

## BALANCED $d$ -LATTICES ARE COMPLEMENTED \*

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### Abstract

We characterize  $d$ -lattices as those bounded lattices in which every maximal filter/ideal is prime, and we show that a  $d$ -lattice is complemented iff it is balanced iff all prime filters/ideals are maximal.

**Keywords:** balanced congruence, balanced lattice,  $d$ -lattice, prime ideal, maximal ideal.

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According to Chajda and Eigenthaler ([1]), a  $d$ -lattice is a bounded lattice  $L$  satisfying for all  $a, c \in L$  the implications

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- (i)  $(a, 1) \in \theta(0, c) \rightarrow a \vee c = 1$ ;
- (ii)  $(a, 0) \in \theta(1, c) \rightarrow a \wedge c = 0$ ;

where  $\theta(x, y)$  denotes the least congruence on  $L$  containing the pair  $(x, y)$ . Every bounded distributive lattice is a  $d$ -lattice. The 5-element nonmodular lattice  $N_5$  is a  $d$ -lattice.

**Theorem 1.** *A bounded lattice is a  $d$ -lattice if and only if all maximal ideals and maximal filters are prime.*

**Proof.** Let  $I$  be a maximal ideal in a  $d$ -lattice  $L$ . Let  $x, y \in L \setminus I$ . We need to show that  $x \wedge y \in L \setminus I$ . Since  $I$  is maximal, there are  $c_1, c_2 \in I$  such that  $c_1 \vee x = c_2 \vee y = 1$ . For  $c = c_1 \vee c_2 \in I$  we have  $c \vee x = c \vee y = 1$ . Then  $(x, 1) = (0 \vee x, c \vee x) \in \theta(0, c)$  and similarly  $(y, 1) \in \theta(0, c)$ , hence  $(x \wedge y, 1) \in \theta(0, c)$ . By (i) we have  $(x \wedge y) \vee c = 1$ , hence  $x \wedge y \notin I$ . The primality of maximal filters can be proved similarly.

Conversely, assume that all maximal ideals and filters in  $L$  are prime. To show (i), assume that  $a, c \in L$ ,  $a \vee c \neq 1$ . By the Zorn lemma, there exists a maximal ideal  $I$  containing  $a \vee c$ . By our assumption,  $I$  is prime. Then  $\alpha = I^2 \cup (L \setminus I)^2$  is a congruence on  $L$ . Since  $c \in I$ , we have  $(0, c) \in \alpha$ , which implies that  $\theta(0, c) \subseteq \alpha$ . Since  $a \in I$ , we have  $(a, 1) \notin \alpha$ , hence  $(a, 1) \notin \theta(0, c)$ . This shows (i). The proof of (ii) is similar. ■

By [1], a bounded lattice is called “balanced”, if the 0-class of any congruence determines the 1-class, and conversely. They showed that complemented lattices are balanced, and they asked:

- (\*) Is there a  $d$ -lattice which is balanced but not complemented?

We use the above characterization of  $d$ -lattices to answer this question.

If  $A$  is a subset of an algebra, write  $\theta_A$  for the smallest congruence that identifies all elements of  $A$ ; if  $\phi$  is a congruence,  $x$  an element, write  $x/\phi$  for the  $\phi$ -congruence class of  $x$ .

Further, a congruence  $\phi$  (on an algebra with constants 0 and 1) is called balanced if  $0/\phi = 0/\theta_{(1/\phi)}$  and  $1/\phi = 1/\theta_{(0/\phi)}$ ; an algebra is called balanced iff all its congruence relations are balanced, or equivalently if: for any congruence relations  $\phi, \phi'$  we have:

$$0/\phi = 0/\phi' \text{ iff } 1/\phi = 1/\phi'.$$

Fix a  $d$ -lattice  $(L, \vee, \wedge, 0, 1)$ . For  $a \in L$  we denote  $F_a := \{x : x \vee a = 1\}$ , and  $I_a := \{x : x \wedge a = 0\}$ .

