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LEXICO EXTENSION AND A CUT COMPLETION OF A HALF l-GROUP

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Abstract

The cut completi on of an hl-group G with the abelian increasing part is investigated under the assumption that G is a lexico extension of its hl-subgroup.

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0. Introduction

The notion of a half l-group as a generalization of the notion of an l-group was introduced and studied by M. Giraudet and F. Lucas [4].

R.N. Ball [1] has defined the notion of a cut completion of an l-group.

In this paper we define the notions of a cut completion and a lexico extension of a half l-group. We prove a theorem on a cut completion of a half l-group having an abelian increasing part which can be expressed as a nontrivial lexico extension. A particular case of this theorem is a result of J. Jakubík [5] dealing with a cut completion of an abelian l-group.

1. Preliminaries

Let G be an abelian l-group. G is called a *lexico extension of its l-subgroup* $A \neq \{0\}$ if

(i) A is a convex l-subgroup of G,

(ii) if $0 < g \in G,\ g \notin A$, then $g > a$ for each $a \in A$.

If G is a lexico extension of A, we shall write $G = \langle A \rangle$. If $G = \langle A \rangle$, then A is an l-ideal of G and (cf. [3] and [2])

(a) A is comparable to all convex l-subgroups of G (i.e., if A' is a convex l-subgroup of G then either $A' \subseteq A$ or $A \subseteq A'$).

(b) G/A is a linearly ordered group.

Let G be a group and a partially ordered set. Set

$$G\uparrow = \{g \in G : x \leq y \Rightarrow g + x \leq g + y \text{ for all } x, y \in G\},$$

$$G\downarrow = \{g \in G : x \leq y \Rightarrow g + x \geq g + y \text{ for all } x, y \in G\}.$$

$G\uparrow$ ($G\downarrow$) is called the *increasing* (*decreasing*) part of G.

G is said to be a *half l-group* (abbreviated to an *hl-group*) if the following conditions are satisfied (cf. [4]):

(i) the partial order $\leq$ on G is non-trivial,

(ii) if $x, y, g \in G$ and $x \leq y$, then $x + g \leq y + g$,

(iii) $G = G\uparrow \cup G\downarrow$,

(iv) $G\uparrow$ is an l-group.

If $G\uparrow$ is a linearly ordered group, then hl-group G will be called a *half linearly ordered group.*

Every l-group $G \neq \{0\}$ is a special case of an hl-group with $G\downarrow = \emptyset$.

We denote by $\mathcal{HL}$ the class of all hl-groups that fail to be l-groups.

The following results will be applied in the next.

Proposition 1.1 (cf. [4]). *Let* $G \in \mathcal{HL}$. *Then*

(i) $G\uparrow$ *is a subgroup of the group* G *and* $G\uparrow$ *has index* 2,

(ii) $G\uparrow$ *and* $G\downarrow$ *are isomorphic groups and also dually isomorphic lattices,*

(iii) *if* $x \in G\uparrow$ *and* $y \in G\downarrow$, *then* x *and* y *are incomparable,*

(iv) *the set* $\{g \in G:\ g \neq 0,\ 2g = 0\}$ *is nonempty.*

Let G be an hl-group. A subgroup $A \neq \{0\}$ of G is called a half l-subgroup (abbreviated to an hl-subgroup) if $A\uparrow = A \cap G\uparrow$ is an l-subgroup of $G\uparrow$. If A is an hl-subgroup (proper hl-subgroup) of G we use the notation $A \leq G$ ($A < G$). We say that an hl-subgroup A of G is convex in G if $A\uparrow$ is convex in $G\uparrow$. A convex hl-subgroup A of G is said to be an hl-ideal of G if $A\uparrow$ is a normal subgroup of G. According to 1.1 $G\uparrow$ is an hl-ideal of G.

Let G be an hl-group, $G \in \mathcal{HL}$ and $A\uparrow$ an hl-ideal of G, $A \in \mathcal{HL}$. We can form the factor group $\overline{G} = G/A\uparrow$. For elements $g_1 + A\uparrow$, $g_2 + A\uparrow \in \overline{G}$, we put $g_1 + A\uparrow \leq g_2 + A\uparrow$ if and only if there exist $g_1' \in g_1 + A\uparrow$ and $g_2' \in g_2 + A\uparrow$ with $g_1' \leq g_2'$. Then $\overline{G}$ is a partially ordered set and to each $g_1' \in g_1 + A\uparrow$ there exists $g_2' \in g_2 + A\uparrow$ such that $g_1' \leq g_2'$. It can be easily verified that if $A < G$, then $\overline{G}$ is an hl-group with the increasing part $\overline{G}\uparrow = \{g + A\uparrow : g \in G\uparrow\}$ and decreasing part $\overline{G}\downarrow = \{g + A\uparrow : g \in G\downarrow\}$.

