THE ORDER OF NORMAL FORM HYPERSUBSTITUTIONS OF TYPE (2)

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Abstract
In [2] it was proved that all hypersubstitutions of type $\tau = (2)$ which are not idempotent and are different from the hypersubstitution which maps the binary operation symbol $f$ to the binary term $f(y, x)$ have infinite order. In this paper we consider the order of hypersubstitutions within given varieties of semigroups. For the theory of hypersubstitution see [3].

Keywords: hypersubstitutions, terms, idempotent elements, elements of infinite order.

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1 Preliminaries
In [1] hypersubstitutions were defined to make the concept of a hyperidentity more precise. In this paper we consider the type $\tau = (2)$ and the binary operation symbol $f$. Type (2) hypersubstitutions seem to be simple enough to be accessible, yet rich enough to provide an interesting structure.
An identity $s \approx t$ of type $\tau = (2)$ is called a hyperidentity of a variety $V$ of this type if for every substitution of terms built up by at most two variables (binary terms) for $f$ in $s \approx t$, the resulting identity holds in $V$. This shows that we are interested in mappings 

$$\sigma : \{f\} \to W(X_2),$$

where $W(X_2)$ is the set of all terms constructed by $f$ and the variables from the two-element alphabet $X_2 = \{x, y\}$. Any such mapping is called a hypersubstitution of type $\tau = (2)$. By $\sigma_t$ we denote the hypersubstitution $\sigma : \{f\} \to \{t\}$.

A hypersubstitutions $\sigma$ can be uniquely extended to a mapping $\hat{\sigma}$ on $W(X)$ (the set of all terms built up by $f$ and variables from the countably infinite alphabet $X = \{x, y, z, \cdots\}$) inductively defined by

(i) if $t = x$ for some variable $x$, then $\hat{\sigma}[t] = x$,

(ii) if $t = f(t_1, t_2)$ for some terms $t_1, t_2$, then $\hat{\sigma}[t] = \sigma(f)(\hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

By $Hyp$ we denote the set of all hypersubstitutions of type $\tau = (2)$. For any two hypersubstitutions $\sigma_1, \sigma_2$ we define a product

$$\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$$

and obtain together with $\sigma_{id} = \sigma_{xy}$, i.e., $\sigma_{id}(f) = xy$, a monoid $Hyp = (Hyp; \circ_h, \sigma_{id})$. We will refer to this monoid as to $Hyp$. In [2] Denecke and Wismath described all idempotent elements of $Hyp$.

We use the following denotation: Let $W_x$ denote the set of all words using only the letter $x$, and dually for $W_y$. We set

$$E_x = \{\sigma_{xu} | u \in W_x\}, \quad E_y = \{\sigma_{vy} | v \in W_y\}, \quad E = E_x \cup E_y,$$

where $xu$ abbreviates $f(x, u)$.

Clearly, for any element $xu$ with $u \in W_x$ we have

$$\sigma_{xu} \circ_h \sigma_{xu} = \sigma_{xu},$$

and for any element $vy$ with $v \in W_y$ we have

$$\sigma_{vy} \circ_h \sigma_{vy} = \sigma_{vy}.$$

This shows that all elements of $E$ are idempotent. The hypersubstitutions $\sigma_x, \sigma_y$ mapping the binary operation symbol $f$ to $x$ and to $y$, respectively, and the identity hypersubstitution are also idempotent.
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The hypersubstitution \( \sigma_{yx} \) satisfies the equation

\[ \sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy}. \]

Further we have:

**Proposition 1.1** (see [2]). If \( \sigma_s \circ_h \sigma_t = \sigma_{id} \), then either \( \sigma_s = \sigma_t = \sigma_{id} \) or \( \sigma_s = \sigma_t = \sigma_{yx} \).

In the following theorem we will use the concept of the length of a term as number of occurrences of variables in the term.

In [2] was proved

**Theorem 1.2.**

(i) If \( \sigma \in Hyp \) is an idempotent, then \( \sigma \in E \cup \{ \sigma_x, \sigma_y, \sigma_{xy} \} \).

(ii) If \( \sigma \in Hyp \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx} \}) \), then \( \sigma^n \neq \sigma^{n+1} \) for all \( n \in \mathbb{N} \) with \( n \geq 1 \) (i.e. \( \sigma \) has infinite order).

(iii) If \( \sigma \in Hyp \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx} \}) \), then the length of the word \((\sigma \circ_h \sigma)(f)\) is greater than the length of \( \sigma(f) \).

