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ESTIMATES FOR THE DISTRIBUTION OF THE FIRST EXIT TIME OF α-STABLE PROCESSES

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Abstract

The Varopoulos-Hardy-Littlewood theory and the spectral analysis are used to estimate the tail of the distribution of the first exit time of α-stable processes.

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1. Preliminaries

Let $\xi(t)$ $(t > 0)$ be an α-stable process on $\mathbf{R}^d$ $(d \geq 3)$ i.e., it is a Markov process with the strong Markov property, the transition kernel of which is given by the convolution with a function $p_t(x)$ and the Fourier transform of $p_t(x)$ has a form

$$\hat{p}_t(y) = \exp(-t\|y\|^{\alpha}), \ (1 \leq \alpha \leq 2).$$

The uniformity in time of α–stable processes implies that the family of operators

$$T_t f(x) = Ef(\xi_x(t)), \ (f \in L^p(\mathbf{R}^d))$$

forms a semigroup of contractions.

Let $f \in L^p(\mathbf{R}^d)$ $(1 \le p \le \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. The form of $\hat{p}_t$ gives us the following estimation:

$$\|T_t f\|_\infty \le \|p_t\|_q \|f\|_p.$$

Because

$$p_t(x) = \frac{1}{t^{d/\alpha}} p_1(\frac{x}{t^{1/\alpha}}),$$

we obtain that

$$\|p_t\|_q = t^{\frac{-dp}{\alpha}} \|p_1\|_q,$$

for any $1 \le q \le +\infty$. This implies

$$\|T_t f\|_\infty \le C t^{\frac{-dp}{\alpha}} \|f\|_p \ (p \ge 1), \tag{1}$$

where the constant C depends only on p.

Let U be a bounded domain in $\mathbf{R}^d$. Let ξ starts in $x \in U$ (let us denote this fact by $\xi_x(t)$). Trajectories of such processes are right–continuous and have left–hand side limits, so we can define the first exit time from U by

$$\tau_x = \inf\{t \mid \xi_x(t) \notin U\}.$$

Lemma 1 . *Let U be a domain in $\mathbf{R}^d$ and $\xi_x(t)$ be an α–stable process which starts in $x \in U$. Let us denote by*

$$I_x(t) = \mathbf{1}_{\{t < \tau_x\}}$$

the indicator of the set of all trajectories for which $t < \tau_x$. Let $f \in L^p(U)$, $1 \le p \le \infty$. The family of operators defined by

$$S_t f(x) = E f(\xi_x(t)) I_x(t) \tag{2}$$

forms a semigroup of contractions on $L^p(U)$.

Proof. By using of the strong Markov property of $\xi(t)$ we have

$$I_x(t)I_{\xi_x(t)}(s) = I_x(t+s).$$

Hence

$$S_t \circ S_s f(x) = E\, S_s f(\xi_x(t)) I_x(t) =$$

$$= E(E\, f(\xi_{\xi_x(t)}(s)) I_{\xi_x(t)}(s)) I_x(t) =$$

$$= E f(\xi_x(t)) I_x(t+s).$$

So, $S_t \circ S_s = S_{t+s}$. ■

The estimation (1) implies

Lemma 2. *Let $f \in L^p(U)$, $1 \le p \le \infty$. Then*

$$\|S_t f\|_\infty \le C t^{\frac{-dp}{\alpha}} \|f\|_p, \tag{3}$$

where constant C depends only on p, d, α.

Let D be the infinitesimal generator of S_t and let $N(\lambda)$ denote the dimension of its spectral projector $P(-\infty, \lambda)$.

Lemma 3. *For any bounded domain U there exists a constant C such that for every $\lambda \ge 0$*

$$N(\lambda) \le C \lambda^{\frac{d}{\alpha}}.$$

Proof. By (3) and the Varopoulos theory ([2], Theorem 1) we have

$$\|f\|_{2d/(d-\alpha)} \le C\|D^{\frac{1}{2}} f\|_2,\ f \in \mathrm{Dom}\left(D^{\frac{1}{2}}\right).$$

So, by the Levin–Solomyak generalization of the CLR inequality (see [1]), for any $V \geq 0$, $V \in L^{\frac{d}{\alpha}}(U)$ the number of the negative eigenvalues of the operator $D - V$ has an upper bound equals to

$$C_1 \int_U V^{\frac{d}{\alpha}} dx,$$

with some constant C_1 which depends only on α and U. Hence, for any $\lambda > 0$, the operator $D - \lambda Id$ has a finite number of the negative eigenvalues and, because this number is equal to $N(\lambda)$, we have

$$N(\lambda) \leq C_1 \mid U \mid \lambda^{\frac{d}{\alpha}}.$$

■

Let (X, μ) be a σ–finite measure space and let Q_t $(t > 0)$ be a submarkovian (strongly continuous) symmetric semigroup, i.e., for all $t > 0$, $Q_t : L^2(X) \longrightarrow L^2(X)$ is a symmetric operator, and for all $f \in L^2$ with $0 \leq f \leq 1$ we have $0 \leq Q_t f \leq 1$ (we note that S_t is such a semigroup).

