ON THE STABILITY FOR PANCYCLICITY

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Abstract

A property $P$ defined on all graphs of order $n$ is said to be $k$-stable if for any graph of order $n$ that does not satisfy $P$, the fact that $uv$ is not an edge of $G$ and that $G + uv$ satisfies $P$ implies $d_G(u) + d_G(v) < k$. Every property is $(2n - 3)$-stable and every $k$-stable property is $(k+1)$-stable. We denote by $s(P)$ the smallest integer $k$ such that $P$ is $k$-stable and call it the stability of $P$. This number usually depends on $n$ and is at most $2n - 3$. A graph of order $n$ is said to be pancyclic if it contains cycles of all lengths from 3 to $n$. We show that the stability $s(P)$ for the graph property "$G$ is pancyclic" satisfies

$$\max\left(\left\lceil \frac{4n}{3} \right\rceil - 2, n + t\right) \leq s(P) \leq \max\left(\left\lceil \frac{6n}{5} \right\rceil - 5, n + t\right)$$

where $t = 2\left\lceil \frac{n+1}{2} \right\rceil - (n + 1)$.

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1. Introduction

We use [3] for terminology and notation not defined here and consider simple graphs only. For any integer $k$, denote by $C_k$ a cycle of length $k$. A graph of order $n$ is said to be pancyclic if it contains cycles of all lengths from 3 to $n$.

In [2], Bondy and Chvátal introduced the closure of a graph and the stability of a graph property. The $k$-closure $C_k(G)$ of a graph $G$ is obtained by recursively joining pairs of non_adjacent vertices whose degree sum is at least $k$, until no such pair remains.
A property $P$ defined on all graphs of order $n$ is said to be $k$-stable if for any graph of order $n$ that does not satisfy $P$, the fact that $uv$ is not an edge of $G$ and that $G + uv$ satisfies $P$ implies $d_G(u) + d_G(v) < k$. Vice versa, if $uv \notin E(G), d_G(u) + d_G(v) \geq k$ and $G + uv$ has property $P$, then $G$ itself has property $P$. Every property is $(2n - 3)$-stable and every $k$-stable property is $(k + 1)$-stable. We denote by $s(P)$ the smallest integer $k$ such that $P$ is $k$-stable and call it the stability of $P$. This number usually depends on $n$ and is at most $2n - 3$.

**Theorem 1** [2]. The property $P$: "$G$ contains a cycle $C_k$" satisfies $s(P) = 2n - k$ for $4 \leq k \leq n$ and $s(P) = 2n - k - 1$ for $4 \leq k < n$ if $k$ is even.

**Question 1.** What is the stability for the property "$G$ is pancyclic"?

In 1971 Bondy [1] has posed the interesting "metaconjecture".

**Conjecture 1** (metaconjecture). Almost any non-trivial condition on a graph which implies that the graph is hamiltonian also implies that the graph is pancyclic (except for maybe a simple family of exceptional graphs).

By Theorem 1, $s(P) = n$ for the property "$G$ is hamiltonian". The complete bipartite graphs $K_{\frac{n}{2}, \frac{n}{2}}$ for $n$ even, $n \geq 4$, and $K_{\frac{n+1}{2}, \frac{n-1}{2}}$ for $n$ odd, $n \geq 5$, show that the stability $s(P)$ for the property "$G$ is pancyclic" satisfies $s(P) \geq n + t$ for all $n \geq 4$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n + 1)$. In [5] the following Theorem was proved.

**Theorem 2.** Let $G$ be a hamiltonian graph of order $n \geq 32$ and $u$ and $v$ two nonadjacent vertices with $d(u) + d(v) \geq n + t$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n + 1)$. Then $G$ contains all cycles of length $k$ where $3 \leq k \leq \frac{n+13}{5}$.

Moreover, examples were presented showing one cannot expect $G$ to contain cycles of length considerably longer than $\frac{n}{3}$ with the assumption of Theorem 2.

For the property $P$: "$G$ is pancyclic" we will prove the following Theorem.

**Theorem 3.** Let $P$ be the property "$G$ is pancyclic". Then the stability $s(P)$ satisfies $\max(\lfloor \frac{4n}{5} \rfloor - 5, n + t) \leq s(P) \leq \max(\lfloor \frac{4n}{5} \rfloor - 2, n + t)$, where $t = 2\lceil \frac{n+1}{2} \rceil - (n + 1)$.  

2. Exact Values and the Lower Bound

For a graph \( G \) of order \( n \) denote by \( s(P, n) \) the stability of the property "\( G \) is pancyclic". Then it is not very difficult to check that \( s(P, n) = n + t \) for \( 3 \leq n \leq 9 \), where \( t = 2\left\lceil \frac{n+1}{2} \right\rceil - (n+1) \).