If we set \( G := Hyp \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yx} \}) \), then \( G \) does not form a subsemigroup of \( Hyp \). In fact, we consider the hypersubstitution \( \sigma_{wx} \) where \( w \) is a term different from \( x \) and from \( y \). Then \( \sigma_{wx} \in G \). Let \( u \in W_x \) and let \( xu \in W_x \) be the term formed from \( xu \) by substitution of all occurrences of the letters \( x \) by \( y \), then \( \sigma_{xu} \in G \). But then we see

\[ \sigma_{xu} \circ_h \sigma_{wx} = \sigma_{xy} \]

and the product of these elements from \( G \) is outside of \( G \).

If we want to check whether an equation \( s \approx t \) is satisfied as a hyperidentity in a given variety \( V \) of semigroups, it is not necessary to test all hypersubstitutions from \( Hyp \). Depending on the identities satisfied in \( V \) we may restrict ourselves to a smaller subset of \( Hyp \). By definition of a binary operation on this subset, we will define a new algebra which, in general is not a monoid and will determine the order of elements of those algebras.

### 2 Normal Form hypersubstitutions

In [4] J. Plonka defined a binary relation on the set of all hypersubstitutions of an arbitrary type with respect to a variety of this type.
Definition 2.1. Let $V$ be a variety of semigroups, and let $\sigma_1, \sigma_2 \in Hyp$. Then
$$\sigma_1 \sim_V \sigma_2 : \Leftrightarrow \sigma_1(f) \approx \sigma_2(f) \in IdV.$$ Clearly, the relation $\sim$ is an equivalence relation on $Hyp$ and has the following properties:

Proposition 2.2 ([3]). Let $V$ be a variety of semigroups and let $\sigma_1, \sigma_2 \in Hyp$.

(i) If $\sigma_1 \sim_V \sigma_2$, then for any term $t$ of type $\tau = (2)$ the equation $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ is an identity of $V$.

(ii) If $s \approx t \in IdV, \hat{\sigma}_1[s] \approx \hat{\sigma}_1[t] \in IdV$ and $\sigma_1 \sim_V \sigma_2 \in IdV$, then $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t] \in IdV$.

In general, the relation $\sim$ is not a congruence relation on $Hyp$. A variety is called solid if every identity in $V$ is satisfied as a hyperidentity. For a solid variety $V$ the relation $\sim$ is a congruence relation on $Hyp$ and the factor monoid $Hyp/\sim$ exists.

In the arbitrary case we form also $Hyp/\sim$ and consider a choice function
$$\varphi : Hyp/\sim \rightarrow Hyp,$$ with $\varphi([\sigma_id]_{\sim}) = \sigma_id,$
which selects from each equivalence class exactly one element. Then we obtain the set $Hyp_{N}(V) := \varphi(Hyp/\sim)$ of all normal form hypersubstitutions with respect to $V$ and $\varphi$.

On the set $Hyp_{N}(V)$ we define a binary operation
$$\circ : Hyp_{N}(V) \times Hyp_{N}(V) \rightarrow Hyp_{N}(V)$$
by $\sigma_1 \circ_N \sigma_2 = \varphi(\sigma_1 \circ_h \sigma_2)$. This mapping is well-defined, but in general not associative. Therefore, $(Hyp_{N}(V); \circ_N, \sigma_id)$ is not a monoid. We call this structure groupoid of normal form hypersubstitutions. We ask, how to characterize the idempotent elements of $Hyp_{N}(V)$ since for practical work normal form hypersubstitutions are more important than usual hypersubstitutions.

Proposition 2.3. Let $V$ be a variety of semigroups and let
$$\varphi : Hyp/\sim \rightarrow Hyp$$ be a choice function. Then
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(i) $\sigma \in Hyp_{N_\varphi}(V)$ is an idempotent element iff $\sigma \circ_h \sigma \sim_V \sigma$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$ if $\sigma_{yx} \in Hyp_{N_\varphi}(V)$.

**Proof.** (i) If $\sigma$ is an idempotent of $Hyp_{N_\varphi}(V)$, then $\sigma \circ_N \sigma = \sigma \sim_V \sigma \circ_h \sigma$. If conversely $\sigma \sim_V \sigma \circ_h \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma$. But then $\sigma \circ_N \sigma = \sigma$ because of $\sigma \in Hyp_{N_\varphi}(V)$.

