

**THE SIZE OF MINIMUM 3-TREES:  
CASES 0 AND 1 MOD 12**

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**Abstract**

A 3-uniform hypergraph is called a minimum 3-tree, if for any 3-coloring of its vertex set there is a heterochromatic triple and the hypergraph has the minimum possible number of triples. There is a conjecture that the number of triples in such 3-tree is  $\lceil \frac{n(n-2)}{3} \rceil$  for any number of vertices  $n$ . Here we give a proof of this conjecture for any  $n \equiv 0, 1 \pmod{12}$ .

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1. INTRODUCTION

A 3-graph is an ordered pair of sets  $G = (V, \Delta)$ . The elements of  $V$  are called *vertices*. The elements of  $\Delta$  are subsets of vertices of cardinality 3 and are called *triples*. Given a 3-graph  $G = (V, \Delta)$  and a vertex  $v$  the trace

$Tr_G(v)$  of  $v$  in  $G$  is the graph with vertex set  $V \setminus \{v\}$ , and a pair  $\{x, y\}$  is an edge of  $Tr_G(v)$  if and only if  $\{v, x, y\}$  is a triple of  $G$ .

A *3-coloring* of a 3-graph is a surjective map from the vertex set onto a set of three elements. A 3-graph is said to be *tight* (see [1]) if any 3-coloring has a heterochromatic triple i.e., a triple whose vertices are colored differently. A tight 3-graph is called a *3-tree* if whenever we delete a triple from it we obtain an untight 3-graph. Different 3-trees on  $n$  vertices may have a different number of triples. From the results of [4], we know that the maximum number of triples in any 3-tree is  $\binom{n-1}{2}$ . It is not difficult to show that the minimum number of triples in such a 3-tree is not less than  $\lceil \frac{n(n-2)}{3} \rceil$ . In [1] it was proved that this bound is sharp for any  $n$  of the form  $\frac{p-1}{2}$  where  $p$  is a prime number, and it was conjectured that the bound is sharp for any  $n$ . In [2] the case when  $n \equiv 3, 4 \pmod{6}$  was solved and in [3] a full proof for the case  $n \equiv 2 \pmod{3}$  is given.

Here we give the proof of the cases  $n \equiv 0, 1 \pmod{12}$ . The case  $1 \pmod{12}$  is solved via a generalization of a construction from [2].

## 2. THE CASE $0 \pmod{12}$

In order to prove the conjecture for any  $n$  it is sufficient to construct a 3-tree with  $\lceil \frac{n(n-2)}{3} \rceil$  triples. In this section we deal only with the case  $n \equiv 0 \pmod{12}$ .

Let us consider the cyclic group  $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ , its elements are the vertices of the 3-graph  $H_n$  defined below.

Of course, we know how to add vertices. If  $e = \{x_1, x_2, x_3\}$  is a triple and  $y$  is a vertex, then  $e+y = \{x_1+y, x_2+y, x_3+y\}$ . If  $F$  is any set of triples and  $S$  any set of vertices then  $F+S = \{f+s \mid f \in F, s \in S\}$ . It is important to observe that all operations must be interpreted in the appropriate cyclic group.

Denote by  $\mathbb{A}_n = \{1, \dots, \frac{n}{6}\} \subset \mathbb{Z}_n$  and  $\mathbb{B}_n = \{1, \dots, \frac{n}{12}\} \subset \mathbb{A}_n$ . For  $a \in \mathbb{A}_n$  and  $b \in \mathbb{B}_n$ , let us consider the following triples:

$$\begin{aligned}\varepsilon_a &= \{0, 2\frac{n}{3}, 2a\}, \\ \zeta_b &= \{0, 2, 3-4b\}, \\ \eta_b &= \{0, 2\frac{n}{3} + 2b, 4b-1\}.\end{aligned}$$

Those triples generate the set of triples of the 3-graph  $H_n$  i.e., any triple will be of the form  $\varepsilon_a + y$  or  $\zeta_b + y$  or  $\eta_b + y$  where  $y \in \mathbb{Z}_n$ . Formally, denote

$$H_n = (\mathbb{Z}_n, (\{\varepsilon_a \mid a \in \mathbb{A}_n\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\}) + \mathbb{Z}_n).$$

Our purpose is to show that  $H_n$  is a 3-tree with  $\frac{n(n-2)}{3}$  triples.

