A NOTE ON DOMINATION PARAMETERS OF THE
CONJUNCTION OF TWO SPECIAL GRAPHS

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Abstract

A dominating set $D$ of $G$ is called a split dominating set of $G$ if the
subgraph induced by the subset $V(G) - D$ is disconnected. The
cardinality of a minimum split dominating set is called the minimum
split domination number of $G$. Such subset and such number was intro-
duced in [4]. In [2], [3] the authors estimated the domination number of
products of graphs. More precisely, they were study products of paths.
Inspired by those results we give another estimation of the domination
number of the conjunction (the cross product) $P_n \land G$. The split dom-
ination number of $P_n \land G$ also is determined. To estimate this number
we use the minimum connected domination number $\gamma_c(G)$.

Keywords: domination parameters, conjunction of graphs.

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1. Definitions and Notations

In this paper we discuss finite connected, undirected simple graphs. For any
graph $G$ we denote $V(G)$ and $E(G)$, the vertex set of $G$ and the edge set of
$G$, respectively. We say that $G$ is of order $n$ if $n$ is a cardinality of $V(G)$. By
$\langle X \rangle_G$ we denote a subgraph of $G$ which is induced by a subset $X \subseteq V(G)$.
A hanging vertex is a vertex of $G$ adjacent to exactly one vertex in $G$. The
complement of $G$ is denoted by $\overline{G}$. A subset $D \subseteq V(G)$ is a dominating set
of $G$ if for every $x \in V(G) - D$ there is a vertex $y \in D$ such that $xy \in E(G)$. We will also write that $x$ is dominated by $D$ or by $y$ in $G$.

In [4] it was introduced the notion of split dominating set of a graph. We say that a dominating set $D \subseteq V(G)$ is a split dominating set of $G$ if the induced subgraph $\langle V(G) - D \rangle_G$ is disconnected. A dominating set $D \subseteq V(G)$ is a connected dominating set of $G$, (see [5]) if the induced subgraph $\langle D \rangle_G$ is connected. The domination number $\gamma(G)$, the connected domination number, respectively. Note that there exists a dominating set if and only if $\gamma(G) \leq \gamma_c(G)$ and also $\gamma(G) \leq \gamma_s(G)$. A dominating set $D$ is called a $(\gamma(G))$-set, $(\gamma_s(G))$-set, $(\gamma_c(G))$-set if $D$ realizes the domination [split domination, connected domination] number, respectively. Note that there exists a $(\gamma_c(G))$-set if and only if $G$ is connected. The conjunction of two graphs $G$ and $H$ is a graph $G \land H$, with $V(G \land H) = V(G) \times V(H)$ and $(g_1, h_1)(g_2, h_2) \in E(G \land H)$ if and only if $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. By $P_n$ we denote an induced path on $n \geq 2$ vertices meant as a graph with $V(P_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(P_n) = \{x_ix_{i+1} : i = 1, 2, \ldots, n - 1\}$. If $V(G) = \{y_1, y_2, \ldots, y_m\}$, then the copy $G^*$ of $G$ is the graph with the vertex set $V(G^*) = \{y_1^*, y_2^*, \ldots, y_m^*\}$ and $y_i^*y_j^* \in E(G^*)$ if and only if $y_iy_j \in E(G)$, where $y_i^*$ corresponds to $y_i$. Further, let $D = \{y_1, y_2, \ldots, y_r\} \subset V(G)$, then the subset $D^* = \{y_1^*, y_2^*, \ldots, y_r^*\} \subset V(G^*)$ is called a duplication of $D$ into the vertex set $V(G^*)$ of the copy $G^*$ or shorter into $G^*$.