If $A = G$ then $\overline{G}$ is trivially ordered. Hence $\overline{G}$ fails to be an hl-group.

A 1-1 mapping φ from an hl-group G onto an hl-group G' is called an hl-isomorphism if φ is a group homomorphism and if $\varphi|G\uparrow$ is a lattice homomorphism of $G\uparrow$ onto $G'\uparrow$.

2. LEXICO EXTENSION OF AN *hl*-SUBGROUP

Let G be an hl-group, $G \in \mathcal{HL}$ with the abelian increasing part $G\uparrow$. Let A be an hl-subgroup of G, $A \in \mathcal{HL}$. If $G\uparrow$ is a lexico extension of $A\uparrow$, then we say that G is a lexico extension of A and we express this situation by writing $G = \langle A \rangle_h$.

Lemma 2.1. *Let* $G = \langle A \rangle_h$. *Then*

(i) A *is an hl-ideal of* G,

(ii) *if* $A < G$, *then* $\overline{G} = G/A\uparrow$ *is a half linearly ordered group.*

Proof. (i) We have to show that $A\uparrow$ is normal in G. Since $A\uparrow$ is a convex l-subgroup of $G\uparrow$, $-g+A\uparrow+g$ $(g\in G)$ is a convex subset of $G\uparrow$. It is a routine to verify that $-g+A\uparrow+g$ is a subgroup of $G\uparrow$. Let $-g+a_1+g,\ -g+a_2+g\in -g+A\uparrow+g,\ g\in G$, and $a_1,a_2\in A\uparrow$. It is easy to verify that in $G\uparrow$ we have $(-g+a_1+g)\vee(-g+a_2+g)=-g+(a_1\vee a_2)+g$ for each $g\in G\uparrow$, $(-g+a_1+g)\vee(-g+a_2+g)=-g+(a_1\wedge a_2)+g$ for each $g\in G\downarrow$ and dually. Hence $-g+A\uparrow+g$ is a sublattice of $G\uparrow$ for each $g\in G$. By summarizing we have that $-g+A\uparrow+g$ is a convex l-subgroup of $G\uparrow$ for each $g\in G$. By (a), $A\uparrow$ and $-g+A\uparrow+g$ are comparable. The fact that $G\uparrow$ is abelian implies $-g+A\uparrow+g=A\uparrow$ for all $g\in G\uparrow$. Suppose that $g\in G\downarrow$ and $A\uparrow\subseteq -g+A\uparrow+g$. Let $a\in A\uparrow$. Then $a=-g+a_0+g$, where $a_0\in A\uparrow$ and hence $-g+a+g=-2g+a_0+2g$. Since $2g\in G\uparrow$, we get $-g+a+g\in A\uparrow$. Thus $-g+A\uparrow+g\subseteq A\uparrow$ for all $g\in G$. Therefore, $A\uparrow$ is normal in G.

(ii) follows from the property (b) of a lexico extension. ■

If $G=\langle A\rangle_h$, then Lemma 2.1 yields that $A\uparrow$ is a normal subgroup of G, but A need not be normal in G.