**Proposition 1.**  $H_n$  has  $\frac{n(n-2)}{3}$  triples.

**Proof.** There are  $n(\frac{n}{6} - 1) + \frac{n}{3}$  triples generated by  $\varepsilon_a$ . The number of triples generated by  $\zeta_b$  and  $\eta_b$  is  $\frac{n^2}{6}$ . Those triples are all different and a straightforward calculation gives the result. ■

Let us construct an auxiliary hypergraph. For this, let  $m \equiv 0 \pmod{3}$  and denote  $\alpha_a = \{0, 2\frac{m}{3}, a\}$ .

The hypergraph  $G_m$  is by definition  $(\mathbb{Z}_m, \{\alpha_a \mid a \in \{1, \dots, \frac{m}{3}\}\} + \mathbb{Z}_m)$ .

Observe that the hypergraph generated by the set of even vertices in  $H_n$  contains a copy of  $G_{n/2}$  and also the hypergraph generated by the set of odd ones by the automorphism  $x \mapsto x + 1$  of  $H_n$ .

**Lemma 2.** Let  $f$  be a non heterochromatic 3-coloring of  $G_m$ . Then, all the cosets of  $\mathbb{Z}_m$  by the subgroup  $\langle \frac{m}{3} \rangle \cong \mathbb{Z}_3$  are monochromatic.

**Proof.** Denote  $t = \frac{m}{3}$ . Let  $f$  be a red-blue-yellow 3-coloring for which the lemma is false. Let  $y \in \mathbb{Z}_m$ , observe that for the 3-coloring  $f + y : a \mapsto f(a + y)$  the lemma is also false. So we can suppose that  $|f(\alpha_t)| = 2$ , and  $f(0) = f(-t) = R$  and  $f(t) = B$ . So for any  $a \in \{1, \dots, t\}$  we have

$$\left. \begin{array}{l} \alpha_a + t = \{t, 0, a + t\} \in G_m \\ \text{and } f(0) = R, f(t) = B. \end{array} \right\} \Rightarrow f(a + t) \neq Y,$$

$$\left. \begin{array}{l} \alpha_a - t = \{-t, t, a - t\} \in G_m \\ \text{and } f(-t) = R, f(t) = B. \end{array} \right\} \Rightarrow f(a - t) \neq Y.$$

Therefore, since any 3-coloring is a surjective map there must be an  $x \in \{1, \dots, t - 1\}$  such that  $f(x) = Y$ . In this case we have

$$\left. \begin{array}{l} \alpha_{t-x} + x = \{x, x - t, t\}, \alpha_x - t = \{-t, t, x - t\} \in G_m \\ \text{and } f(-t) = R, f(t) = B, f(x) = Y. \end{array} \right\} \Rightarrow f(x - t) = B,$$

$$\left. \begin{array}{l} \alpha_{t-x} + x + t = \{x + t, x, -t\}, \alpha_x + t = \{t, 0, x + t\} \in G_m \\ \text{and } f(-t) = f(0) = R, f(t) = B, f(x) = Y. \end{array} \right\} \Rightarrow f(x + t) = R$$

and this is a contradiction because  $\alpha_t + x = \{x, x - t, x + t\} \in G_m$ . ■

Of course, the lemma is equivalent to the fact that any non heterochromatic 3-coloring of  $G_m$  factorizes through a 3-coloring of the quotient hypergraph  $G_m/\langle \frac{m}{3} \rangle$ , i.e., the 3-graph whose vertices are the cosets modulo  $\langle \frac{m}{3} \rangle$  and the triples are the images of the triples in  $G_m$  by the natural map (see [1] for a more formal definition).

Let us prove a key property of the hypergraph  $H_n$ .

**Lemma 3.** *If  $f$  is a non heterochromatic 3-coloring of  $H_n$ , then  $f$  is surjective in the set of odd vertices or is surjective in the set of even vertices.*

**Proof.** For two vertices  $x, y \in \mathbb{Z}_n$  define the distance between them as the minimal natural number  $d$  such that  $(d \bmod n) + x = y$  or  $(d \bmod n) + y = x$ .

Let  $f$  be a non heterochromatic 3-coloring of  $H_n$ . Both cosets,  $\langle 2 \rangle$  and  $\langle 2 \rangle + 1$  can not be monochromatic.