(ii) $\sigma_{yx} \circ_N \sigma_{yx} \sim_V \sigma_{yx} \circ_h \sigma_{yx} = \sigma_{xy} \in Hyp_{N_\varphi}(V)$. Therefore, $\sigma_{yx} \circ_N \sigma_{yx} = \sigma_{xy}$.

As a consequence we have: if $\sigma$ is an idempotent of $Hyp$ and $\sigma \in Hyp_{N_\varphi}(V)$, then it is also an idempotent in $Hyp_{N_\varphi}(V)$ for any variety $V$ of semigroups and any choice function $\varphi$. But in general $Hyp_{N_\varphi}(V)$ has idempotents which are not idempotents in $Hyp$.

### 3 Idempotents in $Hyp_{N_\varphi}(V)$

Now we want to consider the following variety of semigroups: $V = Mod\{ (xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x \}$, i.e., the variety of all medial semigroups satisfying $x^3 \approx x$.

Let $f$ be our binary operation symbol. As usual instead of $f(x, y)$ we will also write $xy$. The elements of $W(X_2)/IdV$ where $X_2 = \{ x, y \}$ is a two-element alphabet, have the following form: $[x^ny^m]_{IdV}, [y^nx^m]_{IdV}, [xy^mx^n]_{IdV}, [yx^my^n]_{IdV}$ where $0 \leq m, n \leq 2$. So we get the set

$W(X_2)/IdV =$

$$= \{ [x]_{IdV}, [x^2]_{IdV}, [xy]_{IdV}, [x^2y]_{IdV}, [y]_{IdV}, [y^2]_{IdV}, [xyx]_{IdV}, [x^2y^2]_{IdV}, [xy^2]_{IdV}, [y^2x]_{IdV}, [yxy]_{IdV}, [y^2x^2]_{IdV}, [y^2x^2]_{IdV}, [y^2x^2]_{IdV}, [y^2y^2]_{IdV} \}$$

From each class we exchange a normal form term using a certain choice function $\varphi$ and obtain the following set of normal form hypersubstitutions: $Hyp_{N_\varphi}(V) = \{ \sigma_x, \sigma_{x^2}, \sigma_{xy}, \sigma_{xy^2}, \sigma_{x^2y}, \sigma_{xy^2}, \sigma_{y^2x}, \sigma_{y^2x^2}, \sigma_y, \sigma_{y^2}, \sigma_{y^2z}, \sigma_{y^2z^2}, \sigma_{y^2z^2} \}$.

The multiplication in the groupoid $(Hyp_{N_\varphi}(V); \circ_N, \sigma_{id})$ is given by the following table.
The table shows that there are many idempotents in $Hyp_{N_v}(V)$ which are not idempotents in $Hyp$.

The following example shows that $(Hyp_N(V); \circ_N, \sigma_{id})$ is not a monoid:

$$(\sigma_{x^2} \circ_N \sigma_{x^2}) \circ_N \sigma_{x^2} = \sigma_{x^2} \circ_N \sigma_{x^2} = \sigma_{x^2},$$

$$\sigma_{x^2} \circ_N (\sigma_{x^2} \circ_N \sigma_{x^2}) = \sigma_{x^2} \circ_N \sigma_x = \sigma_x.$$

All idempotent elements of $Hyp_N(V)$ are

$$\{\sigma_{xy}, \sigma_x, \sigma_{x^2}, \sigma_{x^2y}, \sigma_{x^2y^2}, \sigma_{xy^2}, \sigma_{xy^2x}, \sigma_{xy^2y}, \sigma_{y}, \sigma_y, \sigma_{yx^2y}, \sigma_{yx^2y^2}, \sigma_{yx^2y^2}\}.$$

Now we ask for which varieties at most the idempotents of $Hyp$ are idempotents of $Hyp_{N_v}(V)$.

**Theorem 3.1.** For a variety $V$ of semigroups the following are equivalent:

(i) $\text{Mod}\{ (xy)z \approx x(yz), xy \approx yx \} \subseteq V$,

(ii) $\{ \sigma| \sigma \in Hyp_{N_v}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} = \{ \sigma| \sigma \in Hyp \text{ and } \sigma \circ_h \sigma = \sigma \} \cap Hyp_{N_v}(V)$ for each choice function $\varphi$.

