GENERALIZED MORPHISMS OF ABELIAN \( m \)-ARY GROUPS

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Abstract

We prove that the set of all \( n \)-ary endomorphisms of an abelian \( m \)-ary group forms an \((m, n)\) - ring.

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The terminology and notation used in this paper is standard (see, for example, [7] and [5]). The bibliography of \( m \)-ary groups (till 1982) is given in the survey [3] prepared by K. Glazek.

Let \( \{A_1, A_2, ..., A_{n-1}, A_n\} \) be the sequence of \( m \)-ary groups, where \( m, n \geq 2 \) are fixed. The sequence \( f = \{f_1, f_2, ..., f_{n-1}\} \) of homomorphisms

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} ... \xrightarrow{f_{n-2}} A_{n-1} \xrightarrow{f_{n-1}} A_n
\]

is called an \( n \)-ary homomorphism (cf. [2]).

If \( A_n = A_1 \), then this homomorphism is called an \( n \)-ary endomorphism.

By \( \text{End}(A_1, A_2, ..., A_{n-1}) \) we denote the set of all \( n \)-ary endomorphisms of the sequences \( \{A_1, A_2, ..., A_{n-1}, A_1\} \) of \( m \)-ary groups. It is clear that \( f \) defined in such a way is an \( n \)-ary isomorphism iff all \( f_i \) are isomorphisms.

Let \( f_i = \{f_{i1}, f_{i2}, ..., f_{i(n-1)}\}, i = 1, ..., n \), be an \( n \)-ary homomorphism which corresponds to the sequence

\[
f_i : B_i \xrightarrow{f_{i1}} A_1 \xrightarrow{f_{i2}} ... \xrightarrow{f_{i(n-2)}} A_{n-2} \xrightarrow{f_{i(n-1)}} B_{i+1},
\]
where $B_1, ..., B_{n+1}, A_1, ..., A_{n-2}$ are $m$-ary groups. The \textit{n-ary product} of such\n$n$-ary homomorphisms is defined in the same way as E.L. Post defines the\ncomposition of $m$-ary permutations (cf. [5], p. 249 and [6]).\nNamely:\n\[
g = [f_1 f_2 \cdots f_{n-1} f_n] = \{f_1 f_2 \cdots f_{(n-2)(n-2)} f_{(n-1)(n-1)} f_{n1}, \]
\[
\vdots \]
\[
f_{1k} f_{2(k+1)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)(n-1)} f_{(n-1)(k-1)} f_{nk}, \]
\[
\vdots \]
\[
f_{1(n-1)} f_{21} \cdots f_{(n-2)(n-3)} f_{(n-1)(n-2)} f_{n(n-1)} = \{g_1, g_2, \ldots, g_{n-1}\},\]
i.e., as the \textit{skew product} in the matrix $[f_{ij}]_{m \times (n-1)}$.

Such defined a product is an \textit{n-ary homomorphism} of the sequence\n$\{B_1, A_1, ..., A_{n-2}, B_{n+1}\}$ because

\[
g : B_1 \xrightarrow{g_1} A_1 \xrightarrow{g_2} A_2 \xrightarrow{g_3} \cdots \xrightarrow{g_{n-2}} A_{n-2} \xrightarrow{g_{n-1}} B_{n+1}.
\]
In [2] is proved that $< \text{End}(A_1, A_2, ..., A_{n-1}); [\ ] >$ is an $n$-ary semigroup.\nRemark that some results on $m$-ary transformations of commutative $n$-ary\ngroups are also contained in [7].

Now, let $A_1, A_2, ..., A_{n-1}$ be abelian $m$-ary groups and let $\varphi_j$ be the\nmapping defined by the formula

\[
a^{\varphi_j} = (a^{f_{i1}} a^{f_{i2}} \cdots a^{f_{im}}),
\]
where $\{f_{i1}, ..., f_{i(n-1)}\} = f_i \in \text{End}(A_1, A_2, ..., A_{n-1}), i = 1, ..., m, a \in A_j,$\nj = 1, ..., $n - 1$. Since such defined $\varphi_j$ are homomorphisms (cf. [2]), we have

\[
\{\varphi_1, \varphi_2, ..., \varphi_{n-1}\} = \varphi \in \text{End}(A_1, A_2, ..., A_{n-1}).
\]
This means that in \( \text{End}(A_1, A_2, ..., A_{n-1}) \) is defined an \( m \)-ary operation \(( )\) by the formula

\[
(f_1 f_2 ... f_m) = \varphi.
\]