Examples. Let M be the set of all functions $f:R\to R$; $f(x)=\pm x+k,\ k\in R$. If a binary operation on M is defined as a composition (i.e., $fg(x)=f(g(x))$ for all $x\in R$) and a binary relation $\leq$ on M is defined pointwise, then M is a half linearly ordered group, with $M\uparrow=\{f:f(x)=x+k\}$ and $M\downarrow=\{f:f(x)=-x+k\}$. Now, let $H=\{(f_1,f_2):f_1,f_2\in M\uparrow\}$. For each $(f_1,f_2),(f_1',f_2')\in H$ we put $(f_1,f_2)\leq(f_1',f_2')$ if and only if either $f_1<f_1'$ or $f_1=f_1'$ and $f_2\leq f_2'$. Then H is a linearly ordered set that is called the *lexicographic product* of the two linearly ordered sets $M\uparrow$ and we use the denotation $H=M\uparrow\circ M\uparrow$. Analogously we can construct $K=M\downarrow\circ M\downarrow$. If a binary operation on H is defined componentwise, then H is a linearly ordered group. Therefore, $G=H\cup K$ is a half linearly ordered group with $G\uparrow=H, G\downarrow=K$. Let $A_1=\{(id,g)\in G:g\in M\uparrow\}$ (id is an identity function) and $A_2=\{(g_1,g):g_1$ is a fixed element of $M\downarrow,\ g\in M\downarrow\}$. Then $A=A_1\cup A_2$ is a half linearly ordered group, $A\uparrow=A_1$, $A\downarrow=A_2$. We have $G=\langle A\rangle_h$, but A fails to be normal in G. In fact, for all $f\in M, f\neq id, g_1$ we have $f^{-1}g_1f\neq g_1$, thus $(f,f)^{-1}(g_1,g)(f,f)\notin A_2$ for each $(g_1,g)\in A_2$.

Assume that G is an hl-group, $G\in\mathcal{HL}$ and that $A\in\mathcal{HL}$ is an hl-ideal of G such that A is a normal subgroup of G. Define a partial order on the factor group G/A (and also on the factor group $G\uparrow/A\uparrow$) analogously as above on $G/A\uparrow$. Then G/A is a lattice ordered group. The mapping $f : G\uparrow /A\uparrow \to G/A$ defined by $f(g+A\uparrow) = g+A, g \in G\uparrow$ is an isomorphism of the lattice ordered group $G\uparrow /A\uparrow$ onto G/A.

Suppose that $G = \langle A\rangle_h$ and that A is normal in G. Then, by the property (b) of a lexico extension, we have that G/A is a linearly ordered group.

Lemma 2.2. *Let* $G = \langle A\rangle_h$, $g_1+A\uparrow,\ g_2+A\uparrow \in \overline{G}$, *and let* $g_1+A\uparrow < g_2+A\uparrow$ *Then* $g_1 < g_2$.

Proof. From $g_1 + A\uparrow < g_2 + A\uparrow$ it follows that either $g_1, g_2 \in G\uparrow$ or $g_1, g_2 \in G\downarrow$. Now, let $g_1, g_2 \in G\uparrow$. There exists $g_2' \in g_2 + A\uparrow$ such that $g_1 < g_2'$. Hence $g_2' \in G\uparrow$ and $g_2' - g_1 > 0$. Since $g_2' - g_1 \notin A\uparrow$ and $g_2' - g_2 \in A\uparrow$, we get $g_2' - g_1 > g_2' - g_2$. Hence $-g_1 > -g_2$ and $g_1 < g_2$. Suppose that $g_1, g_2 \in G\downarrow$. There exists $g_2'' \in g_2 + A\uparrow$ with $g_1 < g_2''$. Hence $g_2'' \in G\downarrow,\ g_2'' - g_1 \in G\uparrow,\ g_2'' - g_1 \notin A\uparrow$ and $g_2'' - g_1 > 0$. Further, we have $g_2'' - g_2 \in A\uparrow$. Then $g_2'' - g_1 > g_2'' - g_2$ implies that $-g_1 < -g_2$ and so $g_1 < g_2$. ■

Corollary. *Let* $G = \langle A\rangle_h$. *Then* G *is a half linearly ordered group if and only if* A *is a half linearly ordered group.*

For the remaining part of this section, we assume that G, A and B are hl-groups from $\mathcal{HL}$ such that

(I) $G\uparrow$ and $B\uparrow$ are abelian l-groups,

(II) $G = \langle A\rangle_h,\ A < G$,

(III) A is an hl-subgroup of B,

(IV) $G \cap B = A$.

According to Proposition 1.1, there exists an element $a \in A\downarrow$ of order 2.

Form the set

$$H_0 = \{(g, b) :\ \text{either } g \in G\uparrow, b \in B\uparrow\ \text{ or } g \in G\downarrow, b \in B\downarrow\}.$$

For elements $(g_1, b_1), (g_2, b_2) \in H_0$, we set

$$(g_1, b_1) \equiv (g_2, b_2)$$

if $g_1 - g_2 \in A\uparrow,\ b_1 - b_2 \in A\uparrow,\ g_1 - g_2 = b_2 - b_1$ and if either $g_1, g_2 \in G\uparrow,\ b_1, b_2 \in B\uparrow$ or $g_1, g_2 \in G\downarrow,\ b_1, b_2 \in B\downarrow$.