Definition 1. Let $u(t, x)$ $(x \in X,\ t > 0)$ be a function on $(0, +\infty) \times X$. Let $u(t, \cdot) \in L^1 + L^\infty(X)$. We say that u is a subharmonic function (with respect to the semigroup Q_t) if

$$Q_t u(s, \cdot) \geq u(t + s, \cdot),\ t, s > 0$$

The proof of the following lemma can be found in [2] (Theorem 2).

Lemma 4. *Let $(Q_t;\ t > 0)$ be a submarkovian symmetric semigroup and let C, $n > 0$ and $1 \leq p < +\infty$ be such that*

$$\|Q_t f\|_\infty \leq C t^{-n/2p} \|f\|_p;\ \ t > 0,\ \ f \in L^p.$$

Then for every subharmonic function $u(t, x)$ and every $0 < s < r \leq +\infty$ we have

$$t^{n/2s} \|u(t, \cdot)\|_r \leq c t^{n/2r} \sup_t \|u(t, \cdot)\|_s, \tag{4}$$

where constant c depends only on C, n, p, r, s.

2. The main result

Lemma 3 implies that there exists a sequence of positive numbers $w_1 \leq w_2 \leq \dots$ and an orthonormal basis $\phi_1, \phi_2, \dots$ in $L^2(U)$ for which

$$D\phi_n = w_n \phi_n.$$

Lemma 5. *There exists $C > 0$ such that*

$$\sup_x \mid \phi_n \mid \leq C w_n^{\frac{2d}{\alpha}}.$$

Proof. Let us notice that

$$S_t \phi_n = \exp(-t w_n)\phi_n.$$

So,

$$u_n(x,t) = \exp(-t w_n)\phi_n(x)$$

is a harmonic function. Lemmas 2 and 4 give us that

$$\sup_x \mid \exp(-t w_n)\phi_n(x) \mid \leq C t^{\frac{-2d}{\alpha}} \sup_t \left(\int_U \mid \exp(-t w_n)\phi_n(x) \mid^2 dx \right)^{1/2}.$$

Thus

$$\sup_x \mid \phi_n(x) \mid \leq C \inf_t \left(t^{\frac{-2d}{\alpha}} \exp(t w_n) \right).$$

Hence

$$\|\phi_n\|_\infty \leq C w_n^{\frac{2d}{\alpha}}.$$

■

Theorem 1. *Let τ_x be a first exit time from a bounded domain U. Let w_1 be the smallest eigenvalue of the infinitesimal generator of S_t. Then for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for every $t > \varepsilon$*

$$P(t < \tau_x) \leq C(\varepsilon) \exp(-w_1 t).$$

Proof. Let us notice that

$$P(t < \tau_x) \leq \|S_t \mathbf{1}_U\|_\infty,$$

and that

$$\mathbf{1}_U = \sum_{n=1}^{\infty} a_n \phi_n,$$

where

$$\sum_{n=1}^{\infty} | a_n |^2 = \|\mathbf{1}_U\|_2^2 = | U |,$$

and the series is convergent in L^2. So,

$$S_t \mathbf{1}_U = \sum_{n=1}^{\infty} a_n \exp(-t w_n) \phi_n. \tag{5}$$

By using Lemma 5 we have that (5) converges uniformly on U and

$$\|S_t \mathbf{1}_U\|_\infty \leq C \sum_{n=1}^{\infty} \exp(-w_n t) w_n^{\frac{2d}{\alpha}}.$$

Since,

$$\sum_{n=1}^{\infty} \exp(-w_n t) w_n^{\frac{2d}{\alpha}} \leq C(\varepsilon) \exp(-t w_1)$$

we ended the proof. ■

Corollary 1. *For every $\varepsilon > 0$*

$$E \exp((w_1 - \varepsilon)\tau_x) < +\infty.$$

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