We consider the conjunction $P_n \land G$, for $n \geq 2$ with a special graph $G$. Before proceeding we introduce some notation with respect to $P_n \land G$. If $y_j \in V(G)$, then the vertex $(x_i, y_j)$ of the conjunction of $P_n \land G$ is simply written as $x^j_i$. For a fixed integer $i$ we put $X_i = \{x^j_i : 1 \leq j \leq |V(G)|\}$. A set $B$ of all vertices belonging to $k$ consecutive sets $X_{i+1}, \ldots, X_{i+k}$ is called a block of a graph $P_n \land G$ of size $k \times |V(G)|$. For a convenience, the set $X_i$ we will call the $i$-th column of a graph $P_n \land G$. Any other terms not defined in this paper can be found in [1].

2. Introduction

In this section we introduce some basic facts which will be useful in further investigations. It was proved in [4], that

**Theorem 1** [4]. $\gamma_s(P_n) = \left\lceil \frac{n}{3} \right\rceil$, for $n \geq 3$. 
Theorem 2 [4]. For any noncomplete graph $G$ with at least one hanging vertex
\[ \gamma_s(G) = \gamma(G). \]

Next, it is easy to check that

Proposition 3. There is no a split dominating set of $\overline{P_n}$, for $i = 1, 2, 3$.

Proposition 4. $\gamma_s(P_4) = 2$, since $\overline{P_4} \cong P_4$.

Now, we calculate a split domination number of $\overline{P_n}$ if $n \geq 5$.

Theorem 5. $\gamma_s(\overline{P_n}) = n - 3$, for $n \geq 5$.

Proof. Let $V(P_n) = \{x_1, x_2, \ldots, x_n\}$, such that $d_{P_n}(x_1) = d_{P_n}(x_n) = 1$ and $d_{P_n}(x_i) = 2$, for $i = 2, 3, \ldots, n - 1$. At the beginning we can observe that $d_{\overline{P_n}}(x_1) = d_{\overline{P_n}}(x_n) = n - 2$ and $d_{\overline{P_n}}(x_i) = n - 3$, for $i = 2, 3, \ldots, n - 1$.

Now, we show that the induced subgraph $H = \langle\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\}\rangle_{\overline{P_n}}$ is connected, when $n_1 < n_2 < \ldots < n_k$, for $k \geq 4$. Since $n_3 - n_1 \geq 2$, $n_4 - n_1 \geq 2, \ldots, n_k - n_1 \geq 2$, then $x_{n_1}$ is adjacent to $x_{n_s}$ in $\overline{P_n}$, for $s = 3, 4, \ldots, k$. Hence $H_1 = \langle\{x_{n_1}, x_{n_3}, x_{n_4}, \ldots, x_{n_k}\}\rangle_{\overline{P_n}}$ is a connected subgraph. Arguing as above we prove that $H_2 = \langle\{x_{n_2}, x_{n_4}, x_{n_5}, \ldots, x_{n_k}\}\rangle_{\overline{P_n}}$ also is connected. Since $k \geq 4$, then $V(H_1) \cap V(H_2) \neq \emptyset$ and $H = \langle V(H_1) \cup V(H_2)\rangle_{\overline{P_n}}$ is connected.

This means that there is no a disconnected subgraph of $\overline{P_n}$ of order at least $n - 4$. To complete the proof we construct a split dominating set $D$ of $\overline{P_n}$, such that $|D| = n - 3$. Let $D$ consists of vertices $x_i$, for $i = 4, 5, \ldots, n$. Since $n \geq 5$, thus $D \neq \emptyset$ and $V(\overline{P_n}) - D = \{x_1, x_2, x_3\}$. Moreover, vertices $x_1, x_2$ are adjacent to $x_5 \in D$ in $\overline{P_n}$ and $x_3$ is adjacent to $x_5 \in D$ in $\overline{P_n}$. Furthermore, $x_2$ is an isolated vertex in $\langle V(\overline{P_n}) - D\rangle_{\overline{P_n}}$. All this together gives that $D$ is the minimum split dominating set of $\overline{P_n}$ of order $n - 3$, as required.