Recall (cf. for example [1]) that a non-empty set \( A \) with two operations \(( ) : A^m \to A\) and 
\([ \ ] : A^n \to A\) is said to be an \((m, n)\)-ring if

1) \( < A; ( ) > \) is an abelian \( m \)-ary group;
2) \( < A; [ ] > \) is an \( n \)-ary semigroup;
3) \( [a_1^{i-1}b_1^n a_{i+1}^n] = ([a_1^{i-1}b_1 a_{i+1}^n] ... [a_1^{i-1}b_m a_{i+1}^n]) \) for all  \( i = 1, ..., n \)
and \( a_1, ..., a_n, b_1, ..., b_m \in A. \)

**Theorem.** If all \( m \)-ary groups \( A_1, ..., A_{n-1} \) are abelian, then

\[< \text{End}(A_1, ..., A_{n-1}); ( ), [ ] >\]

is an \((m, n)\)-ring.

In the proof of this theorem we use properties of elements formulated in two easily verified lemmas given below.

Recall that two sequences \( \alpha \) and \( \beta \) of elements from an \( m \)-ary group
\( < A; [ ] > \) are equivalent if there are sequences \( \delta \) and \( \gamma \) of elements from \( A \)
such that \([\gamma, \alpha, \delta] = [\gamma, \beta, \delta].\)

**Lemma 1.** Let \( \varphi : A \to B \) be a homomorphism of an \( m \)-ary groups. If
\( a_1^k \) and \( b_1^{i+k(m-1)} \) are equivalent in \( A \), then \( a_1^\varphi ... a_i^\varphi \) and \( b_1^\varphi ... b_{i+k(m-1)}^\varphi \) are equivalent in \( B. \)

**Lemma 2.** Let \( \varphi : A \to B \) be a homomorphism of an \( m \)-ary groups. If \( a_1^k \) is the inverse sequence for \( a \in A \), then \( a^\varphi a_1^\varphi ... a_k^\varphi \) and \( a_1^\varphi ... a_k^\varphi a^\varphi \) are neutral sequences in \( B. \)

**Proof of Theorem.** In [2] it is proved that \( < \text{End}(A_1, ..., A_{n-1}); [ ] > \) is an \( n \)-ary semigroup.

Now, we prove that \( < \text{End}(A_1, ..., A_{n-1}); ( ) > \) is an \( m \)-ary group.
Let
\[(f_1 f_2 \ldots f_m) f_{m+1} \ldots f_{2m-1}) = g = \{g_1, g_2, \ldots, g_{n-1}\};\]
\[(f_1 \ldots f_i (f_{i+1} \ldots f_{i+m}) f_{i+m+1} \ldots f_{2m-1}) = h\]
\[= \{h_1, h_2, \ldots, h_{n-1}\}, \ i = 1, 2, \ldots, m - 1;\]
\[(f_1 f_2 \ldots f_m) = \varphi = \{\varphi_1, \varphi_2, \ldots, \varphi_{n-1}\};\]
and
\[(f_{i+1} \ldots f_{i+m}) = \psi = \{\psi_1, \psi_2, \ldots, \psi_{n-1}\},\]
where \(f_j = \{f_{j1}, f_{j2}, \ldots, f_{j(n-1)}\}, \ j = 1, 2, \ldots, 2m - 1.\)
Moreover the brackets \((\ )\) will be also denoted the derived (extended) operation.

At first, we prove the associativity of the \(m\)-ary operation \((\ )\). Observe that
\[a^{\varphi_j} = (a^{f_{j1}} a^{f_{j2}} \ldots a^{f_{jm}}),\] where \(a \in A_j, j = 1, \ldots, n - 1,\) implies
\[a^{g_j} = \left( a^{\varphi_j} a^{f_{(m+1)j}} \ldots a^{f_{(2m-1)j}} \right) = \]
\[= \left( \left( a^{f_{j1}} a^{f_{j2}} \ldots a^{f_{jm}} \right) a^{f_{(m+1)j}} \ldots a^{f_{(2m-1)j}} \right) = \]
\[= \left( a^{f_{j1}} a^{f_{j2}} \ldots a^{f_{(2m-1)j}} \right).\]
Hence,
\[a^{g_j} = \left( a^{f_{j1}} a^{f_{j2}} \ldots a^{f_{(2m-1)j}} \right). \tag{1}\]

