\sum\limits_{i=1}^{r+1-(w+1)} t_{w+1,i}=j_{w+1}\right)}\frac{(-1)^{\sum\limits_{i=1}^{w+1}k_i+j_i-t_{i,1}}}{(2a_1)^{j_1}}\times$$

$$\prod_{v=2}^{w+1}\left(\frac{b_v^{+}!}{b_v!\prod\limits_{i=1}^{v-1}t_{i,v+1-1}!}\left(\frac{1}{2a_v}-d(\underset{\sim}{\mathbf{k}}^{v-1})\right)^{j_v-b_v^{+}-1}\right)\times$$

$$\frac{e^{-d(\mathbf{k}^{w+1})y}y^{\sum\limits_{i=1}^{w+1}t_{i,1}}}{\prod\limits_{i=1}^{w+1}t_{i,1}!}\prod_{i=w+2}^{r}\frac{e^{-d(\mathbf{k}^{w+1})x_i}x_i^{\sum\limits_{v=1}^{w+1}t_{v,i+1-v}}}{\prod\limits_{v=1}^{w+1}t_{v,i+1-v}!}$$

which completes the proof. ∎

We can now establish

Proposition 1. *We have*

$$F(u|\underset{\sim}{\mathbf{c}}_1^r,\underset{\sim}{\mathbf{c}}_2^s,2\underset{\sim}{\mathbf{m}}^r,\underset{\sim}{\mathbf{g}}_2^s)=\frac{1}{\prod\limits_{v=2}^{r}(2a_v)^{b_v+1}}\sum_{k_1=0}^{1}\sum_{j_1=0}^{k_1b_1^{+}}\sum_{\left(\sum\limits_{i=1}^{r}t_{1,i}=j_1\right)}\cdots\sum_{k_r=0}^{1}\sum_{j_r=0}^{k_rb_r^{+}}$$

$$\sum_{\left(\sum\limits_{i=1}^{1}t_{r,i}=j_r\right)}\frac{(-1)^{\sum\limits_{i=1}^{r}(k_i+j_i-t_{i,1})}}{(2a_1)^{j_1}}\prod_{i=2}^{r}\left(\frac{b_i^{+}!}{b_i!\prod\limits_{v=1}^{i-1}t_{v,i+1-v}!}\left(\frac{1}{2a_i}-d(\underset{\sim}{\mathbf{k}}^{i-1})\right)^{j_i-b_i^{+}-1}\right)\times$$

$$\frac{\left(\sum_{i=1}^{r} t_{i,1}\right)! u^{\sum_{v=1} r t_{v,1}}}{\prod_{i=1}^{r} t_{i,1}!} \sum_{\left(\sum_{i=r+1}^{r+s} l_i = \sum_{v=1}^{r} t_{v,1}\right)}$$

$$\prod_{i=r+1}^{r+s} \left(\frac{\Gamma\left(l_i + \frac{g_i}{2}\right)}{l_i! \Gamma\left(\frac{g_i}{2}\right)} \cdot \frac{1}{(2a_i)^{\frac{g_i}{2}}} \left(\frac{1}{2a_i} + d(\underset{\sim}{\mathbf{k}}^r) u \right)^{-\left(l_i + \frac{g_i}{2}\right)} \right).$$

Proof. We can use Lemma 1 to write

$$F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, 2\underset{\sim}{\mathbf{m}}^r, \underset{\sim}{\mathbf{g}}_2^s) = \int_0^{+\infty} \cdots \int_0^{+\infty} L_{r,r}\left(u \sum_{i=r+1}^{r+s} x_i\right) \prod_{i=r+1}^{r+s} f_i(x_i) dx_i$$

$$= \int_0^{+\infty} \cdots \int_0^{+\infty} \left(\frac{1}{\prod_{v=2}^{r} (2a_v)^{b_v+1}} \sum_{k_1=0}^{1} \sum_{j_1=0}^{k_1 b_1^+} \sum_{\left(\sum_{i=1}^{r} t_{1,i} = j_1\right)} \cdots \sum_{k_r=0}^{1} \sum_{j_r=0}^{k_r b_r^+} \right.$$

