MODIFICATIONS OF CSÁKÁNY’S THEOREM

IVAN CHAJDA

Department of Algebra and Geometry, Palacký University of Olomouc
Tomkova 40, Cz-779 00 Olomouc, Czech Republic

e-mail: chajda@risc.upol.cz

Abstract

Varieties whose algebras have no idempotent element were characterized by B. Csákány by the property that no proper subalgebra of an algebra of such a variety is a congruence class. We simplify this result for permutable varieties and we give a local version of the theorem for varieties with nullary operations.

Keywords: congruence class, idempotent element, permutable variety, Mal’cev condition.

1991 Mathematics Subject Classification: 8B05, 08A30.

1 Introduction

B. Csákány [2] proved the following statement:

Proposition. For a variety \( \mathcal{V} \), the following conditions are equivalent:

(i) None of algebras in \( \mathcal{V} \) having at least two elements have idempotent elements;

(ii) No algebra \( A \in \mathcal{V} \) has a proper subalgebra whose carrier is a class of some \( \theta \in \text{Con } A \);

(iii) There exist \( n \in \mathbb{N} \), ternary terms \( p_1, \ldots, p_n \), and unary terms \( f_1, \ldots, f_n, g_1, \ldots, g_n \) such that the identities

\[
\begin{align*}
  x &= p(f_1(x), x, y), \\
  p_i(g_i(x), x, y) &= p_{i+1}(f_{i+1}(x), x, y), \quad \text{for } i = 1, \ldots, n - 1, \\
  y &= p_n(g_n(x), x, y)
\end{align*}
\]

hold in \( \mathcal{V} \).
It was recognized by J. Kollár [3] that each of the equivalent conditions of
the Proposition is equivalent to

(iv) For all $A \in \mathcal{V}$, the greatest congruence $\iota_A$ on $A$ is a compact element
of $\text{Con } A$.

Analyzing the proof of Proposition, we find out that these conditions
are also equivalent to

(v) $\theta(F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)) = F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$ in $\text{Con } F_{\mathcal{V}}(x, y)$, where $F_{\mathcal{V}}(x_1, \ldots, x_n)$ denotes the free algebra of $\mathcal{V}$ generated by the set
$\{x_1, \ldots, x_n\}$ of free generators, and $\theta(M \times M)$ denotes the least congruence
containing the set $M \times M$.

From this point of view, (v) can be modified by several ways. We can
consider a variety $\mathcal{V}$ with constants (i.e. nullary operations) and we can
omit a free variable on the left-hand side of (v) to obtain

(vi) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$ in $\text{Con } F_{\mathcal{V}}(x)$.

This condition used in [1] it is equivalent to the property that $\iota_A$ is generated
by the set of nullary operations for each $A \in \mathcal{V}$.

Further, we can also

(a) simplify Csákány’s original result for permutable varieties;

(b) omit one free variable in both sides of (v) to obtain

(vii) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$ in $\text{Con } F_{\mathcal{V}}(x)$.

In the second case we obtain a local version of Csákány’s theorem. These
modifications are treated in this paper.

2 Results

Theorem 1. Let $\mathcal{V}$ be a permutable variety. The following conditions are
equivalent:

(i) None of algebras in $\mathcal{V}$ having at least two elements have idempotent
elements;

(ii) No algebra $A \in \mathcal{V}$ has a proper subalgebra whose carrier is a class
of some $\theta \in \text{Con } A$;

(iii) There exist $n \in \mathbb{N}$ and a $(2 + n)$-ary term $p$ and unary terms
$v_1, \ldots, v_n, w_1, \ldots, w_n$ such that the identities

$$x = p(x, y, v_1(x), \ldots, v_n(x)),$$

$$\theta(F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)) = F_{\mathcal{V}}(x, y) \times F_{\mathcal{V}}(x, y)$$

in $\text{Con } F_{\mathcal{V}}(x, y)$, where $F_{\mathcal{V}}(x_1, \ldots, x_n)$ denotes the free algebra of $\mathcal{V}$ generated by the set
$\{x_1, \ldots, x_n\}$ of free generators, and $\theta(M \times M)$ denotes the least congruence
containing the set $M \times M$.

From this point of view, (v) can be modified by several ways. We can
consider a variety $\mathcal{V}$ with constants (i.e. nullary operations) and we can
omit a free variable on the left-hand side of (v) to obtain

(vi) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$ in $\text{Con } F_{\mathcal{V}}(x)$.

This condition used in [1] it is equivalent to the property that $\iota_A$ is generated
by the set of nullary operations for each $A \in \mathcal{V}$.