Suppose that  $f(\langle 2 \rangle + 1) = Y$ , then  $f(\langle 2 \rangle) = \{R, B\}$  and since  $x \mapsto x + 2$  is an automorphism of  $H_n$  we also may assume that  $f(0) = R$  and  $f(2) = B$ . Therefore the triple  $\zeta_1 = \{0, 2, -1\}$  contradicts the fact that  $f$  is non heterochromatic. So, both cosets are bichromatic.

Let  $Y$  be the common color to both cosets. Let  $x$  and  $y$  be vertices such that  $f(\{x, y\}) = \{R, B\}$  and the distance between  $x$  and  $y$  is minimal. Since  $x \mapsto x + 1$  is an automorphism of  $H_n$  we may assume that  $y = 0$ ,  $f(0) = R$  and  $f(x) = B$ . Therefore,  $f(\langle 2 \rangle) = \{R, Y\}$ ,  $f(\langle 2 \rangle + 1) = \{B, Y\}$  and  $x \in \{\frac{n}{2} + 1, \frac{n}{2} + 3, \dots, -1\}$ . Of course, by the minimality of the distance between  $x$  and  $y$ , for all  $z \in \{x + 1, x + 2, \dots, -1\}$  we have  $f(z) = Y$ .

For  $x \in \{\frac{n}{2} + 1, \frac{n}{2} + 3, \dots, 2\frac{n}{3} - 1\}$ , let  $d$  be the solution in  $\mathbb{B}_n$  of  $2\frac{n}{3} - 2d + 1 = x$ . In this case the triple  $\eta_d + 1 - 4d = \{1 - 4d, 0, x\} \in H_n$  is heterochromatic and this is a contradiction.

On the other hand, let  $x \in \{2\frac{n}{3} + 1, 2\frac{n}{3} + 3, \dots, -1\}$ .

If  $x \equiv 1 \pmod{4}$  then let us consider the solution  $d$  in  $\mathbb{B}_n$  of  $1 - 4d = x$ . In this case, the triple  $\zeta_d - 2 = \{-2, 0, x\} \in H_n$  gives a contradiction.

If  $x \equiv 3 \pmod{4}$  then let  $d$  be the solution in  $\mathbb{B}_n$  of  $3 - 4d = x$ . In this case we have

$$\left. \begin{array}{l} \eta_d + x = \{x, x + 2\frac{n}{3} + 2d, 2\} \in H_n, \\ f(x) = B, f(x + 2\frac{n}{3} + 2d) = Y, f(2) \neq B. \end{array} \right\} \Rightarrow f(2) = Y$$

and the triple  $\zeta_d = \{0, 2, x\} \in H_n$  is heterochromatic, which is impossible.  $\blacksquare$

**Lemma 4.** *If  $f$  is a non heterochromatic 3-coloring of  $H_n$ , then  $f$  is surjective in the set of odd vertices and is also surjective in the set of even vertices.*

**Proof.** Let  $f$  be a non heterochromatic 3-coloring of  $H_n$  then by the preceding lemma we may suppose that  $f(\langle 2 \rangle) = \{R, B, Y\}$  and  $R \notin f(\langle 2 \rangle + 1)$ .

Since the hypergraph generated by  $\langle 2 \rangle$  is isomorphic to  $G_{n/2}$ , hence by Lemma 2, for all  $\alpha \in \langle 2 \rangle$  the coset  $\langle \frac{n}{3} \rangle + \alpha$  must be monochromatic. So, we can suppose that  $f(\langle \frac{n}{3} \rangle) = R$  and  $f(\langle \frac{n}{3} \rangle + 2) = B$ .

For a better understanding, we urge the reader to remember (see the beginning of Section 2) that we can add a set of vertices to a triple thus obtaining in this way a set of triples.

For all  $b \in \mathbb{B}_n$  we have that

$$\left. \begin{array}{l} \zeta_b + \langle \frac{n}{3} \rangle = \{0, 2, 3 - 4b\} + \langle \frac{n}{3} \rangle, \\ f(\langle \frac{n}{3} \rangle) = R, f(\langle \frac{n}{3} \rangle + 2) = B \\ \text{and } R \notin f(\langle \frac{n}{3} \rangle + 3 - 4b). \end{array} \right\} \Rightarrow f(\langle \frac{n}{3} \rangle + 3 - 4b) = B.$$

Observe that

$$\bigcup_{b \in \mathbb{B}_n} (\langle \frac{n}{3} \rangle + 3 - 4b) = \bigcup_{b \in \mathbb{B}_n} (\langle \frac{n}{3} \rangle + 4b - 1)$$

and therefore for any  $b \in \mathbb{B}_n$ ,  $f(\langle \frac{n}{3} \rangle + 4b - 1) = B$  holds.

