Discussiones Mathematicae
General Algebra and Applications 22 (2002) 153–159

FROBENIUS n-GROUP ALGEBRAS

Biljana Zeković

Faculty of Science, University of Montenegro
P.O. Box 211
8–1000 Podgorica
Serbia and Montenegro
e-mail: biljanaz@cg.yu

Abstract

Frobenius algebras play an important role in the representation theory of finite groups. In the present work, we investigate the (quasi) Frobenius property of n-group algebras. Using the (quasi-) Frobenius property of ring, we can obtain some information about constructions of module category over this ring ([2], p. 66–67).

Keywords: n-ary group (n-group, polyadic group), $(2, n)$-ring, n-group-ring (algebra), (quasi-) Frobenius property, Artinianity property, regular bilinear from, descending chain condition for left (right) ideals, univeral enveloping (or covering) group, annhilator.

2000 Mathematics Subject Classification: 20N15, 20C05, 20C07, 16S34, 17A40.

1. Introduction

Let G be an *n-group* with an n-ary operation

$$(a_1, a_2, ..., a_n) \mapsto a_1, a_2, ..., a_n$$

(for the theory of n-groups or polyadic groups, in another terminology, see, e.g., [1], [3] and [4]). One can construct an *n-group ring* in a similar way as usual group rings are constructed. Then the n-group ring (over the n-group G), denoted here by RG, is a special $(2, n)$-*ring* defined in the following way:

Elements of RG can be written in the unique way as finite sums:

$$\sum_i r_i g_i \quad (\text{where } \ r_i \in R, g_i \in G)\,,$$

where R is an associative ring with unit element and G is an n-group. The addition in RG is treated as the addition of the coefficients r_i, and a multiplication is n-ary. Elements $g_i \in G$ are multiplied as in the n-group G (see [5]).

n-Group G can be embeded into universal enveloping group G^*. An *evenloping group* is often called a coverning group or, in another terminology, a *Post corvening group*. There exists a normal subgroup G_0 of the group G^* such that

$$G_0 = \{g_1 ... g_{n-1} : \quad g_i \in G, \ \ i = 1, ..., n\}$$

(see [1], [3] and [4]) and G^*/G_0 is a cyclic group of order $n-1$. Besides,

$$G_0 = \left\{z^{-1}t : \quad t \in G\right\},$$

where z is a fixed element from G. The subgroup I of the addition group of n-group ring RG is called (k)-*ideal* if

$$(RG)^{k-1} I (RG)^{n-k} \subseteq I,$$

for any $k \in \{1, 2, ..., n\}$.

If the above inclusion holds for each $k \in \{1, 2, ..., n\}$, then I is called an *ideal*. For $k = n$ $\ (k = 1)$, (n)-ideal ((1)-ideal, resp.) I is called a *left* (*right*, resp.) ideal of n-group ring RG. n-Group ring RG is said to be *Artinian* (left or right) if the descending chain condition for ideals (left or right, respectively) is satisfied.

All additional necessary notations and definitions can be founded in the papers listed in References.

1. Quasi-Frobenius n-group rings

Definition 1.1. Artinian n-group ring RG is *quasi-Frobenius* (or QF-*ring*) if for any left ideal L and right ideal H the following two conditions hold

$$l(r(L)) = L \quad \text{and} \quad r(l(H)) = H,$$

where

$$r(L) = \{x \in RG_0 : \quad Lx = 0\}$$

is the *right annihilator of the left ideal* L and

$$l(H) = \{x \in RG_0 : \quad xH = 0\}$$

is the *left annihilator of the right ideal* H.
Note that

$$l(r(L)) \supseteq L$$

and

$$r(l(H)) \supseteq H.$$

Lemma 1.1. *Let I be a left ideal in the group ring RG_0 and J a left ideal in the n-group ring RG. Then,*

$$z^{-1}r(I)z = r(J),$$

for any $z \in G$.

Proof. Suppose that $x \in r(J)$. Then,

$$Jx = 0, \qquad x \in RG_0.$$

Hence,

$$Izx = 0,$$

because

$$I = Jz^{-1}$$

(see [5]), i.e.

$$Izxz^{-1} = 0, \quad zxz^{-1} \in RG_0.$$

Therefore,

$$zxz^{-1} \in r(I),$$

i.e.

$$x \in z^{-1}r(I)z.$$

Conversely, let
$$x \in z^{-1}r(I)z.$$
Then,
$$x = z^{-1}yz, \quad y \in r(I)$$
and
$$Iy = 0,$$
i.e.
$$Jz^{-1}y = 0,$$
because
$$I = Jz^{-1}$$
and
$$Jz^{-1}yz = 0, \quad z^{-1}yz \in RG_0.$$
Therefore,
$$x = z^{-1}yz \in r(J).$$
■

Lemma 1.2. *Let I be left ideal of RG_0 and J be left ideal of RG. Then:*
$$l(r(I)) = I \Leftrightarrow l(r(J)) = J.$$

Proof. Let $l(r(I)) = I$ and $x \in l(r(J))$, where $x \in RG$. Then,
$$xr(J) = 0, \quad \text{i.e.} \quad xz^{-1}r(I)z = 0.$$
Hence,
$$xz^{-1}r(I) = 0.$$
We have
$$xz^{-1} \in l(r(I)) = I, \quad xz^{-1} \in RG_0,$$
and
$$x \in Iz = J.$$
Therefore, we obtain
$$l(r(J)) \subseteq J$$
and the equallity follows, because $l(r(J)) \supseteq J$. Conversely, let $l(r(J)) = J$ and $x \in l(r(I))$, for $x \in RG_0$.

