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ON SOME FINITE GROUPOIDS WITH DISTRIBUTIVE SUBGROUPOID LATTICES

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Abstract

The aim of the paper is to show that if $\mathcal{S}(\mathcal{G})$ is distributive, and also $\mathcal{G}$ satisfies some additional condition, then the union of any two subgroupoids of $\mathcal{G}$ is also a subgroupoid (intuitively, $\mathcal{G}$ has to be in some sense a unary algebra).

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Obviously properties of the subalgebra lattice of an algebra have influence on the algebra (see e.g. [4] and [5]). Analogously, properties of all subalgebra lattices of algebras within a given variety force some properties of the variety (see [2] and [3]).

In the present paper necessary and sufficient conditions are found for some finite groupoids $\mathcal{G}$ to have the subgroupoid lattice $\mathcal{S}(\mathcal{G})$ distributive. It is a simple exercise to see that if $\mathcal{G} = \langle G, \vee \rangle$ is a semilattice (not necessarily finite), then $\mathcal{S}(\mathcal{G})$ is distributive iff for any elements $x, y \in \mathcal{G}$, $x \vee y = x$ or $x \vee y = y$. Since every element forms a one-element subsemilattice, the right-hand side of the equivalence is equivalent with the condition that the union of any two subsemilattices is also a subsemilattice of $\mathcal{G}$. Now we generalize this result to some finite groupoids ($\mathcal{G}(g_1, \ldots, g_n)$ denotes *the subgroupoid of a groupoid* $\mathcal{G}$ *generated by elements* $g_1, \ldots, g_n \in \mathcal{G}$).

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Theorem 1. *Let $\mathcal{G} = \langle G, \circ \rangle$ be a finite groupoid such that*

(∗) *for each two different elements g and h of $\mathcal{G}$,*

$$\text{if } g, h \in \mathcal{G}(g \circ h), \text{ then } g \circ h \in \mathcal{G}(g) \text{ or } g \circ h \in \mathcal{G}(h).$$

Then $\mathcal{S}(\mathcal{G})$ is a distributive lattice iff the (set-theoretical) union of any two subgroupoids is a subgroupoid of $\mathcal{G}$.

Note (although it is not used below) that the condition (∗) is equivalent to the following:

(∗′) *for each two different elements $g, h \in \mathcal{G}$,*

$$\text{if } g, h \in \mathcal{G}(g \circ h), \text{ then } h \in \mathcal{G}(g) \text{ or } g \in \mathcal{G}(h).$$

The implication $(*') \Longrightarrow (*)$ is obvious. On the other hand, take elements g, h of $\mathcal{G}$ such that $g, h \in \mathcal{G}(g \circ h)$. Then by (∗), $g \circ h \in \mathcal{G}(g)$ or $g \circ h \in \mathcal{G}(h)$. Hence, $g, h \in \mathcal{G}(g \circ h) \subseteq \mathcal{G}(g)$ or $g, h \in \mathcal{G}(g \circ h) \subseteq \mathcal{G}(h)$.

Proof. $\Longleftarrow$ is obvious, because then the operations of supremum and infimum in $\mathcal{S}(\mathcal{G})$ are just the set-theoretical union and intersection.

$\Longrightarrow$. Assume that there are two subgroupoids $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{G}$ such that their union $\mathcal{H}_1 \cup \mathcal{H}_2$ is not a subgroupoid of $\mathcal{G}$. Then there are elements $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$ such that $h_1 \circ h_2 \notin \mathcal{H}_1 \cup \mathcal{H}_2$ or $h_2 \circ h_1 \notin \mathcal{H}_1 \cup \mathcal{H}_2$. In particular, $h_1 \circ h_2$ or $h_2 \circ h_1$ is not contained in $\mathcal{G}(h_1) \cup \mathcal{G}(h_2)$.

Now take the set $\mathcal{A}$ of all ordered pairs $\langle g, h \rangle$ of elements of $\mathcal{G}$ such that $g \circ h$ or $h \circ g$ does not belong to $\mathcal{G}(g) \cup \mathcal{G}(h)$. Observe first that if $\langle g, h \rangle \in \mathcal{A}$, then $g \neq h$. Secondly, $\mathcal{A}$ is non-empty. Thirdly, for any $\langle g, h \rangle \in \mathcal{A}$,

$$g \notin \mathcal{G}(h) \quad \text{and} \quad h \notin \mathcal{G}(g). \tag{1}$$

In particular,

$$\mathcal{G}(g) \subsetneqq \mathcal{G}(g, h) \quad \text{and} \quad \mathcal{G}(h) \subsetneqq \mathcal{G}(g, h).$$

For any pair $\langle e_1, e_2 \rangle \in \mathcal{A}$, $\mathcal{G}(e_1) \cup \mathcal{G}(e_2)$ has finitely many elements, since $\mathcal{G}$ is a finite groupoid. Thus we can take the set $\mathcal{B}$ of all pairs $\langle e_1, e_2 \rangle \in \mathcal{A}$ such that $\mathcal{G}(e_1)$ has the least number of elements. Next, we take a pair $\langle g, h \rangle \in \mathcal{B}$ such that $\mathcal{G}(h)$ has the least number of elements. In the rest of the paper, g and h denote these elements.