$$\sum_{\left(\sum_{i=1}^{1} t_{r,i} = j_r\right)} \frac{(-1)^{\sum_{v=1}^{r} k_v + j_v - t_{v,1}}}{(2a_1)^{j_1}} \prod_{w=2}^{r} \left(\frac{b_w^+!}{b_w! \prod_{v=1}^{w-1} t_{v,w+1-v}!} \right)$$

$$\left. \left(\frac{1}{2a_w} - d(\underset{\sim}{\mathbf{k}}^{w-1}) \right)^{j_v - b_v^+ - 1} \frac{e^{-d(\mathbf{k}^r) u} u^{\sum_{i=1}^{r} t_{i,1}}}{\prod_{i=1}^{r} t_{i,1}!} \right) \prod_{i=r+1}^{r+s} f_i(x_i) dx_i$$

(we point out that $\prod_{i=r+1}^{r} \frac{e^{d(\mathbf{k}^r)x_i} x_i^{\sum_{i=1}^{r} t_{i,1}}}{\prod_{i=1}^{r} t_{i,1}!} = 1$ since the lower index exceeds the upper one) thus getting

$$F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, 2\underset{\sim}{\mathbf{m}}^r, \underset{\sim}{\mathbf{g}}_2^s) =$$

$$\frac{1}{\prod_{v=2}^{r}(2a_v)^{b_v+1}} \sum_{k_1=0}^{1} \sum_{j_1=0}^{k_1 b_1^+} \sum_{\left(\sum_{i=1}^{r} t_{1,i}=j_1\right)} \cdots \sum_{k_r=0}^{1} \sum_{j_r=0}^{k_r b_r^+} \sum_{\left(\sum_{i=1}^{1} t_{r,i}=j_r\right)} \frac{(-1)^{\sum_{v=1}^{r}(k_v+j_v-t_{v,1})}}{(2a_1)^{j_1}} \times$$

$$\prod_{w=2}^{r} \left(\frac{b_w^+!}{b_w! \prod_{v=1}^{w-1} t_{v,w+1-v}!} \right) \left(\frac{1}{2a_w} - d(\underset{\sim}{\mathbf{k}}^{w-1}) \right)^{j_v - b_v^+ - 1} \int_0^{+\infty} \cdots \int_0^{+\infty} \times$$

$$e^{-d(\mathbf{k}^r)u \sum_{i=r+1}^{r+s} x_i} \frac{\left(u \sum_{i=r+1}^{r+s} x_i\right)^{\sum_{i=1}^{r} t_{i,1}}}{\prod_{i=1}^{r} t_{i,1}!} \prod_{i=r+1}^{r+s} f_i(x_i) dx_i.$$

Now

$$\int_0^{+\infty} \cdots \int_0^{+\infty} e^{-d(\mathbf{k}^r)u \sum_{i=r+1}^{r+s} x_i} \frac{\left(u \sum_{i=r+1}^{r+s} x_i\right)^{\sum_{i=1}^{r} t_{i,1}}}{\prod_{i=1}^{r} t_{i,1}!} \prod_{i=r+1}^{r+s} f_i(x_i) dx_i$$

$$= \sum_{\left(\sum_{i=r+1}^{r+s} l_i = \sum_{v=1}^{r} t_{v,1}\right)} \frac{\left(\sum_{v=1}^{r} t_{v,1}\right)! u^{\sum_{v=1}^{r} t_{v,1}}}{\prod_{v=1}^{r} t_{v,1}! \prod_{v=r+1}^{r+s} l_i!} \prod_{i=r+1}^{r+s} \int_0^{+\infty} \frac{e^{-\left(\frac{1}{2a_i}+d(\underset{\sim}{\mathbf{k}}^r)\right)x_i}}{\Gamma\left(\frac{g_i}{2}\right)(2a_i)^{\frac{g_i}{2}}} \times$$