**Proof.** "(i)⇒(ii)" Let $\varphi$ be an arbitrary choice function and let $\sigma \in Hyp_{N_v}(V)$ be an idempotent element of $Hyp_{N_v}(V)$. Then $\sigma = \sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma$. Let $u$ and $v$ be the words corresponding to $\sigma$ and to $\sigma \circ_h \sigma$, respectively. By $\ell(u)$ we denote the length of $u$. Assume that $\sigma \notin E \cup \{ \sigma_{id}, \sigma_x, \sigma_y \}$.

By Theorem 1.2 (iii) the length of $v$ is greater than that of $u$ since $\sigma \neq \sigma_{f(y:x)}$ by Theorem 2.3 (ii). But then $u \approx v \notin \text{IdMod}\{ x(yz) \approx (xy)z, xy \approx yx \}$ since from the associative and the commutative identity one can derive only identities $u \approx v$ with $\ell(u) = \ell(v)$. But by assumption, $u \approx v \in \text{IdV} \subseteq \text{IdMod}\{ (xy)z \approx x(yz), xy \approx yx \}$, a contradiction. This shows

$$\{ \sigma| \sigma \in Hyp_{N_v}(V) \text{ and } \sigma \circ_N \sigma = \sigma \} \subseteq (E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \}) \cap Hyp_{N_v}(V).$$

If conversely $\sigma$ is an idempotent of $Hyp$, i.e. $\sigma \circ_h \sigma = \sigma$, then $\sigma \circ_N \sigma \sim_V \sigma \circ_h \sigma = \sigma$ and thus $\sigma \circ_N \sigma = \sigma$, since $\sigma \in Hyp_{N_v}(V)$ and $\sigma$ is an idempotent of $Hyp_{N_v}(V)$. Therefore we have equality.

"(ii)⇒(i)" Assume that $\text{Mod}\{ (xy)z \approx x(yz), xy \approx yx \} \not\subseteq V$. Then there exists an identity $x^k \approx x^n \in \text{IdV}$ with $1 \leq k < n \in \mathbb{N}$. Now we construct an idempotent element of $Hyp_{N_v}(V)$ which is not in $E \cup \{ \sigma_x, \sigma_y, \sigma_{id} \}$. We set $m := n - k$ and $w := x^2u$ for some word $u \in W_x$ with $\ell(u) = 3km - 2$. 
Clearly, $\sigma_w \not\in E \cup \{\sigma_x, \sigma_y, \sigma_id\}$. It is easy to see that the length of $w$ is $3km$ and the length of the word $v$ corresponding to $\sigma_w \circ_h \sigma_w$ is $(3km)^2$. In fact, from $x^k \approx x^n \in IdV$ it follows $x^a \approx x^{a+bm} \in IdV$ for all natural numbers $a \geq k$ and $b \geq 1$ and in particular we have $x^{3km} \approx x^{3km+(9k^2m-3km)} = x^{(3km)^2}$. Thus

$$(\sigma_w \circ_h \sigma_w)(f) \approx x^{(3km)^2} \approx x^{3km} \approx f(x, x, u) = \sigma_w(f).$$

Therefore, $\sigma_w \circ_h \sigma_w \sim_V \sigma_w$ and $\sigma_w \circ_N \sigma_w \sim_V \sigma_w \circ_h \sigma_w \sim_V \sigma_w$. Let $\varphi$ be a choice function with $\sigma_w \in Hyp_{N_\varphi}(V)$. Then from $\sigma_w \circ_N \sigma_w \sim_V \sigma_w$ it follows $\sigma_w \circ_N \sigma_w = \sigma_w$, a contradiction.

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4 Elements of infinite order

We remember that the order of an element of a groupoid is the cardinality of the subgroupoid generated by this element if this cardinality is finite and the order is infinite otherwise. By $O(\sigma)$ we denote the order of the hypersubstitution $\sigma \in Hyp_{N_\varphi}(V)$. By Theorem 1.2 (ii), the hypersubstitution $\sigma_{f(x,f(y,x))}$ has infinite order in $Hyp$, but in $Hyp_{N_\varphi}(V) = \{\sigma_x, \sigma_x^2, \sigma_{xy}, \sigma_{xy^2}, \sigma_{xy^2x}, \sigma_{xy^2x^2}, \sigma_{xy^2y}, \sigma_{xy^2y^2}, \sigma_{xy^2y^2x}, \sigma_{xy^2y^2x^2}, \sigma_{xy^2y^2x^2y}, \sigma_{xy^2y^2x^2y^2} \}$, where $V = Mod\{(xy)z \approx x(yz), xyuv \approx xuyv, x^3 \approx x\}$ we have

