APPENDIX TO THE PAPER
”OSGOOD TYPE CONDITIONS FOR AN mTH ORDER DIFFERENTIAL EQUATION”

STANISLAW SZUFLA

Department of Mathematics and Informatics
Adam Mickiewicz University
Poznań, Poland

Keywords: initial value problems, measures of noncompactness.

2000 Mathematics Subject Classification: 34G20.

Assume that \( I = [0, a] \), \( E \) is a Banach space, \( B = \{ x \in E : \| x \| \leq b \} \) and \( f : I \times B \rightarrow E \) is a bounded continuous function. Let \( M = \sup\{ \| f(t, x) \| : t \in I, x \in B \} \). We choose a positive number \( d \) such that \( d \leq a \) and

\[
\sum_{j=1}^{m-1} \| \eta_j \| \frac{d^j}{j!} + M \frac{d^m}{m!} \leq b
\]

for given \( \eta_1, \ldots, \eta_{m-1} \in E \). Put \( J = [0, d] \) and

\[
F(x)(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds \quad (t \in J).
\]

Denote by \( C(J, E) \) the Banach space of continuous functions \( x : J \mapsto E \) with usual supremum norm. Let \( \hat{B} \subset C(J, E) \) be the subset of those functions with values in \( B \). It is known that \( F \) is a continuous mapping \( \hat{B} \mapsto \hat{B} \) and

(1) \[ \| F(x)(t) - F(x)(\tau) \| \leq K |t - \tau| \quad \text{for } t, \tau \in J \text{ and } x \in \hat{B}, \]

where \( K = \sum_{j=1}^{m-1} \| \eta_j \| \frac{d^{j-1}}{(j-1)!} + M \frac{d^m}{(m-1)!} \). Moreover, a continuous function \( x : J \mapsto B \) is a solution to the Cauchy problem

\[
\frac{d^m x}{dt^m} = f(t, x) \quad \text{with } x(0) = x_0.
\]
S. Szulafa

\[
x^{(m)} = f(t, x) \\
x(0) = 0, x'(0) = \eta_1, \ldots, x^{(m-1)}(0) = \eta_{m-1}
\]

if \( x \) is a fixed point of \( F \).

The purpose of this paper is to show that the following theorem is true:

**Theorem.** Let \( w : [0, 2b] \to \mathbb{R}_+ \) be a continuous nondecreasing function such that \( w(0) = 0 \), \( w(r) > 0 \) for \( r > 0 \) and

\[
\int_{0^+} \frac{dr}{\eta^m w(r)} = \infty.
\]

If

\[
\|f(t,x) - f(t,y)\| \leq w(\|x - y\|) \quad \text{for} \quad t \in I, \ x, y \in B,
\]

then the successive approximations \( u_n \), defined by

\[
u_0 = 0, u_{n+1} = F(u_n) \quad \text{for} \quad n \in N,
\]

converge uniformly on \( J \) to the unique solution \( u \) of (2).

**Proof.** First, similarly as in the proof of Theorem III. 9.1 in [1], we shall show that

\[
\lim_{n \to \infty} \|u_n(t) - u_{n-1}(t)\| = 0 \quad \text{for} \quad t \in J.
\]

Put \( \lambda(t) = \lim_{n \to \infty} \|u_n(t) - u_{n-1}(t)\| \). From (1) and (2) it is clear that

\[
\|u_n(t_1) - u_{n-1}(t_1)\| \leq \|u_n(t_2) - u_{n-1}(t_2)\| + 2K|t_1 - t_2|.
\]

For any \( \varepsilon > 0 \) there is \( n_0 \in N \) such that

\[
\|u_n(t_2) - u_{n-1}(t_2)\| \leq \lambda(t_2) + \varepsilon \quad \text{for} \quad n \geq n_0.
\]

Therefore

\[
\|u_n(t_1) - u_{n-1}(t_1)\| \leq \lambda(t_2) + \varepsilon + 2K|t_1 - t_2| \quad \text{for} \quad n \geq n_0
\]

and consequently,

\[
\lambda(t_1) \leq \lambda(t_2) + \varepsilon + 2K|t_1 - t_2|.
\]
As $\varepsilon$ is arbitrary, we get
\[
\lambda(t_1) \leq \lambda(t_2) + 2K|t_1 - t_2|.
\]
This implies
\[
|\lambda(t_1) - \lambda(t_2)| \leq 2K|t_1 - t_2| \quad \text{for} \quad t_1, t_2 \in J,
\]
which proves the continuity of $\lambda(\cdot)$.