From the structure of $P_n$, $\overline{P_n}$ and from the definition of the connected domination number it follows immediately

Proposition 6.
\[ \gamma_c(P_n) = n - 2, \quad \text{for} \quad n \geq 3 \quad \text{and} \]
\[ \gamma_c(\overline{P_n}) = 2, \quad \text{for} \quad n \geq 4. \]
From Theorem 1, Theorem 5 and Proposition 6 it follows the Nordhaus-Gaddum type result

**Theorem 7.**

\[
\gamma_s(P_n) + \gamma_s(\overline{P_n}) = \left\lceil \frac{n}{3} \right\rceil + n - 3, \quad \text{for } n \geq 5,
\]

\[
\gamma_c(P_n) + \gamma_c(\overline{P_n}) = n, \quad \text{for } n \geq 4.
\]

3. Main Results

**Proposition 8.** For any graph \( G \), \( \gamma(P_2 \land G) \leq 2\gamma(G) \).

**Proof.** Let \( D = \{x_1, x_2, \ldots, x_s\} \) be a minimum dominating set of \( G \). Duplicating \( D \) into two columns \( P_2 \land G \) we obtain a subset \( A_2 = \{x^1_1, x^1_2, \ldots, x^1_s, x^2_1, x^2_2, \ldots, x^2_s\} \subset V(P_2 \land G) \). We show that \( A_2 \) is a dominating set of \( P_2 \land G \). Let \( x^1_j \in (V(P_2 \land G) - A_2) \). Since \( D \) is a dominating set of \( G \), there exists a vertex \( x_k \) of \( D \) in \( G \), such that \( x_kx^1_j \in E(G) \). Further, by the definition of \( P_2 \land G \) and by a construction of the subset \( A_2 \) we have that \( x^1_jx^2_k \in E(P_2 \land G) \) and \( x^2_k \in A_2 \), respectively. Hence \( x^1_j \) is dominated by \( A_2 \) in \( P_2 \land G \). Similarly, we can show that the vertex \( x^2_j \in (V(P_2 \land G) - A_2) \) is dominated by \( A_2 \) in \( P_2 \land G \). All this together gives that \( A_2 \) is a dominating set of \( P_2 \land G \) and \( \gamma(P_2 \land G) \leq |A_2| = 2\gamma(G) \), as required.

It follows immediately from the obvious inequality \( \gamma(G) \leq \gamma_c(G) \) and from the above proposition that

**Corollary 9.** For any connected graph \( G \), \( \gamma(P_2 \land G) \leq 2\gamma_c(G) \).

**Proposition 10.** For any graph \( G \) with \( \gamma_c(G) \geq 2 \),

\[
\gamma(P_3 \land G) \leq 2\gamma_c(G).
\]

**Proof.** Let \( D \) be a \( \gamma_c(G) \)-set. Put \( A_3 = \{x^2_j, x^3_j : \text{for all } x_j \in D\} \). Now we show that \( A_3 \) is a dominating set of \( P_3 \land G \). Arguing as in a proof of Proposition 8, we see that \( A_3 \) dominates vertices \( x^2_j, x^3_j \in (V(P_3 \land G) - A_3) \) in \( P_3 \land G \). To complete the proof we must show that any vertex from \( X_1 \) is dominated by \( A_3 \) in \( P_3 \land G \). We recall that \( X_1 \) is the first column of the graph \( P_3 \land G \) as it was mentioned earlier. Let \( x^1_j \in X_1 \). If \( x_j \in V(G) - D \), then it

\[
\vdots
\]
is dominated by a vertex $x_k \in D$ and in a consequence $x_j^1$ is dominated by $x_k^2 \in A_3$. Assume that $x_j \in D$. Since $(D)_G$ is connected and $|D| = \gamma_c(G) \geq 2$, thus there exists a vertex $x_k \in D$ different from $x_j$, such that $x_jx_k \in E(G)$. Moreover, $x_j^1x_k^2 \in E(P_3 \wedge G)$. This means that $x_j^1$ is dominated by $A_3$ in $P_3 \wedge G$ because of $x_k^2 \in A_3$. Hence $A_3$ is a dominating set of $P_3 \wedge G$. Since