**Fact 2.**  $F_a$  is a filter,  $I_a$  is an ideal.

**Proof.** Let  $x, y \in F_a$ . Similarly as in the proof of Theorem 1,  $(x, 1) \in \theta(0, a)$ ,  $(y, 1) \in \theta(0, a)$ , hence  $(x \wedge y, 1) \in \theta(0, a)$ , which by the definition of a  $d$ -lattice implies  $x \wedge y \in F_a$ . The proof for  $I_a$  is similar. ■

**Fact 3.** If  $I$  is an ideal disjoint to  $F_a$ , and  $a \notin I$ , then also the ideal generated by  $I \cup \{a\}$  is disjoint to  $F_a$ .

**Proof.** If  $x \leq i \vee a$  for some  $i \in I$ , and  $x \in F_a$ , then also  $i \vee a \in F_a$ , hence  $i \vee a = (i \vee a) \vee a = 1$ . Thus,  $i \in F_a$ , so  $F_a \cap I \neq \emptyset$ . ■

**Fact 4.** If  $f : L_1 \rightarrow L_2$  is a homomorphism from  $L_1$  onto  $L_2$ , and  $L_1$  is balanced, then  $L_2$  is balanced.

**Proof.** In fact, this holds “level-by-level”: If  $\phi$  is an unbalanced congruence on  $L_2$ , then the preimage of  $\phi$  is unbalanced on  $L_1$ . ■

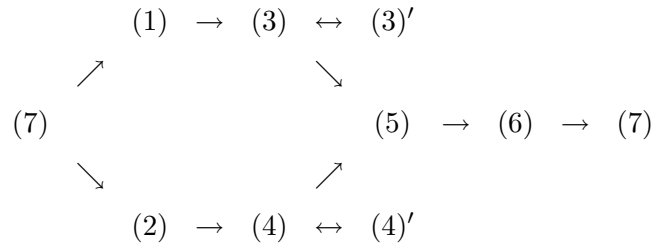
**Theorem 5.** The following are equivalent (for a  $d$ -lattice  $L$ ):

- (1) There is a maximal (hence prime) filter whose complement is not a maximal ideal.
- (2) There is a maximal (hence prime) ideal whose complement is not a maximal filter.
- (3) There are two prime ideals in  $L$ , one properly containing the other.
- (3)' There is a prime ideal in  $L$  which is not maximal.
- (4) There are two prime filters in  $L$ , one properly containing the other.
- (4)' There is a prime filter in  $L$  which is not maximal.
- (5) There is a homomorphism from  $L$  onto the 3-element lattice  $\{0, d, 1\}$ .

- (6)  $L$  is not balanced.  
 (7)  $L$  is not complemented.

In particular a  $d$ -lattice is balanced iff it is complemented.

**Proof.**



(1)  $\rightarrow$  (3): By Theorem 1, the complement of a maximal filter is a (necessarily prime) ideal. If this ideal is not maximal, it can be properly extended to a maximal (hence prime) ideal. The proof of (2)  $\rightarrow$  (4) is similar (dual).

(3)  $\rightarrow$  (3)' is trivial, and (3)'  $\rightarrow$  (3) follows from Zorn's lemma and Theorem 1. Similarly we get (4)  $\leftrightarrow$  (4)'.

(3)  $\rightarrow$  (5): Let  $I_1 \subset I_2 \subset L$  be prime ideals. Map  $I_1$  to 0,  $I_2 \setminus I_1$  to  $d$ , and  $L \setminus I_2$  to 1. Check that this is a lattice homomorphism. The proof of (4)  $\rightarrow$  (5) is dual.

(5)  $\rightarrow$  (6) follows from Fact 4, since the three-element lattice is not balanced.

(6)  $\rightarrow$  (7) is from [1].

Now we show (7)  $\rightarrow$  (1). (Again, (7)  $\rightarrow$  (2) is dual.) Assume that  $L$  is not complemented, so there is some  $a$  such that  $F_a \cap I_a = \emptyset$ . Let  $F'$  be the filter generated by  $F_a \cup \{a\}$ . We have  $F' \cap I_a = \emptyset$  by the dual of Fact 3, so  $F'$  is proper. By the Zorn lemma,  $F'$  can be extended to a maximal filter  $F$ . Let  $I' = L \setminus F$ . It is enough to see that  $I'$  is not maximal. Let  $I$  be the ideal generated by  $I' \cup \{a\}$ . By Fact 3,  $I \cap F_a = \emptyset$ , so  $I$  is a proper ideal properly extending  $I'$ .  $\blacksquare$

## REFERENCES

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