On the other hand

$$\left. \begin{array}{l} \zeta_b + 4b - 3 + \langle \frac{n}{3} \rangle = \{4b - 3, 4b - 1, 0\} + \langle \frac{n}{3} \rangle, \\ f(\langle \frac{n}{3} \rangle) = R, f(\langle \frac{n}{3} \rangle + 4b - 1) = B \\ \text{and } R \notin f(\langle \frac{n}{3} \rangle + 4b - 3). \end{array} \right\} \Rightarrow f(\langle \frac{n}{3} \rangle + 4b - 3) = B.$$

Since every odd vertex is either in some coset of the form  $\langle \frac{n}{3} \rangle + 4b - 1$  or in some coset of the form  $\langle \frac{n}{3} \rangle + 4b - 3$ , hence  $f(\langle 2 \rangle + 1) = B$ .

Let  $x \in \langle 2 \rangle$  a vertex colored yellow. Recall that  $f(\langle \frac{n}{3} \rangle + x) = Y$  so we can suppose that  $x \in \{2, 4, \dots, \frac{n}{3} - 2\} = 2\mathbb{B}_n \cup (\frac{n}{3} - 2\mathbb{B}_n)$ . If  $x = 2b$ ,  $b \in \mathbb{B}_n$  we have the heterochromatic triple  $\eta_b = \{0, x - \frac{n}{3}, 4b - 1\} \in H_n$ . In any other case,  $x = \frac{n}{3} - 2b$ ,  $b \in \mathbb{B}_n$  and the triple  $\eta_b + x = \{x, 0, \frac{n}{3} + 2b - 1\} \in H_n$  is heterochromatic and this is a contradiction. ■

**Lemma 5.** *If  $f$  is a non heterochromatic 3-coloring of  $H_n$ , then all the cosets of  $\mathbb{Z}_n$  by the subgroup  $\langle \frac{n}{3} \rangle \cong \mathbb{Z}_3$  are monochromatic.*

**Proof.** Let  $f$  be a non heterochromatic 3-coloring of  $H_n$ , then by Lemma 4  $f$  is surjective in the set of odd vertices and in the set of even vertices. Both sets of vertices induce hypergraphs that are isomorphic to  $G_{n/2}$ . By Lemma 2 the cosets mod  $(n/6)$  in  $G_{n/2}$  are monochromatic but these cosets are precisely the cosets mod  $(n/3)$  in  $\mathbb{Z}_n$  (by the two isomorphisms). ■

**Lemma 6.**  *$H_n$  is tight if and only if  $H_n / \langle \frac{n}{3} \rangle$  is tight.*

**Proof.** Any non heterochromatic 3-coloring  $f'$  of  $H_n / \langle \frac{n}{3} \rangle$  lifts to a non heterochromatic 3-coloring  $f$  of  $H_n$ . On the other hand (by the preceding lemma) any non heterochromatic 3-coloring  $f$  of  $H_n$  factorizes (i.e.,  $f = f' \circ \text{nat}$ ) through a non heterochromatic 3-coloring  $f'$  of  $H_n / \langle \frac{n}{3} \rangle$ . ■

**Theorem 7.**  *$H_n$  is tight.*

**Proof.** Denote  $\widehat{H}_n = H_n / \langle \frac{n}{3} \rangle$ . Let  $f'$  be a non heterochromatic 3-coloring of  $H_n$ . As in the preceding lemma the map  $f'$  factorizes through a non heterochromatic 3-coloring  $f$  of  $\widehat{H}_n$ , moreover by Lemma 4  $f'$  (and so  $f$ ) is surjective in the set of odd and in the set of even vertices. Denote by  $t = \frac{n}{3}$  and recall that  $f : \mathbb{Z}_n / \langle \frac{n}{3} \rangle \cong \mathbb{Z}_t \rightarrow \{R, B, Y\}$  is a non heterochromatic red-blue-yellow 3-coloring of  $\widehat{H}_n$ .