\[\gamma(P_3 \wedge G) \leq |A_3| = 2\gamma_c(G),\]

thus the theorem is true. \hfill \blacksquare

**Remark 1.** It is easy to see that $A_3$ also is a dominating set of $P_3 \wedge G$, where $G$ is a graph with $\gamma_c(G) \geq 2$. Hence $\gamma(P_3 \wedge G) \leq 2\gamma_c(G)$ with $\gamma_c(G) \geq 2$.

**Proposition 11.** For any graph $G$ with $\gamma_c(G) \geq 2$

\[\gamma(P_3 \wedge G) \leq 3\gamma_c(G).\]

**Proof.** Let $D = \{x_1, \ldots, x_m\}$ be a minimum connected dominating set of $G$. Duplicating $D$ into 2-nd, 3-rd, 4-th column of $P_3 \wedge G$ we obtain a subset

\[A_5 = \{x_j^i : i = 2, 3, 4 \text{ and } j = 1, 2, \ldots, m\} \subset V(P_3 \wedge G).\]

Simple observation shows that $A_5$ is a dominating set of $P_3 \wedge G$. Thus $\gamma(P_3 \wedge G) \leq |A_5| = 3\gamma_c(G)$ and proof is complete. \hfill \blacksquare

In [2] it was presented the following result

**Proposition 12** [2]. For $n > 1$ and every graph $G$ we have

\[\gamma(P_n \wedge G) \leq 2\gamma(G)\left(\left\lfloor\frac{n}{4}\right\rfloor + 1\right).\]

**Counterexample.** Let $P_n = P_3$ and $G = P_5$, then $P_n \wedge G = P_3 \wedge P_5$ has two connected components, say $Y_1$ and $Y_2$. Further, this must be that $\gamma(P_3 \wedge P_5) = \gamma(Y_1) + \gamma(Y_2)$. It is easy to observe that $\gamma(Y_1) = 2$ and $\gamma(Y_2) = 3$, thus $\gamma(P_3 \wedge P_5) = 5$. Now, using the estimation from Proposition 12 we obtain

\[\gamma(P_3 \wedge P_5) \leq 4\left(\left\lfloor\frac{3}{4}\right\rfloor + 1\right) = 4,\]

since $\gamma(G) = \gamma(P_5) = 2$. It is not true, since as we noticed $\gamma(P_3 \wedge P_5) = 5$.

Using above facts we give another estimation for $\gamma(P_n \wedge G)$.

**Theorem 13.** Let $G$ be a graph with $\gamma_c(G) \geq 2$. Then, for $n \geq 2$ we have

\[\gamma(P_n \wedge G) \leq \begin{cases} 
\gamma_c(G)(2\left\lfloor\frac{n-1}{4}\right\rfloor + 1), & \text{if } n \equiv 1(\text{mod } 4), \\
\gamma_c(G)(2\left\lfloor\frac{n-1}{4}\right\rfloor + 2), & \text{otherwise}.
\end{cases}\]
\textbf{Proof.} Let \( n = 4q + r, \quad q \geq 1, \quad 0 \leq r < 4, \quad r \neq 1. \) Partition the set \( V(P_n \land G) \) into \( q \) blocks \( B_1, \ldots, B_q \) of size \( 4 \times |V(G)| \) and one block \( B_{q+1} \) of size \( r \times |V(G)| \) (it can be that \( B_{q+1} = \emptyset \)). Put \( A_j^i \) be a duplication of \( A_j \) into the block \( B_i \), for \( i = 1, 2, \ldots, q \) and \( j = 2, 3, 5 \), where \( A_j \) is the subset defined in the proofs of above propositions.