(2) For each pair of subgroupoids $\mathcal{H}_1, \mathcal{H}_2$ of $\mathcal{G}$,

$$\text{if } \mathcal{H}_1 \subseteq \mathcal{G}(g) \text{ and } g \notin \mathcal{H}_1, \text{ then } \mathcal{H}_1 \cup \mathcal{H}_2 \text{ is a subgroupoid of } \mathcal{G}.$$

Take elements $e_1 \in \mathcal{H}_1$ and $e_2 \in \mathcal{H}_2$. Then we have $g \notin \mathcal{G}(e_1) \subseteq \mathcal{G}(g)$, because $\mathcal{G}(e_1) \subseteq \mathcal{H}_1$. Thus $\mathcal{G}(e_1)$ has fewer elements than $\mathcal{G}(g)$. Hence, $\langle e_1, e_2\rangle \notin \mathcal{A}$, and consequently $e_1 \circ e_2, e_2 \circ e_1 \in \mathcal{G}(e_1) \cup \mathcal{G}(e_2) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$. This fact implies that $\mathcal{H}_1 \cup \mathcal{H}_2$ is a subgroupoid.

(3) For each subgroupoid $\mathcal{H}$ of $\mathcal{G}$,

$$\text{if } \mathcal{H} \subseteq \mathcal{G}(h) \text{ and } h \notin \mathcal{H}, \text{ then } \mathcal{H} \cup \mathcal{G}(g) \text{ is a subgroupoid of } \mathcal{G}.$$

Take an element $e \in \mathcal{H}$. Then $h \notin \mathcal{G}(e) \subseteq \mathcal{G}(h)$, because $\mathcal{G}(e) \subseteq \mathcal{H}$. Thus $\mathcal{G}(e)$ has fewer elements than $\mathcal{G}(h)$. Hence, $\langle g, e\rangle \notin \mathcal{A}$, and consequently $g \circ e, e \circ g \in \mathcal{G}(e) \cup \mathcal{G}(g) \subseteq \mathcal{H} \cup \mathcal{G}(g)$.

Next, take $f \in \mathcal{G}(g)$ and assume that $g \in \mathcal{G}(f)$, i.e. $\mathcal{G}(f) = \mathcal{G}(g)$. Then $\langle f, e\rangle \notin \mathcal{A}$, because otherwise $\langle f, e\rangle$ would belong to $\mathcal{B}$, so the choice of the pair $\langle g, h\rangle$ would imply that $\mathcal{G}(e)$ and $\mathcal{G}(h)$ have the same number of elements; it is a contradiction. Thus $f \circ e, e \circ f \in \mathcal{G}(e) \cup \mathcal{G}(g) \subseteq \mathcal{H} \cup \mathcal{G}(g)$.

Finally, take $f \in \mathcal{G}(g)$ and assume that $g \notin \mathcal{G}(f)$. Then $\mathcal{G}(f)$ has fewer elements than $\mathcal{G}(g)$. Hence, $\langle f, e\rangle \notin \mathcal{A}$, and again $f \circ e, e \circ f \in \mathcal{G}(e) \cup \mathcal{G}(f) \subseteq \mathcal{H} \cup \mathcal{G}(g)$.

Now let $i \in \{g \circ h, h \circ g\}$ be an element of $\mathcal{G}$ such that

$$i \notin \mathcal{G}(g) \cup \mathcal{G}(h). \tag{4}$$

Then $(*)$ implies

$$g \notin \mathcal{G}(i) \quad \text{or} \quad h \notin \mathcal{G}(i).$$

In particular,

$$\mathcal{G}(i) \subsetneqq \mathcal{G}(g, h). \tag{5}$$

Assume that $g \in \mathcal{G}(i)$. Then $h \notin \mathcal{G}(i)$, in particular, $\mathcal{G}(h)$ and $\mathcal{G}(i)$ are not comparable in $\mathcal{S}(\mathcal{G})$.

Let

$$\mathcal{I} = \mathcal{G}(h) \cap \mathcal{G}(i) \quad \text{and} \quad \mathcal{J} = \mathcal{I} \cup \mathcal{G}(g).$$

Since $h \notin \mathcal{I} \subseteq \mathcal{G}(h)$ and $g \in \mathcal{J}$, we first have by (1)

$$\mathcal{I} \subsetneqq \mathcal{G}(h) \quad \text{and} \quad \mathcal{I} \subsetneqq \mathcal{J}.$$

Secondly, (3) implies that $\mathcal{J}$ is a subgroupoid of $\mathcal{G}$.