The relation $\equiv$ is an equivalence. It is clear that the relation $\equiv$ is reflexive and symmetric. To establish the transitivity, suppose that $(g_1, b_1) \equiv (g_2, b_2)$, $(g_2, b_2) \equiv (g_3, b_3)$. We will consider only the following case. Let $g_1, g_2 \in G \downarrow$, and $b_1, b_2 \in B \downarrow$. Then $g_3 \in G \downarrow$, and $b_3 \in B \downarrow$. We have $g_1 - g_2 \in A \uparrow$, $b_2 - b_1 \in A \uparrow$, $g_1 - g_2 = b_2 - b_1$, $g_2 - g_3 \in A \uparrow$, $b_3 - b_2 \in A \uparrow$, and $g_2 - g_3 = b_3 - b_2$. By (I) and (II), $A \uparrow$ is abelian. Then $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) = (b_2 - b_1) + (b_3 - b_2) = (b_3 - b_2) + (b_2 - b_1) = b_3 - b_1$. Hence $g_1 - g_3 \in A \uparrow$ and $b_3 - b_1 \in A \uparrow$. Therefore $(g_1, b_1) \equiv (g_3, b_3)$

Denote

$$\overline{(g,b)} = \{(g', b') \in H_0 : \ (g,b) \equiv (g', b')\},$$
$$H = \{\overline{(g,b)} : \ (g,b) \in H_0\}.$$

Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$. We put

$$\overline{(g_1, b_1)} + \overline{(g_2, b_2)} = \overline{(g_1 + g_2, b_1 + b_2)}.$$

The binary operation $+$ on H is correctly defined; $\overline{(0,0)}$ is a neutral element and $\overline{(-g, -b)}$ is an inverse to $\overline{(g,b)}$.

We have

Lemma 2.3. *$(H, +)$ is a group.* ■

Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$. We put

$$\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$$

if either $g_1 < g_2$ and $g_1 - g_2 \notin A \uparrow$ or $g_1 - g_2 \in A \uparrow$ and $g_1 - g_2 \leq b_2 - b_1$.

The definition implies that either $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Now we verify that the relation $\leq$ is correctly defined. Let $\overline{(g_1', b_1')} = \overline{(g_1, b_1)}$, $\overline{(g_2', b_2')} = \overline{(g_2, b_2)}$.

Assume that $g_1 < g_2$, $g_1 - g_2 \notin A \uparrow$. Then $g_1 + A \uparrow < g_2 + A \uparrow$ and $g_1, g_2 \in G \uparrow$ or $g_1, g_2 \in G \downarrow$. Since $g_1 - g_1' \in A \uparrow$ and $g_2 - g_2' \in A \uparrow$, we get $g_1 + A \uparrow = g_1' + A \uparrow$ and $g_2 + A \uparrow = g_2' + A \uparrow$. With respect to Lemma 2.2, we get $g_1' < g_2'$. Suppose that $g_1' - g_2' \in A \uparrow$. Then $g_1 - g_2 = (g_1 - g_1') + (g_1' - g_2') + (g_2' - g_2) \in A \uparrow$, a contradiction. Hence $g_1' - g_2' \notin A \uparrow$.

Assume that $g_1 - g_2 \in A\uparrow, g_1 - g_2 \leq b_2 - b_1$. Then $g'_1 - g'_2 = (g'_1 - g_1) + (g_1 - g_2) + (g_2 - g'_2) \leq (b_1 - b'_1) + (b_2 - b_1) + (b'_2 - b_2) = (b_1 - b'_1) + (b'_2 - b_2) + (b_2 - b_1) = (b_1 - b'_1) + (b'_2 - b_1) = (b'_2 - b_1) + (b_1 - b'_1) = b'_2 - b'_1$. We also have shown that $g'_1 - g'_2 \in A\uparrow$.

It is evident that the relation $\leq$ is reflexive.