$$x_i^{l_i+\frac{g_i}{2}-1} dx_i = \sum_{\left(\sum_{i=r+1}^{r+s} l_i = \sum_{v=1}^{r} t_{v,1}\right)} \frac{\left(\sum_{v=1}^{r} t_{v,1}\right)! u^{\sum_{v=1}^{r} t_{v,1}}}{\prod_{v=1}^{r} t_{v,1}! \prod_{v=r+1}^{r+s} l_i!} \times$$

$$\prod_{i=r+1}^{r+s} \left(\frac{\Gamma\left(l_i + \frac{g_i}{2}\right)}{\Gamma\left(\frac{g_i}{2}\right)(2a_i)^{\frac{g_i}{2}}} \left(\frac{1}{2a_i} + d(\underset{\sim}{\mathbf{k}}^r)u\right)^{-\left(l_i+\frac{g_i}{2}\right)} \right).$$

To compute the last integrals it suffices to use the transformation $z_i = \left(\frac{1}{2a_i} + d(\underset{\sim}{\mathbf{k}}^r)u\right) x_i$, $i = r+1, ..., r+s$, and use the well known gamma function. To complete the proof we need only to substitute the result we just obtained in the expression of $F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, 2\underset{\sim}{\mathbf{m}}^r, \underset{\sim}{\mathbf{g}}_2^s)$. ■

3. Monte carlo methods

Monte Carlo methods can be used to table distributions $F(u|\underset{\sim}{\mathbf{c}}_1^r, \underset{\sim}{\mathbf{c}}_2^s, \underset{\sim}{\mathbf{g}}_1^r, \underset{\sim}{\mathbf{g}}_2^s)$ in a straightforward way since it is easy to generate independent chi-squares. The exact formula derived above may be used to check the accuracy of these methods.

For instance, in a balanced variance components model in which the first factor crosses with the second that nests a third, the variance component $\sigma^2(\beta)$ associated with the second factor is not the difference between two ANOVA mean squares, see [2] pg. 35. Thus, a usual F test cannot be derived for the nullity of $\sigma^2(\beta)$, but using the known results on that model it is straightforward to get a generalized F test with $r = s = 2$.

The coefficients will be $c_{11} = p_1$, $c_{12} = 1 - p_1$, $c_{21} = p_2$ and $c_{22} = 1 - p_2$ with $\max\{p_2, 1-p_2\} \leq p_1$. Moreover, if the factors have three levels, we will have $g_1 = 2$, $g_2 = 12$, $g_3 = 4$ and $g_4 = 6$, so that both evenness conditions hold. In this case the exact expression for the distribution will be

$$F(z|p_1,p_2) = 1 - \frac{\left(1+\frac{3p_2}{(1-p_1)}z\right)^{-2}\left(1+\frac{2(1-p_2)}{(1-p_1)}z\right)^{-3}}{2}\sum_{i=0}^{5}\frac{(12z)^i}{i!\,(2(1-p_1))^i}$$

$$\sum_{j=0}^{i}\binom{i}{j}\left(\frac{2(1-p_1)\,p_2}{4(1-p_1)+12p_2 z}\right)^{j}(2+j-1)!\left(\frac{2(1-p_1)(1-p_2)}{6(1-p_1)+12(1-p_2)z}\right)^{i-j}\times$$

$$(3+i-j-1)! - \left(1-\frac{(1-p_1)}{6p_1}\right)^{-6}\left(1+\frac{p_2}{2p_1}z\right)^{-2}\left(1+\frac{(1-p_2)}{6p_1}z\right)^{-3}$$

$$+\frac{\left(1-\frac{(1-p_1)}{6p_1}\right)^{-6}\left(1+\frac{p_2}{2p_1}z\right)^{-2}\left(1+\frac{(1-p_2)}{6p_1}z\right)^{-3}}{120}\sum_{k=0}^{7}\frac{(12p_1-2(1-p_1))^k z^k}{k!\,(2p_1(1-p_1))^k}$$

$$\sum_{l=0}^{k}\binom{k}{l}\left(\frac{2(1-p_1)\,p_2}{4(1-p_1)+12p_2 z}\right)^{l} l!\left(\frac{2(1-p_1)(1-p_2)}{6(1-p_1)+12(1-p_2)z}\right)^{k-l}(2+k-l)!,$$

where for the sake of clarity of notation we omitted the degrees of freedom. In the next table we present the values given by the exact expression and by Monte Carlo methods (10000 runs for each value), the first value in each pair being given by the exact expression. For each pair we consider the values 1, 5, 10 and 20 for z.