If \( n = 4q \), then \( D = \bigcup_{i=1}^q A_{3}^{i} \) is a dominating set of \( P_n \land G \) and

\[
|D| = 2q\gamma_c(G) = 2 \left( \left\lfloor \frac{4q - 1}{4} \right\rfloor + 1 \right) \gamma_c(G) = \gamma_c(G) \left( 2 \left\lfloor \frac{n - 1}{4} \right\rfloor + 2 \right).
\]

If \( n = 4q + 2 \), then \( D = \bigcup_{i=1}^q A_{3}^{i} \cup A_{2}^{q+1} \) is a dominating set of \( P_n \land G \) and

\[
|D| = 2q\gamma_c(G) + 2\gamma_c(G) = \left( 2 \left\lfloor \frac{4q + 1}{4} \right\rfloor + 2 \right) \gamma_c(G) = \gamma_c(G) \left( 2 \left\lfloor \frac{n - 1}{4} \right\rfloor + 2 \right).
\]

If \( n = 4q + 3 \), then \( D = \bigcup_{i=1}^{q+1} A_{3}^{i} \) is a dominating set of \( P_n \land G \) and

\[
|D| = 2(q + 1)\gamma_c(G) = 2 \left( \left\lfloor \frac{4q + 2}{4} \right\rfloor + 1 \right) \gamma_c(G) = \gamma_c(G) \left( 2 \left\lfloor \frac{n - 1}{4} \right\rfloor + 2 \right).
\]

Assume that \( n = 4q + 1 \). Thus we create \( q - 1 \) blocks of size \( 4 \times |V(G)| \), say \( B_1, \ldots, B_{q-1} \) and one block \( B_q \) of size \( 5 \times |V(G)| \). Let \( D = \bigcup_{i=1}^{q-1} A_{3}^{i} \cup A_{5}^{q} \), then \( D \) is a dominating set of \( P_n \land G \) with

\[
|D| = 2(q - 1)\gamma_c(G) + 3\gamma_c(G) = (2q + 1)\gamma_c(G)
\]

\[
= \left( 2 \left\lfloor \frac{4q}{4} \right\rfloor + 1 \right) \gamma_c(G) = \left( 2 \left\lfloor \frac{n - 1}{4} \right\rfloor + 1 \right) \gamma_c(G).
\]

Therefore, since \( \gamma(P_n \land G) \leq |D| \), the result holds, for \( n \geq 4 \) as it was assumed at the beginning of the proof.

Since \( 2\gamma_c(G) = (2 \left\lfloor \frac{n-1}{4} \right\rfloor + 2)\gamma_c(G) \), for \( n = 2, 3, 4 \) and \( 3\gamma_c(G) = (2 \left\lfloor \frac{5-1}{4} \right\rfloor + 1)\gamma_c(G) \), then Theorem 13 was proved for any \( n \geq 2. \) \( \blacksquare \)

Moreover, since \( \gamma_c(P_k) = 2 \), for \( k \geq 4 \), then the last result and a simple calculation lead to the following conclusion.
Corollary 14 [2]. For \( n \geq 2 \) and \( k \geq 4 \),

\[
\gamma(P_n \land T_k) \leq \begin{cases} 
n, & \text{if } n \equiv 0(\text{mod } 4), 
n + 1, & \text{if } n \equiv 1(\text{mod } 4) \text{ and } n \equiv 3(\text{mod } 4), 
n + 2, & \text{if } n \equiv 2(\text{mod } 4). 
\end{cases}
\]

Mention that for the graph \( P_3 \land P_5 \), considered after Proposition 12, using the estimation from Theorem 13 we have \( 5 = \gamma(P_3 \land P_5) \leq \gamma_c(P_5)(2 \lfloor \frac{3-1}{4} \rfloor + 2) = 6 \).