Further, we can also

(a) simplify Csákány’s original result for permutable varieties;

(b) omit one free variable in both sides of (v) to obtain

(vii) $\theta(F_{\mathcal{V}}(\emptyset) \times F_{\mathcal{V}}(\emptyset)) = F_{\mathcal{V}}(x) \times F_{\mathcal{V}}(x)$ in $\text{Con } F_{\mathcal{V}}(x)$.

In the second case we obtain a local version of Csákány’s theorem. These
modifications are treated in this paper.

2 Results

Theorem 1. Let $\mathcal{V}$ be a permutable variety. The following conditions are
equivalent:

(i) None of algebras in $\mathcal{V}$ having at least two elements have idempotent
elements;

(ii) No algebra $A \in \mathcal{V}$ has a proper subalgebra whose carrier is a class
of some $\theta \in \text{Con } A$;

(iii) There exist $n \in \mathbb{N}$ and a $(2 + n)$-ary term $p$ and unary terms
$v_1, \ldots, v_n, w_1, \ldots, w_n$ such that the identities

$$x = p(x, y, v_1(x), \ldots, v_n(x)),$$
\[ y = p(x, y, w_1(x), \ldots, w_n(x)) \]

hold in \( V \).

**Proof.** The equivalence of (i) and (ii) is proven by the Proposition. Prove (ii)\(\Rightarrow\)(iii): Set \( A = F_V(x, y) \) and \( B = F_V(x) \). Let \( \theta = \theta(B \times B) \in \text{Con } A \) (where \( B \) is the carrier of \( B \)). Since \( B \) is a subalgebra of \( A \), we have \( \theta(B \times B) = \theta = \iota_A \) by (ii). However, \( V \) is permutable; thus \( \theta(B \times B) = R(B \times B) \), the least reflexive and compatible relation on \( A \) containing the set \( B \times B \). It follows \( \iota_A = R(B \times B) \) which yields \( \langle x, y \rangle \in R(B \times B) \). Hence, there exists a \((2 + n)\)-ary term \( p \) and elements \( b_1, \ldots, b_n, b'_1, \ldots, b'_n \in B \) such that \( x = p(x, y, b_1, \ldots, b_n) \) and \( y = p(x, y, b'_1, \ldots, b'_n) \).

Since \( b_i, b'_i \in F_V(x) \), there are unary terms \( v_i, w_i \) with \( b_i = v_i(x) \) and \( b'_i = w_i(x) \), \( i = 1, \ldots, n \).

For (iii)\(\Rightarrow\)(i) let \( A \in V \) with \( |A| > 1 \) and suppose that \( a \in A \) is an idempotent element. Let \( b \in A \setminus \{a\} \). Then \( v_i(a) = a = w_i(a) \) and, by (iii), \( a = p(a, b, v_i(a), \ldots, v_n(a)) = p(a, b, a, \ldots, a) = p(a, b, w_1(a), \ldots, w_n(a)) = b \), a contradiction. \( \blacksquare \)

**Example 1.** For a variety \( \mathcal{R} \) of rings with 1, we can set \( n = 2, v_1(x) = 1 = w_2(x), v_2(x) = 0 = w_1(x) \) and \( p(x, y, a, b) = ax + by \). Hence, it follows that the reduct of a ring with 1, determined by the terms 0, 1, \( x - y + z \), and \( xy \), generates a permutable variety with no idempotent elements.

In this section we consider only varieties \( V \) having a nullary operation which will be denoted by 0; it is usually called a constant. For \( A \in V \), this constant will be denoted by \( 0_A \). We do not restrict the number of nullary operations of \( V \) but this 0 will be considered to be fixed.

**Theorem 2.** Let \( V \) be a variety with 0. The following conditions are equivalent:

(i) No \( A \in V \) having at least two elements has \( 0_A \) as an idempotent element;

(ii) For each \( A \in V \) and each \( \theta \in \text{Con } A \), \([0_A]_{\theta}\) is not a proper subalgebra of \( A \);

(iii) There exist \( n \in \mathbb{N} \), binary terms \( q_1, \ldots, q_n \), and unary terms \( v_1, \ldots, v_n, w_1, \ldots, w_n \) such that the identities

\[ x = q_1(x, v_1(0)), \]
\[ q_1(x, w_i(0)) = q_{i+1}(x, v_{i+1}(0)), \quad i = 1, \ldots, n - 1, \]

\[ 0 = q_n(x, w_n(0)) \]

hold in \( \mathcal{V} \).