$$\sigma_{xy^2} \circ_N \sigma_{xy^2} = \sigma_{xy^2x^2}$$

and

$$\sigma_{xy^2} \circ_N \sigma_{xy^2x^2} \circ_N \sigma_{xy^2} = \sigma_{xy^2x^2} = \sigma_{xy^2x^2} \circ_N \sigma_{xy^2},$$

thus

$$\sigma_{xy^2x^2}^3 = \sigma_{xy^2x}^2$$

and $\sigma_{xy^2}$ has finite order. Now we characterize elements of infinite order with respect to varieties of semigroups which contain the variety of commutative semigroups.

By $\langle \sigma \rangle_{N_\varphi}$ we denote the subgroupoid of $Hyp_{N_\varphi}(V)$ generated by the hypersubstitution $\sigma$.

**Theorem 4.1.** Let $V$ be a variety of semigroups. Then the following are equivalent:
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(i) \( \text{Mod}\{xy\}z \approx x(yz), xy \approx yx \subseteq V \)

(ii) \( \{\sigma\mid \sigma \in \text{Hyp}_{N}\phi(V) \text{ and the order of } \sigma \text{ is infinite}\} = \text{Hyp}_{N}\phi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_id, \sigma_yx\} \cup A_1 \cup A_2) \), where \( A_1 = \{\sigma\mid \sigma \in \text{Hyp}_{N}\phi(V) \cap \{\sigma_v \mid v \in W_2\} \setminus (E \cup \{\sigma_x\}) \text{ and } \phi_{\circ N} \cap \{\sigma_{xy} u \mid u \in W(X_2)\} \neq \emptyset\} \) and \( A_2 = \{\sigma\mid \sigma \in \text{Hyp}_{N}\phi(V) \cap \{\sigma_v \mid v \in W_2\} \setminus (E \cup \{\sigma_y\}) \text{ and } \phi_{\circ N} \cap \{\sigma_{xy} u \mid u \in W(X_2)\} \neq \emptyset\} \) for each choice function \( \phi \).

**Proof.** "(i)\(\Rightarrow(ii)\): Let \( \phi \) be a choice function. Let \( \sigma \) be an element of \( \text{Hyp}_{N}\phi(V) \) with \( O(\sigma) = \infty \). By Theorem 3.1 and Proposition 2.3, \( \sigma \notin E \cup \{\sigma_x, \sigma_y, \sigma_id, \sigma_yx\} \).

If we assume that \( \sigma \) belongs to \( A_1 \), then there exists a word \( u \in W(X_2) \) such that \( \sigma_{xy} \in \langle \phi_{\circ N} \rangle \). Clearly, there exists a natural number \( n \geq 1 \) such that \( \ell(\sigma_{xy}) = n \). Moreover, we have

\[
\sigma \circ_N \sigma_{xy} \sim_V \sigma \circ_N \sigma_{xy} = \sigma,
\]

since the word corresponding to \( \sigma \) is in \( W_x \). Because of \( \sigma \in \text{Hyp}_{N}\phi(V) \) we get

\[
\sigma \circ_N \sigma_{xy} = \sigma
\]

and \( \ell(\sigma) + \ell(\sigma_{xy}) = n + 1 \). But this means, \( O(\sigma) \leq n \). Thus \( \sigma \notin A_1 \). In a similar way we show \( \sigma \notin A_2 \). This shows \( \{\sigma\mid \sigma \in \text{Hyp}_{N}\phi(V) \text{ and the order of } \sigma \text{ is infinite}\} \subseteq \text{Hyp}_{N}\phi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_id, \sigma_yx\} \cup A_1 \cup A_2) \).

Suppose that \( \sigma \in \text{Hyp}_{N}\phi(V) \setminus (E \cup \{\sigma_x, \sigma_y, \sigma_id, \sigma_yx\} \cup A_1 \cup A_2) \). Let \( u \) be the word corresponding to \( \sigma \).