Let $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$, $\overline{(g_2, b_2)} \leq \overline{(g_1, b_1)}$. Then $g_1 - g_2 \in A\uparrow$, $g_1 - g_2 \leq b_2 - b_1$ and $g_2 - g_1 \leq b_1 - b_2$. Hence $g_1 - g_2 = b_2 - b_1$ and so $\overline{(g_1, b_1)} = \overline{(g_2, b_2)}$. The antisymmetry is satisfied.

Let $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$, $\overline{(g_2, b_2)} \leq \overline{(g_3, b_3)}$.

(α) Assume that $g_1 < g_2$, $g_1 - g_2 \notin A\uparrow$, $g_2 < g_3$, and $g_2 - g_3 \notin A\uparrow$. We will consider only the case that $g_1, g_2 \in G\downarrow$. Then also $g_3 \in G\downarrow$ and $g_1 < g_3$. Assume that $g_1 - g_3 \in A\uparrow$. Then $g_1 + A\uparrow = g_3 + A\uparrow$. Since $g_1 + A\uparrow$ is a convex subset of $G\downarrow$ and $g_1 < g_2 < g_3$, we obtain $g_2 \in g_1 + A\uparrow$. Hence $g_1 - g_2 \in A\uparrow$, a contradiction.

(β) Assume that $g_1 - g_2 \in A\uparrow$, $g_1 - g_2 \leq b_2 - b_1$, $g_2 - g_3 \in A\uparrow$, and $g_2 - g_3 \leq b_3 - b_2$. Then $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) \in A\uparrow$, and $g_1 - g_3 = (g_1 - g_2) + (g_2 - g_3) \leq (b_2 - b_1) + (b_3 - b_2) = (b_3 - b_2) + (b_2 - b_1) = b_3 - b_1$.

(γ) Assume that $g_1 < g_2$, $g_1 - g_2 \notin A\uparrow$, $g_2 - g_3 \in A\uparrow$, and $g_2 - g_3 \leq b_3 - b_2$. We will consider only the case $g_1, g_2 \in G\downarrow$. Hence $g_3 \in G\downarrow$, $g_2 - g_1 > 0$ and $g_2 - g_1 \notin A\uparrow$. From this it follows that $g_2 - g_1 > g_2 - g_3$, $-g_1 < -g_3$ and $g_1 < g_3$. Suppose that $g_1 - g_3 \in A\uparrow$. Then $g_1 - g_2 = (g_1 - g_3) + (g_3 - g_2) \in A\uparrow$, a contradiction.

(δ) Suppose that $g_1 - g_2 \in A\uparrow$, $g_1 - g_2 \leq b_2 - b_1$, $g_2 < g_3$, and $g_2 - g_3 \notin A\uparrow$. The case is analogous to (γ).

In all cases (α)-(δ) we get $\overline{(g_1, b_1)} \leq \overline{(g_3, b_3)}$, i.e the relation $\leq$ is transitive.

We have shown that the following lemma is valid.

Lemma 2.4. *$(H, \leq)$ is a partially ordered set.* ■

Lemma 2.5. *Let $\overline{(g_1, b_1)}, \overline{(g_2, b_2)}, \overline{(g_3, b_3)} \in H$, $\overline{(g_1, b_1)} \leq \overline{(g_2, b_2)}$. Then $\overline{(g_1, b_1)} + \overline{(g_3, b_3)} \leq \overline{(g_2, b_2)} + \overline{(g_3, b_3)}$.*

Proof. We will consider only the case that $g_1, g_2, g_3 \in G\downarrow$.

Suppose that $g_1 < g_2$ and $g_1 - g_2 \notin A\uparrow$. Then $g_1 + g_3 < g_2 + g_3$ and $(g_1 + g_3) - (g_2 + g_3) = g_1 - g_2 \notin A\uparrow$.

Assume that $g_1 - g_2 \in A\uparrow$ and $g_1 - g_2 \le b_2 - b_1$. Then $(g_1 + g_3) - (g_2 + g_3) \in A\uparrow$ and $(g_1 + g_3) - (g_2 + g_3) = g_1 - g_2 \le b_2 - b_1 = (b_2 + b_3) - (b_1 + b_3)$. Therefore $\overline{(g_1, b_1)} + \overline{(g_3, b_3)} \le \overline{(g_2, b_2)} + \overline{(g_3, b_3)}$. ■

Form the sets

$$H\uparrow = \{\overline{(g,b)} : \ g \in G\uparrow, b \in B\uparrow\}, \ H\downarrow = \{\overline{(g,b)} : \ g \in G\downarrow, b \in B\downarrow\}.$$