At the end, we consider the minimum split domination number of the conjunction of \( P_n \) and a graph \( G \) with a special property. First, we assume that \( G \) has at least two hanging vertices, then we have

**Proposition 15.** Let \( G \) be a graph with at least one hanging vertex. Then

\[
\gamma_s(P_n \land G) = \gamma(P_n \land G), \text{ for } n \geq 2.
\]

**Proof.** Let \( G \) be a graph as in the statement of the corollary. Since \( G \) has at least one hanging vertex, thus by the definition of \( P_n \land G \), we obtain that \( P_n \land G \) has at least one hanging vertex (note that it has at least two hanging vertices, since \( n \geq 2 \)). Then according to Theorem 2 we have that \( \gamma_s(P_n \land G) = \gamma(P_n \land G) \), as desired. 

Further, we assume that \( G \) is a connected graph with the minimum domination number equal to half its order.

The following result was given in [3].

**Theorem 16** [3]. A connected graph \( G \) of order \( 2n \geq 4 \) has \( \gamma(G) = n \) if and only if either \( G \cong C_4 \) or \( G \) satisfies: the vertex set of a graph \( G \) can be partitioned into two sets \( V_1 \) and \( V_2 \), such that \( |V_1| = |V_2| = n \) with only matching between \( V_1 \) and \( V_2 \) and satisfying \( \langle V_1 \rangle_G \cong K_n \) and \( \langle V_2 \rangle_G \) is connected.

From the above theorem it follows that the graph \( G \) different from \( C_4 \) has at least two hanging vertices. Moreover, according to Proposition 15, we observe that \( \gamma_s(P_n \land G) = \gamma(P_n \land G) \), for \( G \) mentioned in Theorem 16. Now, we give the estimation for the split domination number with respect to the conjunction of \( P_n \) and a graph \( G \) with the minimum domination number equal to half its order. But first we find a relationship between domination parameters in \( G \).
Theorem 17. Let $G$ be a connected graph of order $2n \geq 4$ with $\gamma(G) = n$. Then $\gamma_s(G) = \gamma_c(G) = \gamma(G)$.

Proof. Assume that $G \cong C_4$. The subset containing exactly two adjacent [not adjacent] vertices realizes $\gamma(G) = 2$ and it is a minimum connected [a minimum split dominating] set of $C_4$. Thus the result holds, for $C_4$. Now, assume that $G$ is different from $C_4$. By Theorem 16 we have that $V(G)$ can be partitioned into two sets $V_1$ and $V_2$ of order $n$, such that $(V_2)_{\gamma}$ is connected and $(V_1)_{\gamma} \cong \overline{K_n}$. This means that the subset $V_1$ is a set of all hanging vertices of $G$. Let $D = V_2$, since there is a matching between $V_1 = V(G) - D$ and $D$ in $G$. It means that $D$ is a minimum dominating set of $G$. To complete this theorem, we show that $D$ is a $\gamma_c(G)$-set and also a $\gamma_s(G)$-set. Because of $(D)_{\gamma}$ is connected, as it was stated in Theorem 13, then $D$ is a $\gamma_c(G)$-set. Moreover, since $(V(G) - D)_{\gamma} \cong \overline{K_n}$, $n \geq 2$ is disconnected, thus we $D$ is a $\gamma_s(G)$-set, proving the theorem.

Finally, using this theorem, Theorem 13 and Proposition 15 we obtain the following estimation for a split dominating number of $P_n \land G$.

Corollary 18. Let $G$ be a connected graph of order $2m \geq 4$ with $\gamma(G) = m$. Then

$$\gamma_s(P_n \land G) = \gamma(P_n \land G) \leq \begin{cases} \gamma(G)(2 \left\lfloor \frac{n-1}{4} \right\rfloor + 1), & \text{if } n \equiv 1(\text{mod } 4), \\ \gamma(G)(2 \left\lfloor \frac{n-1}{4} \right\rfloor + 2), & \text{otherwise}. \end{cases}$$

References


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