Next we will give a proof for the lower bound given in Theorem 3.

**Proof.** As mentioned in the introduction the complete bipartite graphs \( K_{\frac{n}{2}, \frac{n}{2}} \) for \( n \) even, \( n \geq 4 \), and \( K_{\frac{n+1}{2}, \frac{n+1}{2}} \) for \( n \) odd, \( n \geq 5 \), show that \( s(P, n) \geq n + t \) for all \( n \geq 4 \), where \( t = 2\left\lceil \frac{n+1}{2} \right\rceil - (n+1) \).

1. For \( k \geq 1 \) let \( G_{5k} \) be the graph of order \( n = 5k \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and a Hamilton cycle \( C : v_1 \ldots v_nv_1 \). Define \( u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3} \). Let \( Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\} \) and \( P = \{v_{4k+2}, \ldots, v_{5k}\} \). Define \( N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\} \). Then \( d(u) + d(v) = 6k - 6 = n + \frac{n-30}{5} \) and the graph \( G + uv \) is pancyclic whereas \( G \) misses a cycle of length \( 2k + 3 \).

2. For \( k \geq 1 \) let \( G_{5k+1} \) be the graph of order \( n = 5k + 1 \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and a Hamilton cycle \( C : v_1 \ldots v_nv_1 \). Define \( u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3} \). Let \( Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\} \) and \( P = \{v_{4k+2}, \ldots, v_{5k+1}\} \). Define \( N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\} \). Then \( d(u) + d(v) = 6k - 4 = n + \frac{n-26}{5} \) and the graph \( G + uv \) is pancyclic whereas \( G \) misses a cycle of length \( 2k + 3 \).

3. For \( k \geq 1 \) let \( G_{5k+2} \) be the graph of order \( n = 5k + 2 \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and a Hamilton cycle \( C : v_1 \ldots v_nv_1 \). Define \( u = v_1, v = v_{k+1}, a = v_{2k+1}, b = v_{2k+2}, c = v_{4k+2}, d = v_{4k+3} \). Let \( Q = \{v_2, \ldots, v_k\}, R = \{v_{k+2}, \ldots, v_{2k+2}\}, S = \{v_{2k+3}, \ldots, v_{4k+1}\} \) and \( P = \{v_{4k+2}, \ldots, v_{5k+2}\} \). Define \( N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\} \). Then \( d(u) + d(v) = 6k - 2 = n + \frac{n-22}{5} \) and the graph \( G + uv \) is pancyclic whereas \( G \) misses a cycle of length \( 2k + 3 \).

4. For \( k \geq 1 \) let \( G_{5k+3} \) be the graph of order \( n = 5k + 3 \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and a Hamilton cycle \( C : v_1 \ldots v_nv_1 \). Define \( u = v_1, v = v_{k+4}, a = v_{2k+4}, b = v_{2k+5}, c = v_{4k+4}, d = v_{4k+5} \). Let \( Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+3}\}, S = \{v_{2k+4}, \ldots, v_{4k+3}\} \) and \( P = \{v_{4k+4}, \ldots, v_{5k+3}\} \). Define \( N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\} \).
Lemma 1. Then \(d(u) + d(v) = 6k - 2 = n + \frac{28}{5} \) and the graph \( G + uv \) is pancyclic whereas \( G \) misses a cycle of length \( 2k + 4 \).

5. For \( k \geq 0 \) let \( G_{5k+4} \) be the graph of order \( n = 5k + 4 \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and a Hamilton cycle \( C : v_1 \ldots v_n v_1 \). Define \( u = v_1, v = v_{k+2}, a = v_{2k+2}, b = v_{2k+3}, c = v_{4k+4}, d = v_{4k+5} \). Let \( Q = \{v_2, \ldots, v_{k+1}\}, R = \{v_{k+3}, \ldots, v_{2k+3}\}, S = \{v_{2k+4}, \ldots, v_{4k+3}\} \) and \( P = \{v_{4k+4}, \ldots, v_{5k+4}\} \). Define \( N(u) = Q \cup P \cup R - \{a, b\}, N(v) = Q \cup P \cup R - \{c, d\} \). Then \( d(u) + d(v) = 6k = n + \frac{24}{5} \) and the graph \( G + uv \) is pancyclic whereas \( G \) misses a cycle of length \( 2k + 4 \).

Summarizing we obtain that \( s(P) \geq \max([\frac{5n}{6}] - 5, n+t) \), where \( t = 2[\frac{n+1}{2}] - (n+1) \).

3. The Upper Bound

In this section we will give a proof for the upper bound given in Theorem 3. For this proof we will use the following results.