Then,

$$xr(I) = 0, \quad \text{i.e.} \quad xzr(J)z^{-1} = 0.$$

Hence,

$$xzr(J) = 0.$$

We have

$$xz \in l(r(J)) = J, \quad xz \in RG$$

and

$$x \in Jz^{-1} = I.$$

We obtain

$$l(r(I)) \subseteq I$$

and, since the oposit inclusion is true, the equallity follows. ■

The proof of the next lemma is similar.

Lemma 1.3. *Let I and J be as in Lemma 1.2. Then*

$$r(l(I)) = I \Leftrightarrow r(l(J)) = J.$$

■

Theorem 1.1. *n-Group ring RG is quasi-Frobenius if and anly if the group ring RG_0 is quasi-Frobenius.*

The proof follows from previous lemmas and an existing of a bijection between left (right) ideals in RG and RG_0. The descending chain condition for left (right) ideals, i.e. left (right) Artinianity is satisfied simultanosly in RG and RG_0. ■

Theorem 1.2. *n-Group ring RG is QF-ring if and only if G is a finite n-group and R is a QF-ring.*

The proof follows from Theorem 1.1 and the statement on p. 68 in [2]. If G_0 is a finite group, then group ring RG_0 is QF-ring if and only if R is QF-ring. ■

2. Frobenius n-group algebras

Definition 2.1. n-Group algebra $A = RG$ (of n-group G over the field R) is *Frobenius* if and only if n-group G is finite and there exists a regular bilinear mapping $\varphi : A \times A \to R$ with the "associative property (on RG)":

$$\varphi(a_1...a_n, b) = \varphi\left(a_1, a_2...a_n b\right),$$

for all $a_1, ..., a_n, b \in A$.

Theorem 2.1. *n-Group algebra RG (of n-group G over the field R) is Frobenius if and only if the group algebra RG_0 is Frobenius.*

Proof. Let φ be a regular bilinear mapping

$$\varphi : RG_0 \times RG_0 \to R,$$

with "an associative property (on RG_0)":

$$\varphi(a, bc) = \varphi(ab, c),$$

for all $a, b, c \in RG_0$.

Define a bilinear mapping $\psi : RG \times RG \to R$ in the following way:

$$\psi(a, b) = \varphi\left(z^{-1}a, bz^{-1}\right).$$

Then,

$$\psi\left(a_1...a_n, b\right) = \varphi\left(z^{-1}a_1\left(a_2...a_n\right), bz^{-1}\right) =$$

$$= \varphi\left(z^{-1}a_1, \left(a_2...a_n\right)bz^{-1}\right) = \psi\left(a_1, a_2...a_n b\right).$$

Conversely, let be given a regular bilinear mapping $\psi : RG \times RG \to R$ with the associative property (on RG):

$$\psi\left(a_1...a_n, b\right) = \psi\left(a_1, a_2...a_n b\right).$$

Define a biliner mapping $\varphi : RG_0 \times RG_0 \to R$ as follows:

$$\varphi(x, y) = \psi(zx, yz).$$

Then,

$$\begin{aligned}\varphi(x, yw) = \psi(zx, ywz) &= \\ &= \psi(zx, y_1 ... y_{n-1}(wz)) = \\ &= \psi((zx)y_1 ... y_{n-1}, wz) = \\ &= \psi(zxy, wz) = \varphi(xy, w).\end{aligned}$$

■

REFERENCES

[1] V.A. Artamonov, *Free n-groups* (Russian), Mat. Zametki **8** (1970), 499–507.

[2] L.A. Bokut, I.V. L'vov and V.K. Kharchenko, *Nonkommutative Rings* (Russian), vol. **18** of "Itogi Nauki i Tekhniki", Izdat. VINITI, Moscov 1988.

[3] A.G. Kurosh, *General Algebra. – Lectures of the 1969–1970 Academic Year* (Russian), Izdat, "Nauka", Moscov 1974.

[4] E.L. Post, *Polyadic groups*, Trans. Amer. Math. Soc. **48** (1940), 208–350.

[5] B. Zeković and V.A. Artamonov, *n-Group rings and their radicals* (Russian), Abelian Groups and Modules (Tomsk), **11** (1992), 3–7 (in Russian).

[6] B. Zeković and V.A. Artamonov, *Connections betweem some properties of n-group rings and group rings* (Russian), Math. Montisnigri **11** (1999), 151–158.

Received 1 July 2002
Revised 20 February 2003