Similarly, \[a^{\psi_j} = (a^{f_{(i+1)j}} \ldots a^{f_{(i+m)j}})\] implies
\[a^{h_j} = \left( a^{f_{j1}} \ldots a^{f_{ij} a^{\psi_j} a^{f_{(i+m+1)j}} \ldots a^{f_{(2m-1)j}}} \right) = \]
\[= \left( a^{f_{j1}} \ldots a^{f_{ij}} \left( a^{f_{(i+1)j}} \ldots a^{f_{(i+m)j}} \right) a^{f_{(i+m+1)j}} \ldots a^{f_{(2m-1)j}} \right) = \]
\[= \left( a^{f_{j1}} a^{f_{j2}} \ldots a^{f_{(2m-1)j}} \right),\]
i. e.,
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\[
a^{h_j} = \left( a^{f_{1j}^{j}} a^{f_{2j}^{j}} \ldots a^{f_{(2m-1)j}^{j}} \right).
\]

From (1) and (2), we get \( g_j = h_j \), for all \( j = 1, \ldots, n - 1 \). Therefore \( g = h \) and, in the consequence,

\[
\left( \left( f_1^m \right) f_{m+1}^{2m-1} \right) = \left( f_1^i \left( f_{i+1}^{i+m} \right) f_{i+m+1}^{2m-1} \right)
\]

for all \( i = 1, \ldots, m - 1 \), which proves that \( < \text{End}(A_1, ..., A_{n-1}); (\ ) > \) is an \( m \)-ary semigroup. It is an abelian \( m \)-ary semigroup, because all \( m \)-ary groups \( A_1, ..., A_{n-1} \) are abelian.

Now we prove that the equation

\[
(f_1 f_2 \ldots f_{m-1} u) = \varphi,
\]

(3)

where

\[
f_1, f_2, ..., f_{m-1}, \varphi \in \text{End}(A_1, ..., A_{n-1}),
\]

\[
f_i = \{ f_{i1}, f_{i2}, ..., f_{i(n-1)} \}, \ i = 1, 2, ..., m - 1,
\]

\[
\varphi = \{ \varphi_1, \varphi_2, ..., \varphi_{n-1} \},
\]

has a solution \( u \in \text{End}(A_1, ..., A_{n-1}) \).

Note that \( a_1, ..., a_k \) is the inverse sequence for \( a_j \in A_j \), then the mapping

\[
u_j : a \rightarrow \left( a_1^{f_{1j}^{(m-1)j}} \ldots a_k^{f_{kJ}^{(m-1)j}} \ldots a_1^{f_{1j}^{j}} \ldots a_k^{f_{kJ}^{j} a_{kj}^{j} \varphi_j} \right)
\]

is a homomorphism.

Indeed, if \( b_{11}, ..., b_{ik} \in A_j \) is the inverse sequence for \( b_i \in A_j \) \( (i = 1, 2, ..., m) \) and \( d_1, ..., d_k \in A_j \) is the inverse sequence for \( (b_1 b_2 \ldots b_m) \in A_j \), then

\[
b_{m1}, ..., b_{mk}, ..., b_{21}, ..., b_{2k}, b_{11}, ..., b_{1k}
\]

(4)

is an inverse sequence for \( (b_1 b_2 \ldots b_m) \). Thus \( d_1, ..., d_k \) and (4) are equivalent. By Lemma 1,
are also equivalent sequences.

Using this fact and the abelianity of all $m$-groups $A_1, \ldots, A_{n-1}$, we get

\[
(b_1 b_2 \ldots b_m)^{u_j} = \left( d_1^{f_1(j)} \ldots d_k^{f_k(j)} \right) (b_1 b_2 \ldots b_m)^{\varphi_j} = \\
= \left( d_1^{f_1(j)} \ldots d_k^{f_k(j)} \right) \left( b_1^{f_1(j)} b_2^{f_2(j)} \ldots b_k^{f_k(j)} \right) = \\
= \left( b_1^{u_j} \ldots b_m^{u_j} \right).
\]

This proves that $u_j$ is a homomorphism for every $j = 1, \ldots, n - 1$. Hence $u = \{u_1, \ldots, u_{n-1}\} \in \text{End}(A_1, \ldots, A_{n-1})$. Moreover, by Lemma 2, we get

\[
\left( a^{f_1(j)} \ldots a^{f_{(m-1)}(j)} a^{u_j} \right) = \\
\left( a^{f_1(j)} \ldots a^{f_{(m-1)}(j)} \left( a_1^{f_1(j)} \ldots a_k^{f_k(j)} \ldots a_1^{u_j} \ldots a_k^{u_j} \right) \right) = \\
= \left( a^{f_1(j)} a_1^{u_j} \ldots a_k^{u_j} \ldots a^{f_{(m-1)}(j)} a_1^{u_j} \ldots a_k^{u_j} \right) = a^{\varphi_j}
\]

Therefore, we have (3). Since, the operation $()$ defined on $\text{End}(A_1, \ldots, A_{n-1})$, is abelian, we have that $\langle \text{End}(A_1, \ldots, A_{n-1}); () \rangle$ is an abelian $m$-ary group.