First we shall prove that there is an  $x$  such that  $f(x) = f(x+1)$ . Suppose not. If there is no  $y$  such that  $f(y) = f(y+2)$  then,  $t \equiv 0 \pmod{3}$ , the cosets  $\langle 3 \rangle$ ,  $\langle 3 \rangle + 1$  and  $\langle 3 \rangle + 2$  are monochromatic and the triple  $\zeta_{\frac{t}{4}-1} \pmod{t} = \{0, 2, 7\} \in \widehat{H}_n$  gives a contradiction. So, there exists  $y \in \mathbb{Z}_t$  such that  $f(y) = f(y+2) = R$ . If  $f(y+1) = R$  or  $f(y+3) = R$  then we are done. Let  $f(y+1) = B$ . The triple  $(\zeta_1 \pmod{t}) + y + 1 = \{y+1, y+3, y\} \in \widehat{H}_n$  shows that  $f(y+3) = B$ . Taking as a new  $y$  the vertex  $y+1$  and repeating

this argument the needed number of times we conclude that there is not a yellow vertex which is a contradiction.

Therefore we can suppose that  $f(0) = R$ ,  $f(1) = f(2) = B$ . For all  $b \in \mathbb{B}_n = \{1, \dots, \frac{n}{12}\} \subset \mathbb{Z}_n$  denote  $b' = -4b \bmod t \in \mathbb{Z}_t$ . We have that

$$\left. \begin{array}{l} \zeta_b \bmod t = \{0, 2, b' + 3\} \in \widehat{H}_n, \\ f(0) = R, f(2) = B. \end{array} \right\} \Rightarrow f(b' + 3) \neq Y.$$

Observe that  $\{b' : b \in \mathbb{B}_n\} = \langle 4 \rangle \subset \mathbb{Z}_t$ . Since  $f$  is surjective in the set of odd vertices there must be a vertex  $c' \in \langle 4 \rangle$  such that  $f(c' + 1) = Y$  and  $c' \neq 0$ . Let  $c$  be the element in  $\mathbb{B}_n$  such that  $c' = -4c \bmod t$ . We have that

$$\left. \begin{array}{l} (\zeta_{\frac{n}{12}} \bmod t) - 2 = \{-2, 0, 1\} \in \widehat{H}_n, \\ (\zeta_c \bmod t) - 2 = \{-2, 0, c' + 1\} \in \widehat{H}_n \\ \text{and } f(0) = R, f(1) = B, f(c' + 1) = Y. \end{array} \right\} \Rightarrow f(-2) = R.$$

Now, let  $d$  be the element in  $\mathbb{B}_n$  such that  $c' + 4 = 4d \bmod t$ . We have that

$$\left. \begin{array}{l} (\zeta_d \bmod t) + c' + 1 = \{c' + 1, c' + 3, 0\} \in \widehat{H}_n, \\ \zeta_c \bmod t = \{0, 2, c' + 3\} \in \widehat{H}_n \\ \text{and } f(0) = R, f(2) = B, f(c' + 1) = Y. \end{array} \right\} \Rightarrow f(c' + 3) = R.$$

Since  $f$  is surjective in the set of even vertices there must be a vertex  $x \in \langle 2 \rangle$  such that  $f(x) = B$ . If  $x \in \langle 4 \rangle$  then  $b' = x - c' - 4 \in \langle 4 \rangle$ . In this case the triple

$$(\zeta_b \bmod t) + c' + 1 = \{c' + 1, c' + 3, x\} \in \widehat{H}_n$$

gives a contradiction. If  $x \notin \langle 4 \rangle$  then,  $b' = x - c' - 2 \in \langle 4 \rangle$  and we have

$$\left. \begin{array}{l} (\zeta_b \bmod t) + c' - 1 = \{c' - 1, c' + 1, x\} \in \widehat{H}_n, \\ f(c' - 1) \neq Y, f(c' + 1) = Y, f(x) = B. \end{array} \right\} \Rightarrow f(c' - 1) = B.$$

Therefore, the triple  $(\zeta_d \bmod t) + c' - 1 = \{c' - 1, c' + 1, -2\} \in \widehat{H}_n$  is heterochromatic, which is impossible.  $\blacksquare$

### 3. THE CASE 1 mod 12

When  $n \equiv 1 \pmod{3}$  the bound for the number of triples in a tight 3-graph is  $\frac{n(n-2)+1}{3}$ . This bound can be reached in a 3-graph in which the trace of one

vertex is a cycle and the trace of any other vertex is a tree. Such 3-graph will be called an *almost 3-tree*.

