A NOTE ON PERIODICITY OF THE 2-DISTANCE OPERATOR

Bohdan Zelinka

Department of Applied Mathematics
Technical University of Liberec
Liberec, Czech Republic

To the memory of Ivan Havel

Abstract

The paper solves one problem by E. Prisner concerning the 2-distance operator $T_2$. This is an operator on the class $C_f$ of all finite undirected graphs. If $G$ is a graph from $C_f$, then $T_2(G)$ is the graph with the same vertex set as $G$ in which two vertices are adjacent if and only if their distance in $G$ is 2. E. Prisner asks whether the periodicity $\geq 3$ is possible for $T_2$. In this paper an affirmative answer is given. A result concerning the periodicity 2 is added.

Keywords: 2-distance operator, complement of a graph.

2000 Mathematics Subject Classification: 05C12.

In this paper we consider finite undirected graphs without loops and multiple edges. The vertex set of a graph $G$ is denoted by $V(G)$, its edge set by $E(G)$. The symbol $\overline{G}$ denotes the complement of $G$, i.e., the graph with the same vertex set as $G$ in which two distinct vertices are adjacent if and only if they are not adjacent in $G$.

Let $\phi$ be a graph operator defined on the class $C_f$ of all finite undirected graphs. For every positive integer $r$ we define the power $\phi^r$ so that $\phi^1 = \phi$ and for $r \geq 2$ the operator $\phi^r$ is such that $\phi^r(G) = \phi(\phi^{r-1}(G))$ for each $G \in C_f$. A graph $G \in C_f$ is called $\phi$-periodic, if there exists a positive integer $r$ such that $\phi^r(G) \cong G$. The minimum number $r$ with this property is the periodicity of the graph $G$ in the operator $\phi$.

For an integer $k \geq 2$ the operator $T_k$ on $C_f$ is defined in such a way that for any graph $G \in C_f$ the graph $T_k(G)$ has the same vertex set as $G$.
and two distinct vertices are adjacent in $T_k(G)$ if and only if their distance in $G$ is $k$. The operator $T_k$ is called the $k$-distance operator.

In [2], page 170, E. Prisner asks the following problem:

Is period $\geq 3$ possible for $T_2$?

An affirmative answer is given by the following theorem.

**Theorem.** Let $r$ be an even positive integer. Then there exists a graph $G_r$ whose periodicity in the operator $T_2$ is $r$.

**Proof.** Let $q = 2^r + 1$. Let $V_0, V_1, \ldots, V_{q-1}$ be pairwise disjoint sets of vertices. Let $t$ be an integer, $t \geq 2$ and let $|V_i| = t^i$ for $i = 0, 1, \ldots, q - 1$.

The vertex set of $G_r$ is $V(G_r) = \bigcup_{i=1}^{q-1} V_i$. All sets $V_0, V_1, \ldots, V_{q-1}$ are independent in $G_r$. Let $x \in V_i$, $y \in V_j$ for some $i$ and $j$ from $\{0, 1, \ldots, q - 1\}$.

These vertices are adjacent in $G_r$ if and only if $j \equiv i + 1 \pmod{q}$ or $j \equiv i - 1 \pmod{q}$. This implies that all sets $V_0, V_1, \ldots, V_{q-1}$ induce complete subgraphs in the graph $T_2(G_r)$. If $x \in V_i$, $y \in V_j$, then $x, y$ are adjacent in $T_2(G_r)$ if and only if $j \equiv i - 2 \pmod{q}$ or $j \equiv i - 2 \pmod{q}$. From these facts by induction we obtain that $T_2^n(G)$ for $m \geq 2$ has the following structure.

If $m$ is even, then all sets $V_0, V_1, \ldots, V_{q-1}$ are independent; if $m$ is odd, then they induce complete subgraphs; if $x \in V_i$, $y \in V_j$, then $x, y$ are adjacent in $T_2(G_r)$ if and only if $j \equiv i + 2 \pmod{q}$ or $j \equiv i - 2 \pmod{q}$ in both cases.