Corollary 1 [4]. Let \( G \) be a Hamiltonian graph of order \( n \). If there exist two nonadjacent vertices \( u \) and \( v \) at distance \( d \geq 3 \) on a Hamiltonian cycle of \( G \) such that \( d(u) + d(v) \leq n + d - 2 \), then \( G \) contains cycles of all lengths between \( 3 \) and \( n - d + 1 \).

Lemma 1 [4]. Let \( G \) contain a Hamiltonian path \( P = v_1v_2 \ldots v_n \) such that \( v_1v_n \notin E(G) \) and \( d(v_1) + d(v_n) \geq n + d \) for some integer \( d, 0 \leq d \leq n - 4 \). Then for any \( l, 2 \leq l \leq d + 3 \), there exists a \((v_1, v_n)\)-path of length \( l \).

Theorem 4 [4]. Let \( G \) be a graph of order \( n \). If \( G \) has a Hamiltonian \((u, v)\)-path for a pair of nonadjacent vertices \( u \) and \( v \) such that \( d(u) + d(v) \geq n \), then \( G \) is pancyclic.

Proof of Theorem 3. Suppose there is a graph \( G \) with nonadjacent vertices \( u, v \) such that \( d(u) + d(v) \geq \max([\frac{4n}{5}] - 2, n+t) \), \( G + uv \) is pancyclic, but \( G \) is not. Then \( n \geq 10 \). By Theorem 1, \( G \) is Hamiltonian. Let \( C : v_1 \ldots v_nv_1 \) be a Hamilton cycle in \( G \). Choose the labeling such that \( u = v_1, v = v_{r+2} \) with \( n = r + s + 2 \) and \( r \leq s \). Let \( R = \{v_2, \ldots, v_{r+1}\}, S = \{v_{r+3}, \ldots, v_n\} \) and \( d = d_C(u, v) = r+1 \). Set \( d(u) + d(v) = r + p + s + q \), where \( d_R(u) + d_R(v) = r + p \) and \( d_S(u) + d_S(v) = s + q \). Recall that \( d(u) + d(v) \geq \lceil \frac{4n}{3} \rceil - 2 \). By Theorem 1, \( G \) contains cycles \( C_k \) for \( \lceil \frac{2}{3}n \rceil + 2 \leq k \leq n \).
We distinguish several cases.

**Case 1.** \( d \leq \left[ \frac{n}{3} \right] \).

Since \( n \geq 227 \) we have \( d(u) + d(v) \geq n + 2 \). Thus \( d_S(u) + d_S(v) \geq s + 2 \) for \( 2 \leq d \leq 3 \). By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{s+2} \). Hence \( G \) is pancyclic for \( d = 2 \), a contradiction.

So we may assume that \( d \geq 3 \). By Corollary 1, \( G \) contains cycles \( C_3, \ldots, C_{n-d+1} \). Hence \( G \) is pancyclic since \( n - d + 1 \geq \left[ \frac{2n}{3} \right] + 1 \), a contradiction.

**Case 2.** \( d \geq \left[ \frac{n}{3} \right] + 1 \).

**Subcase 2.1.** \( d_S(u) + d_S(v) \geq s + 2 \).

By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{s+2} \). Note that \( s + 2 \geq \frac{n}{2} + 1 \).

**Subcase 2.1.1.** \( p \geq \left[ \frac{2n}{3} \right] - s \).

By Lemma 1 we can take \((u,v)\)-paths of length \( l \) in \( R \cup \{ u, v \} \) for \( 2 \leq l \leq p + 1 \) and a \((v,u)\)-path of length \( s + 1 \) in \( S \cup \{ u, v \} \). This gives cycles \( C_{s+3}, \ldots, C_{s+p+2} \). Hence \( G \) is pancyclic since \( s + p + 2 \geq \left[ \frac{2n}{3} \right] + 2 \), a contradiction.

**Subcase 2.1.2.** \( p \leq \left[ \frac{2}{3} n \right] - s - 1 \).

Then \( q \geq \left[ \frac{2}{3} \right] - 2 - 2 - p \geq \left[ \frac{2}{3} \right] - \left[ \frac{2n}{3} \right] + s + 1 \geq s + 1 - \left[ \frac{n}{3} \right] \geq 2 \) for \( n \geq 11 \). Take \((v,u)\)-paths of length \( l \) for \( 2 \leq l \leq s - \left[ \frac{n}{3} \right] + 2 \) in \( S \cup \{ u, v \} \). This gives cycles \( C_{n-s+1+2}, \ldots, C_{\left[ \frac{2n}{3} \right]+1} \). Hence \( G \) is pancyclic, a contradiction. It is easy to check that for \( n = 10 \) and \( s = 4 \) \( G \) is also pancyclic and we get a contradiction.