Then we have

Lemma 2.6. $H = (H\uparrow) \cup (H\downarrow)$. ■

Lemma 2.7. *$H\uparrow$ is an increasing part and $H\downarrow$ is a decreasing part of H.*

Proof. Assume that $\overline{(g_1, b_1)}, \overline{(g_2, b_2)} \in H$, $\overline{(g_1, b_1)} \le \overline{(g_2, b_2)}$ and $\overline{(g_3, b_3)} \in H\downarrow$. We intend to show that $H\downarrow$ is a decreasing part of H, i.e., that $\overline{(g_3, b_3)} + \overline{(g_2, b_2)} \le \overline{(g_3, b_3)} + \overline{(g_1, b_1)}$ is valid.

Let $g_1 < g_2$, $g_1 - g_2 \notin A\uparrow$. Then $g_3 + g_1 > g_3 + g_2$. Suppose that $(g_3 + g_2) - (g_3 + g_1) \in A\uparrow$. With respect to Lemma 2.1, $A\uparrow$ is normal in G. Thus $g_2 - g_1 \in -g_3 + A\uparrow + g_3 \subseteq A\uparrow$. Hence $g_1 - g_2 \in A\uparrow$, a contradiction.

Let $g_1 - g_2 \in A\uparrow$ and $g_1 - g_2 \le b_2 - b_1$. By using the normality of $A\uparrow$ in G, we obtain $g_3 + g_2 - (g_3 + g_1) = g_3 + (g_2 - g_1) - g_3 \in A\uparrow$. There exist elements $g_3' \in G\uparrow$ and $b_3' \in B\uparrow$ such that $g_3 = a + g_3'$, $b_3 = a + b_3'$. From $g_2 - g_1 \ge b_1 - b_2$, it follows $a + g_3' - g_3' + g_2 - g_1 + a \le a + b_3' - b_3' + b_1 - b_2 + a$, $(a + g_3' + g_2) - (a + g_3' + g_1) \le (a + b_3' + b_1) - (a + b_3' + b_2)$ and $(g_3 + g_2) - (g_3 + g_1) \le (b_3 + b_1) - (b_3 + b_2)$.

In an analogous way, we show that $H\uparrow$ is an increasing part of H. ■

$H\uparrow$ is a group (subgroup of H) and a partially ordered set (a partial order is inherited from H). Then according to Lemmas 2.5 and 2.7, $H\uparrow$ is a partially ordered group.

Lemma 2.8. *$H\uparrow$ is an l-group.*

Proof. It is sufficient to prove that there exists $\sup\{\overline{(0,0)}, \overline{(g,b)}\}$ for each $\overline{(g,b)} \in H\uparrow$. If $g \notin A\uparrow$ then $g > 0$ or $g < 0$. Hence $\overline{(g,b)}$ and $\overline{(0,0)}$ are comparable. If $g \in A\uparrow$ then $g + b \in B\uparrow$ and $\overline{(g,b)} = \overline{(0, g+b)}$. Let

$b' = \sup\{0, g + b\}$ in $B\uparrow$. By using the same procedure as in the proof of Lemma 2.4 in [5], we obtain $\overline{(0, b')} = \sup\{\overline{(0,0)}, \overline{(g,b)}\}$. ■

From Lemmas 2.3–2.8 it follows

Lemma 2.9. *H is an hl-group, $H \in \mathcal{HL}$.* ■

Recall that there is $a \in A\downarrow$, an element of order 2 (by Proposition 1.1), and that, by (IV), $A \subseteq B$. Define the mapping φ of G into H by $\varphi(g) = \overline{(g, 0)}$ if $g \in G\uparrow$ and $\varphi(g) = \overline{(g, a)}$ if $g \in G\downarrow$. Then φ is an hl-isomorphism of the hl-group G into H.

If we put $\psi(b) = \overline{(0, b)}$ for each $b \in B\uparrow$ and $\psi(b) = \overline{(a, b)}$ for each $b \in B\downarrow$, then ψ is an hl-isomorphism of the hl-group B into H.