This implies that $T_2^n(G_r) \cong G_r$. Now it remains to show that $T_2^n(G)$ is not isomorphic to $G_r$ for $1 \leq m \leq r$. We do it using the independence number $\alpha(G)$. The greatest independent set in $G_r$ is $\bigcup_{i=1}^{1/2(q-1)} V_i$ and thus $\alpha(G_r) = \sum_{i=1}^{1/2(q-1)} t^i = t^2(t^{q-1} - t) / (t^2 - 1)$. If $m$ is odd, then $\alpha(T_2^m(G)) = 1/2(q-1) - 1$. If $m$ is even, $2 \leq m \leq r - 2$, then the set $V_0 \cup V_{q-2} \cup V_{q-1}$ is independent in $T_2^m(G)$ and thus $\alpha(T_2^m(G)) \geq |V_0 \cup V_{q-2} \cup V_{q-1}| = 1 + t^{q-2} + t^{q-1} > t^2(t^{q-1} - 1) / (t^2 - 1) = \alpha(G_r)$; this inequality may be easily proved. Therefore no graph $T_2^n(G_r)$ for $1 \leq m \leq r - 1$ is isomorphic to $G_r$ and thus the periodicity of $G_r$ in $T_2$ is $r$.

We shall remark also the periodicity 2. In [1] F. Harary, C. Hoede and D. Kladlack have proved that if a graph $G$ is self-complementary, i.e., $\overline{G} \cong G$, then $T_2(G) \cong G$ and thus the periodicity of $G$ in $T_2$ is 1. A slight generalization of the result is the following proposition. The diameter of $G$ is denoted by $\text{diam } G$.

**Proposition 1.** Let $G$ be a graph such that $\text{diam } G = \text{diam } \overline{G} = 2$ and $\overline{G}$ is not isomorphic to $G$. Then $G$ is $T_2$-periodic with the periodicity 2.
Proof. If two vertices $x, y$ are adjacent in $G$, then their distance in $G$ is 1 and they are not adjacent in $T_2(G)$. If they are not adjacent in $G$, then their distance in $G$ is 2 and realizes diam $G$. Moreover, $x$ and $y$ are adjacent in $T_2(G)$. Hence $T_2(G) = \overline{G}$. As also diam $\overline{G} = 2$, we have $T_2^2(G) = T_2(T_2(G)) = T_2(\overline{G}) = G$.

We shall create a class of graph which have the property that diam $G = \text{diam } \overline{G} = 2$.

Let $H_1, H_2, H_3, H_4, H_5$ be pairwise disjoint graphs. The graph $G(H_1, H_2, H_3, H_4, H_5)$ contains mentioned graphs as subgraphs and has new edges $xy$ created in the following way. If $x \in V(H_i), y \in V(H_j)$, then $x$ and $y$ are adjacent in $G$ if and only if $j \equiv i + 1 \pmod{5}$ or $j \equiv i + 4 \pmod{5}$. The simplest is the graph $G(K_1, K_1, K_1, K_1, K_1) = C_5$.

**Proposition 2.** For any five graphs $H_1, H_2, H_3, H_4, H_5$ the graph $G(H_1, H_2, H_3, H_4, H_5)$ has the diameter 2 and so has its complement.

**Proof.** Let $x, y$ be two vertices of $G(H_1, H_2, H_3, H_4, H_5)$. Let $i, j$ be such numbers from $\{1, 2, 3, 4, 5\}$ that $x \in V(H_i), y \in V(H_j)$.

If $i = j$, then both $x, y$ are in the graph $H_i$. If they are adjacent in $G$, then their distance is 1. If they are not adjacent, then there exists a path of length 2 connecting them; its inner vertex is in $V(H_{i+1}) \cup V(H_{i+4})$, the subscripts being taken modulo 5. If $j \equiv i + 1 \pmod{5}$ or $j \equiv i + 4 \pmod{5}$, then $x, y$ are adjacent in $G$ and their distance is 1.

If $j \equiv i + 2 \pmod{5}$ or $j \equiv i + 3 \pmod{5}$ then $x, y$ are not adjacent, but there exists a path of length 2 connecting them; its inner vertex is in $V(H_{i+1}) \cup V(H_{i+4})$. Therefore diam $G = 2$. The complement of $G(H_1, H_2, H_3, H_4, H_5)$ is isomorphic to $G(\overline{H_1}, \overline{H_2}, \overline{H_3}, \overline{H_4}, \overline{H_5})$ and thus also diam $\overline{G} = 2$.

**References**


Received 18 February 2000
Revised 5 July 2000