Let  $M$  be a 3-tree with  $n$  vertices with  $n \equiv 0 \pmod{3}$  and suppose that  $M$  has a set  $T$  of  $\frac{n}{3}$  disjoint triples. Let  $C$  be a cycle passing through every vertex of  $M$ . Define the 3-graph  $\widetilde{M}$  obtained from  $M$  by the following procedure:

- add a new vertex  $*$ ,
- add the triples  $\{*, v, w\}$  where  $\{v, w\}$  is an edge of  $C$ ,
- delete all the triples of  $T$ .

It is easy to see, that if all the traces of vertices in  $\widetilde{M}$  are connected then  $\widetilde{M}$  is an almost 3-tree. In particular, if we can prove that  $\widetilde{M}$  is tight then we have a proof of the conjecture on the minimum size of tight 3-graph for the case  $n + 1$ .

In this section we construct a 3-graph  $\widetilde{H}_n$  which is an almost 3-tree and prove that it is tight.

Recall our definition of  $H_n$  from Section 1

$$H_n = (\mathbb{Z}_n, (\{\varepsilon_a \mid a \in \mathbb{A}_n\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\}) + \mathbb{Z}_n),$$

where

$$\begin{aligned} \varepsilon_a &= \{0, 2\frac{n}{3}, 2a\}, \quad \zeta_b = \{0, 2, 3 - 4b\}, \quad \eta_b = \{0, 2\frac{n}{3} + 2b, 4b - 1\}, \\ \mathbb{A}_n &= \{1, \dots, \frac{n}{6}\} \subset \mathbb{Z}_n, \quad \mathbb{B}_n = \{1, \dots, \frac{n}{12}\} \subset \mathbb{A}_n \end{aligned}$$

and  $n \equiv 0 \pmod{12}$ .

Let  $T$  be the set of triples  $\{\varepsilon_{n/6} + \mathbb{Z}_n\}$  and  $C$  be the cycle  $\{\{x, x + 1\} \mid x \in \mathbb{Z}_n\}$ . Let  $\widetilde{H}_n$  be the 3-graph obtained as above, i.e.,

$$\widetilde{H}_n = (\mathbb{Z}_n \cup \{*\}, (\{\varepsilon_a \mid a \in \mathbb{A}_n \setminus \{\frac{n}{6}\}\} \cup \{\zeta_b, \eta_b \mid b \in \mathbb{B}_n\} \cup \{*, 0, 1\}) + \mathbb{Z}_n)$$

where, by definition,  $* + x = *$  for all  $x \in \mathbb{Z}_n$ .

**Theorem 8.**  $\widetilde{H}_n$  is tight.

**Proof.** The proof bellow is not valid for the case  $n = 12$ . However, for that case we can prove that  $\widetilde{H}_{12}$  is tight checking all possible colorings (the number of colorings can be reduced using the symmetries of  $\widetilde{H}_{12}$  and the fact that  $H_{12}$  is tight).

So, let  $s = n/12$ ,  $s \geq 2$  and let  $f$  be a non heterochromatic 3-coloring of  $\widetilde{H}_n$ .



There must be a vertex  $x$  in  $\mathbb{Z}_n$  such that  $f(x) = f(*)$  for if this is not the case, then there are two consecutive vertices  $y, y + 1$  such that  $f(y) \neq f(y + 1)$  and therefore the triple  $\{*, y, y + 1\}$  gives a contradiction.

Then  $f$  is surjective in  $\mathbb{Z}_n$ . By Theorem 7 there must be an heterochromatic triple  $\varepsilon_{2s} + x \in H_n$ . Since  $x \mapsto x + 1$  is an automorphism of  $H_n$  and  $\tilde{H}_n$ , we can suppose that  $x = 0$ . Let  $f(0) = R$ ,  $f(4s) = B$  and  $f(8s) = Y$ .

We divide the proof in two cases when  $f(0) = f(2)$  and otherwise.