If $x \in G \cap B$ then $\varphi(x) = \psi(x)$. In fact, if $x \in (G \cap B)\uparrow = (G\uparrow) \cap (B\uparrow)$, then $\varphi(x) = \overline{(x, 0)} = \overline{(0, x)} = \psi(x)$; and if $x \in (G \cap B)\downarrow = (G\downarrow) \cap (B\downarrow)$, then $\varphi(x) = \overline{(x, a)} = \overline{(a, x)} = \psi(x)$.

In the next, we shall identify elements g and $\varphi(g)$ for each $g \in G$ and also b and $\psi(b)$ for each $b \in B$. Then G and B are hl-subgroups of H.

Lemma 2.10. $H = \langle B \rangle_h$.

Proof. B is an hl-subgroup of H. We have to prove that $H\uparrow = \langle B\uparrow \rangle$. Assume that $\overline{(g, b)} \in H\uparrow$, $\overline{(0, b')} \in B\uparrow$, $\overline{(0,0)} \leq \overline{(g, b)} \leq \overline{(0, b')}$. Then $g \in A\uparrow \subseteq B\uparrow$ and so $g + b \in B\uparrow$, $\overline{(g, b)} = \overline{(0, g + b)} \in B\uparrow$. Hence $B\uparrow$ is a convex l-subgroup of $H\uparrow$. Let $\overline{(0,0)} < \overline{(g, b)} \in H\uparrow$, $\overline{(g, b)} \notin B\uparrow$. Then $g \notin A\uparrow$. Therefore, $g > 0$ and thus $\overline{(0, b')} < \overline{(g, b)}$ for each $\overline{(0, b')} \in B\uparrow$. ■

By using Lemmas 2.10 and 2.1, B is an hl-ideal of H. Therefore, we can form the factor hl-group $\overline{H} = H/B\uparrow$.

Lemma 2.11. *Half l-groups $\overline{G}$ and $\overline{H}$ are hl-isomorphic.*

Proof. Define the mapping $f : \overline{G} \to \overline{H}$ by $f(g + A\uparrow) = g + B\uparrow$. Let $g + A\uparrow = g' + A\uparrow$. Then $g - g' \in A\uparrow \subset B\uparrow$. Thus $g + B\uparrow = g' + B\uparrow$. Therefore the mapping f is correctly defined.

Let $g_1 + A\uparrow,\ g_2 + A\uparrow \in \overline{G}$. Then $f((g_1+A\uparrow)+(g_2+A\uparrow)) = f((g_1+g_2)+A\uparrow\ ) = (g_1+g_2)+B\uparrow = (g_1+B\uparrow)+(g_2+B\uparrow) = f(g_1+A\uparrow)+f(g_2+A\uparrow)$.

Assume that $g_1, g_2 \in G$ and $f(g_1 + A\uparrow) = f(g_2 + A\uparrow)$. From $g_1 + B\uparrow = g_2 + B\uparrow$, we infer that either $g_1, g_2 \in G\uparrow$ or $g_1, g_2 \in G\downarrow$. Hence $g_1 - g_2 \in G\uparrow \cap B\uparrow = A\uparrow$ and so $g_1 + A\uparrow = g_2 + A\uparrow$.

Let $\overline{(g,b)} + B\uparrow \in \overline{H}$. Assume that $\overline{(g,b)} \in H\downarrow$. Hence $g \in G\downarrow$. Recall that g is identified with $\overline{(g,a)}$. As for $\overline{(g,a)} - \overline{(g,b)} = \overline{(0,a-b)} \in B\uparrow$, we have $\overline{(g,a)} + B\uparrow = \overline{(g,b)} + B\uparrow$. Therefore $f(g + A\uparrow) = \overline{(g,b)} + B\uparrow$. If $\overline{(g,b)} \in H\uparrow$, the proof is similar.

We have shown that f is a group isomorphism of $\overline{G}$ onto $\overline{H}$.

Assume that $g_1 + A\uparrow,\ g_2 + A\uparrow \in \overline{G}$ and $g_1 + A\uparrow \leq g_2 + A\uparrow$. If $g_1 + A\uparrow = g_2 + A\uparrow$, then $f(g_1 + A\uparrow) = f(g_2 + A\uparrow)$. Let $g_1 + A\uparrow < g_2 + A\uparrow$. By Lemma 2.2, $g_1 < g_2$. Hence $f(g_1+A\uparrow) = g_1+B\uparrow < g_2+B\uparrow = f(g_2+A\uparrow)$. The converse is similar. ∎