p_2	z	p_1 0.5	0.6	0.7	0.8	0.9
0.1	1	0.506,0.501	0.524,0.526	0.543,0.542	0.576,0.574	0.576,0.574
	5	0.98,0.978	0.977,0.98	0.973,0.973	0.968,0.97	0.964,0.963
	10	0.998,0.998	0.997,0.997	0.996,0.996	0.995,0.996	0.994,0.995
	20	1,1	1,1	1,1	1,1	0.999,0.999
0.2	1	0.516,0.516	0.534,0.537	0.554,0.552	0.584,0.578	0.584,0.578
	5	0.985,0.987	0.982,0.982	0.978,0.977	0.974,0.972	0.97,0.97
	10	0.999,0.999	0.998,0.998	0.998,0.998	0.997,0.996	0.996,0.996
	20	1,1	1,1	1,1	1,1	1,1
0.3	1	0.522,0.519	0.541,0.537	0.56,0.563	0.588,0.591	0.588,0.591
	5	0.987,0.985	0.984,0.984	0.981,0.982	0.977,0.978	0.973,0.975
	10	0.999,0.999	0.999,0.999	0.998,0.998	0.997,0.997	0.997,0.997
	20	1,1	1,1	1,1	1,1	1,1
0.4	1	0.525,0.535	0.543,0.537	0.562,0.552	0.59,0.592	0.59,0.592
	5	0.988,0.987	0.985,0.985	0.982,0.981	0.978,0.974	0.973,0.972
	10	0.999,1	0.999,0.999	0.998,0.998	0.998,0.998	0.997,0.997
	20	1,1	1,1	1,1	1,1	1,1
0.5	1	0.522,0.525	0.541,0.555	0.56,0.554	0.588,0.587	0.588,0.587
	5	0.987,0.987	0.984,0.985	0.981,0.981	0.977,0.976	0.973,0.974
	10	0.999,0.999	0.999,0.999	0.998,0.999	0.997,0.997	0.997,0.996
	20	1,1	1,1	1,1	1,1	1,1
0.6	1	0.516,0.515	0.535,0.543	0.554,0.553	0.584,0.59	0.584,0.59
	5	0.986,0.986	0.982,0.984	0.979,0.98	0.975,0.975	0.97,0.972
	10	0.999,0.999	0.998,0.999	0.998,0.998	0.997,0.998	0.996,0.996
	20	1,1	1,1	1,1	1,1	1,1
0.7	1	0.505,0.502	0.524,0.518	0.544,0.541	0.577,0.578	0.577,0.578
	5	0.982,0.982	0.979,0.979	0.975,0.975	0.97,0.967	0.966,0.966
	10	0.998,0.999	0.998,0.998	0.997,0.997	0.996,0.996	0.995,0.996
	20	1,1	1,1	1,0.999	1,1	1,1
0.8	1	0.492,0.491	0.511,0.522	0.531,0.529	0.567,0.578	0.567,0.578
	5	0.975,0.974	0.972,0.975	0.967,0.97	0.963,0.965	0.958,0.958
	10	0.997,0.996	0.996,0.996	0.995,0.995	0.994,0.994	0.993,0.993
	20	1,1	1,0.999	1,1	0.999,0.999	0.999,0.999
0.9	1	0.479,0.484	0.496,0.502	0.516,0.512	0.554,0.552	0.554,0.552
	5	0.961,0.961	0.958,0.959	0.954,0.952	0.95,0.951	0.945,0.942
	10	0.994,0.994	0.993,0.993	0.991,0.991	0.99,0.99	0.988,0.989
	20	0.999,1	0.999,0.999	0.999,0.999	0.999,0.999	0.998,0.999

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Received 15 October 2002