If  $f(0) = f(2) = R$  then

$$\left. \begin{array}{l} \varepsilon_1 + 8s = \{8s, 4s, 8s + 2\}, \\ \varepsilon_{2s-1} + 2 = \{2, 8s + 2, 4s\}, \\ f(2) = R, f(4s) = B, f(8s) = Y. \end{array} \right\} \Rightarrow f(8s + 2) = B,$$

$$\left. \begin{array}{l} \varepsilon_s = \{0, 8s, 2s\}, \varepsilon_{s-1} + 2 = \{2, 8s + 2, 2s\}, \\ f(0) = R, f(2) = R, f(8s) = Y, f(8s + 2) = B. \end{array} \right\} \Rightarrow f(2s) = R,$$

$$\left. \begin{array}{l} \zeta_1 + 8s = \{8s, 8s + 2, 8s - 1\}, \\ \eta_s + 4s = \{4s, 8s - 1, 2s\}, \\ f(2s) = R, f(4s) = B, f(8s) = Y, f(8s + 2) = B. \end{array} \right\} \Rightarrow f(8s - 1) = B,$$

$$\left. \begin{array}{l} \varepsilon_1 + 4s = \{4s, 0, 4s + 2\}, \\ \varepsilon_{2s-1} + 4s + 2 = \{4s + 2, 2, 8s\}, \\ f(0) = R, f(2) = R, f(4s) = B, f(8s) = Y. \end{array} \right\} \Rightarrow f(4s + 2) = R,$$

$$\left. \begin{array}{l} \varepsilon_{2s-2} = \{0, 8s, 4s - 4\}, \\ \varepsilon_{2s-3} + 2 = \{2, 8s + 2, 4s - 4\}, \\ f(0) = R, f(2) = R, f(8s) = Y, f(8s + 2) = B. \end{array} \right\} \Rightarrow f(4s - 4) = R,$$

$$\left. \begin{array}{l} \varepsilon_{2s-2} + 4s = \{4s, 0, 8s - 4\}, \\ \varepsilon_2 + 8s - 4 = \{8s - 4, 4s - 4, 8s\}, \\ f(0) = R, f(4s - 4) = R, f(4s) = B, f(8s) = Y. \end{array} \right\} \Rightarrow f(8s - 4) = R$$

and

$$\left. \begin{array}{l} \eta_1 + 8s = \{8s, 8s + 3, 4s + 2\}, \\ \eta_2 + 8s - 4 = \{8s - 4, 8s + 3, 4s\}, \\ f(4s) = B, f(4s + 2) = R, f(8s - 4) = R, f(8s) = Y. \end{array} \right\} \Rightarrow f(8s + 3) = R.$$

Moreover, if  $f(8s + 1) = R$  then no matter the color of  $*$  is, some of the triples  $\{*, 8s - 1, 8s\}$ ,  $\{*, 8s, 8s + 1\}$  or  $\{*, 8s + 1, 8s + 2\}$  gives a contradiction. Hence

$$\left. \begin{array}{l} \zeta_s + 8s - 1 = \{8s - 1, 8s + 1, 4s + 2\} \in \widetilde{H}_n, \\ f(4s + 2) = R, f(8s - 1) = B, f(8s + 1) \neq R. \end{array} \right\} \Rightarrow f(8s + 1) = B$$

and the triple  $\zeta_1 + 8s + 1 = \{8s + 1, 8s + 3, 8s\}$  gives a contradiction.

Now, suppose that  $f(0) \neq f(2)$ . If  $f(4s) = f(4s + 2)$  then using the automorphism  $x \mapsto x - 4s$  we reduce the proof to the first case. By the same argument  $f(8s) \neq f(8s + 2)$ . Moreover,

$$\left. \begin{array}{l} \varepsilon_1 = \{0, 8s, 2\} \in \widetilde{H}_n, \\ f(0) = R, f(8s) = Y, f(2) \neq R. \end{array} \right\} \Rightarrow f(2) = Y,$$

$$\left. \begin{array}{l} \varepsilon_1 + 4s = \{4s, 0, 4s + 2\} \in \widetilde{H}_n, \\ f(0) = R, f(4s) = B, f(4s + 2) \neq B. \end{array} \right\} \Rightarrow f(4s + 2) = R,$$

$$\left. \begin{array}{l} \varepsilon_1 + 8s = \{8s, 4s, 8s + 2\} \in \widetilde{H}_n, \\ f(4s) = B, f(8s) = Y, f(8s + 2) \neq Y. \end{array} \right\} \Rightarrow f(8s + 2) = B$$

and

$$\left. \begin{array}{l} \varepsilon_1 + 2 = \{2, 8s + 2, 4\}, \varepsilon_2 = \{0, 8s, 4\} \in \widetilde{H}_n, \\ f(0) = R, f(2) = Y, f(8s) = Y, f(8s + 2) = B. \end{array} \right\} \Rightarrow f(4) = Y.$$

Again, using the automorphism  $x \mapsto x - 2$  we reduce the proof to the first case. ■

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