Now, we prove the identity

\[
\left[ f_i^{-1} (g_i^{m_i}) f_i^{n_i} \right] = \left( \left[ f_i^{-1} g_i f_i^{n_i} \ldots f_i^{-1} g_i f_i^{n_i} \right] \right),
\]

where $i = 1, \ldots, n.$
Let

$$\left[ f_1^{i-1} (g_1^m) f_{i+1}^n \right] = \{ s_1, s_2, \ldots, s_{n-1} \};$$

$$\left( \left[ f_1^{i-1} g_1 f_{i+1}^n \right] \cdots \left[ f_1^{i-1} g_m f_{i+1}^n \right] \right) = \{ r_1, r_2, \ldots, r_{n-1} \};$$

$$f_k = \{ f_{k1}, f_{k2}, \ldots, f_{k(n-1)} \}, \ k = 1, \ldots, n;$$

$$g_j = \{ g_{j1}, g_{j2}, \ldots, g_{j(n-1)} \}, \ j = 1, \ldots, m;$$

$$\left[ f_1^{i-1} g_j f_{i+1}^n \right] = \{ t_{j1}, t_{j2}, \ldots, t_{j(n-1)} \}, \ j = 1, \ldots, m;$$

and

$$\left( g_1^m \right) = \{ \varphi_1, \varphi_2, \ldots, \varphi_{n-1} \}.$$

It is clear that identity (5) is satisfied only in the case when \( s_k = r_k \) for all \( k = 1, \ldots, n - 1. \)

For \( 1 \leq i \leq n - k, \) we have

$$a^{s_k} =$$

$$a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i} \cdots f_{(i+k-1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}$$

$$= (a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i} \cdots f_{(i+k-1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk})$$

$$= \left( a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk} \right)$$

$$\cdots a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{mi} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}$$

and

$$a^{r_k} = \left( a^{t_{1k}} \cdots a^{t_{mk}} \right) =$$

$$\left( a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{1i} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk} \right)$$

$$\cdots a f_{1k} \cdots f_{(i-1)(i+k-2)} g_{mi} f_{(i+1)(i+k)} \cdots f_{(n-k)(n-1)} f_{(n-k+1)1} \cdots f_{(n-1)(k-1)} f_{nk}.$$
Thus, \( a^{sk} = a^{rk} \) and, in the consequence, \( s_k = r_k \).

If \( n - k < i \leq n \), then

\[
a^{sk} = a^{f_{1k}}...a^{f_{(n-k)(n-k+1)}}...a^{f_{(i-1)(i+k-n-1)}}a^{f_{(i)(i+k-n-1)}}a^{f_{(i+1)(i+k-n-1)}}...a^{f_{(n-1)(k-1)}}a^{fnk} =
\]

\[
(a^{f_{1k}}...a^{f_{(n-k)(n-k+1)}}...a^{f_{(i-1)(i+k-n-1)}}a^{f_{(i)(i+k-n-1)}}a^{f_{(i+1)(i+k-n-1)}}...a^{f_{(n-1)(k-1)}}a^{fnk}) =
\]

\[
= (a^{f_{1k}}...a^{f_{(n-k)(n-k+1)}}...a^{f_{(i-1)(i+k-n-1)}}a^{f_{(i)(i+k-n-1)}}a^{f_{(i+1)(i+k-n-1)}}...a^{f_{(n-1)(k-1)}}a^{fnk}) =
\]

\[
...a^{f_{1k}}...a^{f_{(n-k)(n-k+1)}}...a^{f_{(i-1)(i+k-n-1)}}a^{f_{(i)(i+k-n-1)}}a^{f_{(i+1)(i+k-n-1)}}...a^{f_{(n-1)(k-1)}}a^{fnk}
\]

which – similarly as in the previous case – give, \( s_k = r_k \).

This completes the proof.

\[\Box\]

**Corollary 1.** If \( < A_1; +, -, 0 >, ..., < A_{n-1}; \{+, -, 0\} > \) are abelian groups, then \( < \text{End}(A_1, ..., A_{n-1}); \{+, -, \Theta, \} > \) is the multiring, where \( \Theta = (0, ..., 0) \).

**Corollary 2 ([4]).** The set of all endomorphisms of an abelian \( m \)-ary group forms an \( (m, 2) \)-ring.

### References


Generalized morphisms of abelian $m$-ary groups


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