Further, from (4) it follows that
\[
\|u_{n+1}(t) - u_n(t)\| = \|F(u_n)(t) - F(u_{n-1})(t)\|
\]
\[
\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}\|f(s, u_n(s)) - f(s, u_{n-1}(s))\|ds.
\]
By (3) this implies
\[
(6) \quad \|u_{n+1}(t) - u_n(t)\| \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}w(\|u_n(s) - u_{n-1}(s)\|)ds.
\]
Since the sequence $(\|u_n(\cdot) - u_{n-1}(\cdot)\|)$ is equicontinuous and uniformly bounded, from the definition of $\lambda(\cdot)$ and Arzela’s Lemma we deduce that for fixed $t \in J$ there exists a subsequence $(n_k)$ such that $\lim_{k \to \infty} \|u_{n_k+1}(t) - u_{n_k}(t)\| = \lambda(t)$ and $\|u_{n_k}(s) - u_{n_k-1}(s)\| \to \lambda_1(s)$ uniformly in $s \in J$. Replacing $n$ by $n_k$ in (6) and passing to the limit as $k \to \infty$, we obtain the inequality
\[
\lambda(t) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}w(\lambda_1(s))ds.
\]
As $\lambda_1(s) \leq \lim \|u_n(s) - u_{n-1}(s)\| = \lambda(s)$ and $w(r)$ is nondecreasing, we see that
\[
0 \leq \lambda(t) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1}w(\lambda(s))ds \quad \text{for} \quad t \in J.
\]
Applying now Theorem 2 of [3], we conclude that $\lambda(t) \equiv 0$ on $J$, which proves (5).
On the other hand, \((3)\) implies that
\[
\alpha(f(t, X)) \leq w(\alpha(X)) \quad \text{for } t \in J \text{ and } X \subset B,
\]
where \(\alpha\) is the Kuratowski measure of noncompactness. Now we shall show that the sequence \((u_n)\) has a limit point.

Let \(V = \{u_n : n \in N\}\). Then, by \((1)\), \(V\) is a bounded equicontinuous subset of \(\tilde{B}\). Denote by \(v\) the function defined by \(v(t) = \alpha(V(t))\) for \(t \in J\), where \(V(t) = \{u_n(t) : n \in N\}\). It is well known that the function \(v\) is continuous. As \(V = F(V) \cup \{0\}\), we have
\[
V(t) \subset F(V)(t) \cup \{0\}
\]
and consequently \(\alpha(V(t)) \leq \alpha(F(V)(t))\). Since
\[
F(V)(t) \subset \frac{1}{(m-1)!} \left\{ \int_0^t (t-s)^{m-1} f(s, u_n(s))ds : n \in N \right\},
\]
then Heinz’s lemma \([2]\) proves that
\[
\alpha(F(V)(t)) \leq \frac{1}{(m-1)!} \alpha\left( \left\{ \int_0^t (t-s)^{m-1} f(s, u_n(s))ds : n \in N \right\} \right)
\leq \frac{1}{(m-1)!} \int_0^t \alpha(\{(t-s)^{m-1} f(s, u_n(s)) : n \in N\})ds
\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} \alpha(\{f(s, u_n(s)) : n \in N\})ds.
\]
Moreover, in view of \((7)\), we have
\[
\alpha(\{f(s, u_n(s)) : n \in N\}) \leq w(\alpha(V(s))).
\]
Hence
\[
v(t) \leq \alpha(F(V)(t)) \leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(v(s))ds \quad \text{for } t \in J.
\]
Applying now Theorem 2 from \([3]\), we deduce that \(v(t) \equiv 0\) on \(J\). This proves that \(V(t)\) is relatively compact for \(t \in J\) and consequently, by Ascoli’s Theorem, \(V\) is relatively compact in \(C(J, E)\). Hence the sequence \((u_n)\) has a subsequence \((u_{n_k})\) which converges to a limit \(u\). This fact, together with
Appendix to the paper "Osgood type conditions for ..."

(5) and (4), implies that \( u = F(u) \), i.e. \( u \) is a solution of (2). Suppose that \( \overline{u} \) is another solution of (2). Then

\[
\|u(t) - \overline{u}(t)\| = \|F(u)(t) - F(\overline{u})(t)\| \\
\leq \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} w(\|u(s) - \overline{u}(s)\|) \, ds \quad \text{for} \quad t \in J,
\]

and therefore by Theorem 2 of [3] we get \( \|u(t) - \overline{u}(t)\| \equiv 0 \) on \( J \). Thus \( u = \overline{u} \). From the above considerations it is clear that the sequence \((u_n)\) has a unique limit point \( u \), and hence \( \lim_{n \to \infty} u_n(t) = u(t) \) uniformly on \( J \).

References


Received 24 October 2000