If \( u \in W_x \), then \( \langle \phi \rangle_{\text{Hyp}_{N}\phi(V)} \subseteq \{\sigma_v \mid v \in W_2\} \). Otherwise there exists an identity \( a \approx b \in IdV \) such that \( a \in W_x \) and \( b \) uses the letter \( y \). Clearly, \( a \approx b \notin \text{IdMod}\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \subseteq \text{IdMod}\{(xyz)z \approx x(yz), xy \approx yx\} \). Moreover, \( \langle \phi \rangle_{\circ N} \cap \{\sigma_v \mid v \in W(X_2)\} = \emptyset \) and \( \sigma_x \notin \langle \phi \rangle_{\circ N} \). Therefore, for \( \sigma_1, \sigma_2 \in \langle \phi \rangle_{\text{Hyp}_{N}\phi(V)} \) the length of the word corresponding to \( \sigma_1 \circ_N \sigma_2 \) is greater than the length of \( u \). Hence for each \( \sigma' \in \langle \phi \rangle_{\circ N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). Otherwise there would exist an identity \( c \approx d \in IdV \) such that the length of \( d \) is greater than that of \( c \). Clearly, \( c \approx d \notin \text{IdMod}\{(xy)z \approx x(yz), xy \approx yx\} \), what contradicts \( c \approx d \in IdV \subseteq \text{IdMod}\{(xyz)z \approx x(yz), xy \approx yx\} \). Therefore, for all \( \sigma_a, \sigma_b \in \langle \phi \rangle_{\circ N} \) there holds \( \sigma_a \circ_N \sigma_b \neq \sigma \), i.e. \( O(\sigma) = \infty \). If \( u \in W_y \), then we get \( O(\sigma) = \infty \) in the dual way.
If \( u \) uses both letters \( x \) and \( y \), then \( \langle \sigma \rangle_{o_N} \subseteq \{ \sigma_v | v \in W(X_2) \setminus (W_x \cup W_y) \} \). Otherwise there is an identity \( a \approx b \in IdV \) such that \( a \in W_x \cup W_y \) and \( b \) uses both letters \( x \) and \( y \). Clearly, \( a \approx b \notin IdMod\{(xy)z \approx x(yz), xy \approx yx\} \) which contradicts \( a \approx b \in IdV \subseteq IdMod\{(xy)z \approx x(yz), xy \approx yx\} \). The same argumentation as above (using also \( \Hyp_{\sigma} \)) shows that for each \( \sigma' \in \langle \sigma \rangle_{o_N} \) with \( \ell(\sigma') \geq 2 \) the length of the word corresponding to \( \sigma' \) is greater than the length of \( u \). This means there don’t exist hypersubstitutions \( \sigma_a, \sigma_b \in \langle \sigma \rangle_{o_N} \) such that \( \sigma_n \circ \sigma_b = \sigma \) and hence \( O(\sigma) = \infty \). This shows \( \{ \sigma | \sigma \in \Hyp_{N_a}(V) \) and the order of \( \sigma \) is infinite\} \( \supseteq \Hyp_{N_a}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx} \} \cup A_1 \cup A_2) \).

“\( \text{(ii) } \Rightarrow (i) \)”: Assume that \( \text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \). Then there exists an identity \( x^k \approx x^n \in IdV \) with \( 1 \leq k < n \in \mathbb{N} \). We set \( m := n - k \) and \( w := f(f(\ldots f(x,y),\ldots),y) \), where \( w \) has the length \( km + 1 \). It is easy to check that \( (\sigma_w \circ \sigma)(f) = v \approx xy^{(km)} \). In fact, from \( x^k \approx x^n \in IdV \) and \( m := n - k \), it follows \( x^{km} \approx x^c \in IdV \) with \( c = km + (k^2m - k)m = k^2m^2 \). Therefore, \( (\sigma_w \circ \sigma)(f) = v \approx xy^{k^2m^2} \approx xy^{km} \approx \sigma_{w}(f) \), i.e. \( \sigma_w \circ \sigma \approx \sigma_{w} \) with \( \sigma_{w} \circ \sigma \approx \sigma_{w} \circ \sigma \). Let \( \varphi \) be a choice function such that \( \sigma_w \in \Hyp_{N_a}(V) \). Obviously, \( \sigma_w \in \Hyp_{N_a}(V) \setminus (E \cup \{ \sigma_x, \sigma_y, \sigma_{id}, \sigma_{yx} \} \cup A_1 \cup A_2) \) and thus \( O(\sigma) = \infty \). But \( \sigma_w \in \Hyp_{N_a}(V) \) forces \( \sigma_w \circ \sigma = \sigma_w \) and \( O(\sigma) = 2 \), what contradicts \( O(\sigma) = \infty \). Therefore \( \text{Mod}\{(xy)z \approx x(yz), xy \approx yx\} \subseteq V \).

\[ \text{References} \]


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