By the assumption $\mathcal{G}(g) \subseteq \mathcal{G}(i)$, so $\mathcal{J} \subseteq \mathcal{G}(i)$. And by (4), $i \notin \mathcal{J}$. Hence,

$$\mathcal{J} \subsetneqq \mathcal{G}(i).$$

Obviously $\mathcal{G}(g,h)$ is the smallest subgroupoid containing $\mathcal{G}(h) \cup \mathcal{J}$.

By all the above facts, and also (1) and (5), we obtain that the subgroupoids $\mathcal{I}$, $\mathcal{G}(h)$, $\mathcal{J}$, $\mathcal{G}(i)$, $\mathcal{G}(g,h)$ form the elementary non-modular lattice $\mathcal{N}_5$. Thus (see [1]), in this case, $\mathcal{S}(\mathcal{G})$ is not distributive.

If $h \in \mathcal{G}(i)$, then it can be shown that $\mathcal{G}(g) \cap \mathcal{G}(i)$, $\mathcal{G}(g)$, $(\mathcal{G}(g) \cap \mathcal{G}(i)) \cup \mathcal{G}(h)$ (it follows from (2) that this union is a subgroupoid of $\mathcal{G}$), $\mathcal{G}(i)$, $\mathcal{G}(g,h)$ form the lattice $\mathcal{N}_5$. For this purpose it is sufficient to replace g by h and vice versa in the above proof; therefore details of this proof are omitted.

Thus now we can assume

$$g, h \notin \mathcal{G}(i). \tag{6}$$

Assume also $g \notin \mathcal{G}(h,i)$. Then, by (4), $\mathcal{G}(g)$ and $\mathcal{G}(h,i)$ are not comparable in $\mathcal{S}(\mathcal{G})$. As above, we want to construct a sublattice of $\mathcal{S}(\mathcal{G})$ isomorphic to $\mathcal{N}_5$.

Let

$$\mathcal{I} = \mathcal{G}(g) \cap \mathcal{G}(h,i) \quad \text{and} \quad \mathcal{J} = \mathcal{I} \cup \mathcal{G}(h).$$

Since $g \notin \mathcal{I} \subseteq \mathcal{G}(g)$ and $h \in \mathcal{J}$, taking into account the assumption $g \notin \mathcal{G}(h,i)$ and (1) we first have

$$\mathcal{I} \subsetneqq \mathcal{G}(g) \quad \text{and} \quad \mathcal{I} \subsetneqq \mathcal{J}.$$

Secondly, (2) implies that $\mathcal{J}$ is a subgroupoid of $\mathcal{G}$.

Obviously $\mathcal{J} \subseteq \mathcal{G}(h,i)$, moreover, by (4) we deduce that $i \notin \mathcal{J}$, so

$$\mathcal{J} \subsetneqq \mathcal{G}(h,i).$$

By (5) and the assumption, since $g \in \mathcal{G}(g,h)$, we obtain

$$\mathcal{G}(h,i) \subsetneqq \mathcal{G}(g,h).$$

It trivially follows that $\mathcal{G}(g,h)$ is generated by $\mathcal{G}(g) \cup \mathcal{J}$.

All the above facts and (1) imply that $\mathcal{I}$, $\mathcal{G}(g)$, $\mathcal{J}$, $\mathcal{G}(h,i)$ and $\mathcal{G}(g,h)$ form the elementary non-modular lattice $\mathcal{N}_5$. In particular, $\mathcal{S}(\mathcal{G})$ is not distributive.

If $h \notin \mathcal{G}(g,i)$, then it can be shown that the subgroupoids $\mathcal{G}(h) \cap \mathcal{G}(g,i)$, $\mathcal{G}(h)$, $(\mathcal{G}(h) \cap \mathcal{G}(g,i)) \cup \mathcal{G}(g)$ (it follows from (3) that this union is a subgroupoid of $\mathcal{G}$), $\mathcal{G}(g,i)$ and $\mathcal{G}(g,h)$ form the lattice $\mathcal{N}_5$. To this purpose it is sufficient to replace g by h and vice versa in the above proof; therefore details of this proof are omitted.

Thus finally we can assume

$$g \in \mathcal{G}(h,i) \quad \text{and} \quad h \in \mathcal{G}(g,i).$$

Take the subgroupoids

$$\mathcal{I}_g = \mathcal{G}(h) \cap \mathcal{G}(i), \quad \mathcal{I}_h = \mathcal{G}(g) \cap \mathcal{G}(i), \quad \mathcal{I}_i = \mathcal{G}(g) \cap \mathcal{G}(h).$$

By (6), $h \notin \mathcal{I}_g \subseteq \mathcal{G}(h)$, so by (3) we infer that

$$\mathcal{J}_g = \mathcal{G}(g) \cup \mathcal{I}_g$$

is a subgroupoid of $\mathcal{G}$.