**Subcase 2.2.** \( d_S(u) + d_S(v) \leq s + 1 \).

Then \( d_R(u) + d_R(v) \geq r + 1 + \left[ \frac{n}{3} \right] - 2 \). By Theorem 4, \( G \) contains cycles \( C_3, \ldots, C_{r+2} \). Set \( r + 2 = \left[ \frac{n}{3} \right] + 1 + d' \). By Lemma 1 there are \((u,v)\)-paths of lengths \( l \) for \( 2 \leq l \leq \left[ \frac{n}{3} \right] \) in \( R \cup \{ u, v \} \). This gives cycles \( C_{s+1+2}, \ldots, C_{s+1+\left[ \frac{3}{2} \right]} \). So far cycles of lengths \( \left[ \frac{n}{3} \right] + d' + 2, \ldots, \left[ \frac{2n}{3} \right] - d' + 1 \) are missing.

Let \( S = S_1 \cup S_2 \cup S_3 \) with \( S_1 = \{ v_1, \ldots, v_{n-\left[ \frac{n}{3} \right]} \}, S_2 = \{ v_{\left[ \frac{n}{3} \right]+1}, \ldots, v_{2\left[ \frac{n}{3} \right]+d'+1} \} \) and \( S_3 = \{ v_{2\left[ \frac{n}{3} \right]+d'+2}, \ldots, v_n \} \). Then \( |S_1| = n - 2 \left[ \frac{n}{3} \right] - d' - 1 = |S_2| \) and \( |S_2| = d' + 1 + 3 \left[ \frac{n}{3} \right] - n \).

Suppose \( uv_i \in E(G) \) for some \( i \) with \( \left[ \frac{n}{3} \right] + 2 + d' \leq i \leq n \). Then there is a path \( uv_i v_{i-1} \ldots v \) of length \( i - \left( \left[ \frac{n}{3} \right] + d' + 1 \right) + 1 \). Together with the \((u,v)\)-paths in \( R \cup \{ u, v \} \) we obtain cycles of lengths \( i - \left[ \frac{n}{3} \right] - d' + 2, \ldots, i - d' \). Hence, for \( n - \left[ \frac{n}{3} \right] + 1 \leq i \leq n - \left[ \frac{n}{3} \right] + 2d' \), we obtain all missing cycles and \( G \) is pancyclic, a contradiction.
A symmetric argument applies for edges $vv_i$ with $\lceil \frac{n}{3} \rceil + 2 + d' \leq i \leq n$. In this case, for $n - \lceil \frac{n}{3} \rceil - d' + 2 \leq i \leq 2\lceil \frac{n}{3} \rceil + d' + 1$, we obtain all missing cycles and $G$ is pancyclic, a contradiction.

Hence we may assume that $N_{S_2}(u) = N_{S_2}(v) = \emptyset$. Suppose $N_S(u) \cap N_S(v) = \emptyset$. Then $(d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2(\lceil \frac{n}{3} \rceil + d' - 1) + 2(n - 2\lceil \frac{n}{3} \rceil - d' - 1) = 2n - 2\lceil \frac{n}{3} \rceil - 4 \leq n + \lceil \frac{n}{3} \rceil - 4 < \lceil \frac{4n}{3} \rceil - 2$, a contradiction. Hence $N_S(u) \cap N_S(v) \neq \emptyset$. Thus there is a cycle of length $\lceil \frac{n}{3} \rceil + d' + 2$.

Next consider two vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \lceil \frac{n}{3} \rceil$. If $|E(\{x, y\}, \{u, v\})| \geq 3$ then there is a $(u, v)$-path of length $\lceil \frac{n}{3} \rceil + 2$. Together with the $(u, v)$-paths through $R$ we obtain cycles of lengths $\lceil \frac{n}{3} \rceil + 4, \ldots, 2\lceil \frac{n}{3} \rceil + 2$ and $G$ is pancyclic (recall that $d' \geq 1$).

Hence we may further assume that $|E(\{x, y\}, \{u, v\})| \leq 2$ for all pairs of vertices $x \in S_1, y \in S_3$ with $d_C(x, y) = \lceil \frac{n}{3} \rceil$. But then $\lfloor \frac{4n}{3} \rfloor - 2 \leq (d_R(u) + d_R(v)) + (d_S(u) + d_S(v)) \leq 2(\lceil \frac{n}{3} \rceil + d' - 1) + 2(n - 2\lceil \frac{n}{3} \rceil - d' - 1) = 2n - 2\lceil \frac{n}{3} \rceil - 4 \leq n + \lceil \frac{n}{3} \rceil - 4 < \lceil \frac{4n}{3} \rceil - 2$, a final contradiction. \hfill \blacksquare

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References


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