Summarizing the previous results, we have

Theorem 2.12. *Let A, B and G be hl-groups from $\mathcal{HL}$ satisfying* (I)–(IV). *Then there exists an hl-group $H \in \mathcal{HL}$ such that*

(i) $H = \langle B \rangle_h$,

(ii) *G is an hl-subgroup of H,*

(iii) *hl-groups $G/A\uparrow$ and $H/B\uparrow$ are hl-isomorphic.*

3. Cut completion of a lexico extension

Let G be an *hl*-group. A subset X of $G\uparrow$ is said to be a *cut of* $G\uparrow$ if X is an order closed (i.e., $g = \bigvee S$ for $S \subseteq X$ implies $g \in X$) lattice ideal of G such that $g + X \neq X \neq X + g$ for any $0 < g \in G$. A *cut of* $G\downarrow$ is defined in the same way. If X is a cut either of $G\uparrow$ or of $G\downarrow$, then X is called a cut of G.

$G\uparrow$ ($G\downarrow$) is said to be *cut complete* if every cut of $G\uparrow$ ($G\downarrow$) has a supremum in $G\uparrow$ ($G\downarrow$).

Remark that if $Z \subseteq G\uparrow$ ($Z \subseteq G\downarrow$), then $\sup(Z)$ exists in $G\uparrow$ ($G\downarrow$) if and only if $\sup(Z)$ exists in G, and $\sup(Z)$ in G is equal to $\sup(Z)$ in $G\uparrow$ ($G\downarrow$).

G is called *cut complete* provided that every cut of G has a supremum in G.

An hl-subgroup G' of G will be said to be order dense in G if for every element $0 < g \in G$ there exists $g' \in G'$ with $0 < g' \leq g$.

An hl-group G^C is said to be a *cut completion* of G if the following conditions are satisfied:

(i) G^C is cut complete;

(ii) G is an order dense hl-subgroup of G^C;

(iii) if K is an hl-subgroup of G^C such that $G \leq K < G^C$, then K is not cut complete.

Since $G \uparrow$ and $G \downarrow$ are dually isomorphic lattices, we have

Lemma 3.1. *For any hl-group G, $G \in \mathcal{HL}$ is cut complete if and only if $G \uparrow$ is cut complete.* ∎

Lemma 3.2 ([5], Lemma 3.1). *Let G be an abelian l-group, $G = \langle A \rangle$, $A \neq \{0\}$. If A is cut complete, then G is cut complete.* ∎

Lemma 3.3. *Let H and B be hl-groups from $\mathcal{HL}$ such that $H \uparrow$ is abelian and $H = \langle B \rangle_h$. If B is cut complete, then H is cut complete.*

Proof. The assumption that B is cut complete and Lemma 3.1 yield that $B \uparrow$ is cut complete. From $H \uparrow = \langle B \uparrow \rangle, B \uparrow \neq \{0\}$ and Lemma 3.2, we obtain that $H \uparrow$ is cut complete. Hence H is cut complete. ∎

Lemma 3.4. *Let A, B, G satisfy the conditions* (I)–(IV) *and H be such as in Theorem* 2.12. *Suppose that $B = A^C$. Then $H = G^C$.*

Proof. In view of Lemma 2.10, we have $H = \langle B \rangle_h$. A is order dense in B and B is order dense in H. This yields that A is order dense in H. Thus G is order dense in H. From Lemma 3.3, it follows that H is cut complete. Assume that K is an hl-subgroup of H such that $G \leq K < H$. Then $K \uparrow$ is an l-subgroup of $H \uparrow$ with $G \uparrow \leq K \uparrow < H \uparrow$. In the same way as in the proof of Lemma 3.2 in [5], it can be shown that $K \uparrow$ fails to be cut complete. Then, by Lemma 3.1, K is not cut complete. Therefore $H = G^C$. ∎

Theorem 3.5. *Let $G = \langle A \rangle_h$, and $A < G$. Then*

(i) $G^C = \langle A^C \rangle_h$,

(ii) *hl-groups $G/A \uparrow$ and $G^C/A^C \uparrow$ are hl-isomorphic.*

Proof. (i): Put $B = A^C$. Let H be as in Theorem 2.12. Then $H = \langle B \rangle_h$ holds. Applying Lemma 3.4, we get that $G^C = \langle A^C \rangle_h$ is valid.

(ii): Immediately follows from Theorem 2.12. ∎

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