**Proof.** (i)⇒(ii): Let \( \mathcal{A} \in \mathcal{V}, A > 1, \, \theta \in \text{Con} \, \mathcal{A} \) and suppose \([0_A]_\theta \neq A\). Then \( \mathcal{A}/\theta \) has at least two elements, and, of course, \([0_A]_\theta = [0_A]_\theta\). Since \([0_A]_\theta \) is not an idempotent of \( \mathcal{A}/\theta \), \([0_A]_\theta \) cannot be a subalgebra of \( \mathcal{A} \).

(ii)⇒(iii): Set \( \mathcal{A} = \mathcal{F}_\mathcal{V}(x) \) and \( B = \mathcal{F}_\mathcal{V}(\emptyset) \). Let \( \theta = \theta(B \times B) \) in \( \text{Con} \, \mathcal{A} \). Since \( 0_A = 0_B \in B \), the class \([0_A]_\theta \) contains \( B \). Hence, for every \( n \)-ary operation \( f \) of the type of \( \mathcal{V} \), for every \( c_1, \ldots, c_n \in [0_A]_\theta \) and every \( b_1, \ldots, b_n \in B \) we have \( \langle c_i, b_i \rangle \in \theta \quad (i = 1, \ldots, n) \); thus, also \( \langle f(c_1, \ldots, c_n), f(b_1, \ldots, b_n) \rangle \in \theta \). But \( f(b_1, \ldots, b_n) \in B \subseteq [0_A]_\theta \), whence \( f(c_1, \ldots, c_n) \in [0_A]_\theta \). Thus, \([0_A]_\theta \) is a subalgebra of \( \mathcal{A} \). In account of (ii), \([0_A]_\theta = A\); thus, \( (x, 0) \in \theta(B \times B) \). Hence, there exist \( d_0, d_1, \ldots, d_n \in A \) such that \( d_0 = x \), \( d_n = 0 \) and \( \langle d_{i-1}, d_i \rangle = \langle \varphi_i(b_i), \varphi_i(b'_i) \rangle \quad (i = 1, \ldots, n) \) for some \( b_i, b'_i \in B \) and unary polynomials \( \varphi_i \) over \( \mathcal{A} \). Thus \( b_i = v_i(0), b'_i = w_i(0) \) for some unary terms \( v_i, w_i \). Of course, \( \varphi_i(z) = q_i(x, z) \) for some binary terms \( q_1, \ldots, q_n \). The condition (iii) is evident.

(iii)⇒(i): Let \( \mathcal{A} \in \mathcal{V}, A > 1, \ 0_A \neq a \in A \). Suppose that \( 0_A \) is an idempotent of \( \mathcal{A} \). Then \( v_i(0_A) = 0_A = w_i(0_A) \) and

\[ a = q_1(a, v_1(0_A)) = q_1(a, 0_A) = q_1(a, w_1(0_A)) = q_2(a, v_2(0_A)) = q_2(a, 0_A) = q_2(a, w_2(0_A)) = \cdots = 0_A, \]

a contradiction. \( \Box \)

**Example 2.** For a variety \( \mathcal{P} \) of pseudocomplemented semilattices, we set \( n = 1, \ v_1(x) = x^* \), \( w_1(x) = x \) and \( q_1(x, y) = x \land y \). Then

\[ q_1(x, v_1(0)) = x \land 0^* = x, \]

\[ q_1(x, w_1(0)) = x \land 0 = 0. \]

Analogously as previously, we can simplify Theorem 2 for permutable varieties.

**Theorem 3.** Let \( \mathcal{V} \) be a permutable variety with \( 0 \). The following conditions are equivalent:

(i) No \( \mathcal{A} \in \mathcal{V} \) consisting of at least two elements has \( 0_A \) as an idempotent element;
(ii) For each $A \in \mathcal{V}$ and each $\theta \in \text{Con } \mathcal{A}$, $[0_A]_\theta$ is not a proper subalgebra of $A$;

(iii) There exist $n \in \mathbb{N}$ and a $(1 + n)$-ary term $q$ and unary terms $v_1, \ldots, v_n, w_1, \ldots, w_n$ such that the identities

$$x = q(x, v_1(0), \ldots, v_n(0)),$$

$$0 = q(x, w_1(0), \ldots, w_n(0))$$

hold in $\mathcal{V}$.

Example 3. For the variety of Boolean algebras, we can set $n = 1$, $v_1(x) = x'$, $w_1(x) = x$ and $q(x, y) = x \land y$. Then $q(x, v_1(0)) = x \land 0' = x$, and $q(x, w_1(0)) = x \land 0 = 0$.

Acknowledgement

I wish to thank the referee for several comments which enable to improve the preliminary version of this paper.

References