Analogously, by (1) and (6), $g \notin \mathcal{I}_h \subseteq \mathcal{G}(g)$ and $g \notin \mathcal{I}_i \subseteq \mathcal{G}(g)$, so (2) implies

$$\mathcal{J}_h = \mathcal{G}(h) \cup \mathcal{I}_h$$

and

$$\mathcal{J}_i = \mathcal{G}(i) \cup \mathcal{I}_i$$

are subgroupoids of $\mathcal{G}$.
Using (1), (4) and (6) it can be verified that

$$g \in \mathcal{J}_g \quad \text{and} \quad h, i \notin \mathcal{J}_g,$$

$$h \in \mathcal{J}_h \quad \text{and} \quad g, i \notin \mathcal{J}_h,$$

$$i \in \mathcal{J}_i \quad \text{and} \quad g, h \notin \mathcal{J}_i.$$

In particular, $\mathcal{J}_g$, $\mathcal{J}_h$ and $\mathcal{J}_i$ are pairwise non-comparable.

These three facts, and also (1) and (5) imply that

$$\mathcal{J}_g \subsetneqq \mathcal{G}(g,h), \quad \mathcal{J}_h \subsetneqq \mathcal{G}(g,h), \quad \mathcal{J}_i \subsetneqq \mathcal{G}(g,h).$$

Hence, $\mathcal{G}(g,h)$ contains the unions $\mathcal{J}_g \cup \mathcal{J}_h$, $\mathcal{J}_g \cup \mathcal{J}_i$ and $\mathcal{J}_h \cup \mathcal{J}_i$. On the other hand $g, h \in \mathcal{J}_g \cup \mathcal{J}_h$, so $\mathcal{G}(g,h)$ is the smallest subgroupoid containing $\mathcal{J}_g \cup \mathcal{J}_h$. Next, $g, i \in \mathcal{J}_g \cup \mathcal{J}_i$, so the subgroupoid generated by $\mathcal{J}_g \cup \mathcal{J}_i$ contains also h, by the assumption. Thus again, $\mathcal{G}(g,h)$ is generated by the union of $\mathcal{J}_g$ and $\mathcal{J}_i$. Similarly, $h, i \in \mathcal{J}_h \cup \mathcal{J}_i$, so g also belongs to the subgroupoid generated by this union, and consequently, this subgroupoid is equal to $\mathcal{G}(g,h)$.
It is obtained by standard verification that

$$\mathcal{J}_g \cap \mathcal{J}_h = \mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i,$$

$$\mathcal{J}_g \cap \mathcal{J}_i = \mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i,$$

$$\mathcal{J}_h \cap \mathcal{J}_i = \mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i.$$

Note also that $\mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i$ is different from $\mathcal{J}_g$ and $\mathcal{J}_h$ and $\mathcal{J}_i$, because $g, h, i \notin \mathcal{J}_g \cap \mathcal{J}_h = \mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i$.

By all the above facts we obtain that the subgroupoids $\mathcal{I}_g \cup \mathcal{I}_h \cup \mathcal{I}_i$, $\mathcal{J}_g$, $\mathcal{J}_h$, $\mathcal{J}_i$ and $\mathcal{G}(g,h)$ form the elementary non-distributive lattice $\mathcal{M}_5$. Thus $\mathcal{S}(\mathcal{G})$ is not a distributive lattice (see [1]). ∎

For any finite groupoid $\mathcal{G}$ such that the set-theoretical union of any two of its subgroupoids is a subgroupoid we can construct a finite unary algebra $\mathbf{A}$ with its subalgebra lattice $\mathcal{S}(\mathbf{A})$ isomorphic to $\mathcal{S}(\mathcal{G})$. First, the carrier A of $\mathbf{A}$ is equal to the carrier of $\mathcal{G}$. Secondly, for any ordered pair $\langle a, b\rangle$ of elements of A, we define the unary operation $f_{a,b}$ as follows:

$$f_{a,b}(x) = \begin{cases} b & \text{if } x = a \text{ and } b \in \mathcal{G}(a), \\ x & \text{otherwise.} \end{cases}$$

To prove that $\mathcal{S}(\mathbf{A})$ and $\mathcal{S}(\mathcal{G})$ are isomorphic, it is sufficient to show that the subuniverses of $\mathbf{A}$ and that of $\mathcal{G}$ coincide. It follows from the assumption that the union of any two subgroupoids of $\mathcal{G}$ is also a subgroupoid of $\mathcal{G}$. Simple details of this proof are omitted.

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