BLOCK GRAPHS WITH LARGE PAIRED DOMINATION MULTISUBDIVISION NUMBER

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Abstract

The paired domination multisubdivision number of a nonempty graph $G$, denoted by $\text{msd}_{pr}(G)$, is the smallest positive integer $k$ such that there exists an edge which must be subdivided $k$ times to increase the paired domination number of $G$. It is known that $\text{msd}_{pr}(G) \leq 4$ for all graphs $G$. We characterize block graphs with $\text{msd}_{pr}(G) = 4$.

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1. Introduction

The study of changes that occur in domination-related parameters of a graph when its edges are subdivided\(^1\) was initiated in [11]. If $\pi$ is a domination-type parameter of $G$, the smallest number of edges that must be subdivided, where each edge of $G$ can be subdivided at most once, in order to increase $\pi$ is called

\(^1\)See Section 2 for definitions of terms used in this section.
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the $\pi$-subdivision number, denoted by $sd_\pi(G)$. Subdivision numbers have been studied for the domination number [6, 11], as well as for connected [4], double [1], Roman [10], total [7, 9] and paired domination numbers [5].

Instead of subdividing multiple edges once each, one may wish to subdivide a single edge multiple times. The smallest number of times that a single edge of $G$ must be subdivided to increase $\pi$ is called the $\pi$-multisubdivision number, denoted by $msd_\pi(G)$. Domination and paired domination multisubdivision numbers were studied in [3] and [2], respectively. In particular, it was shown in [2] that the paired domination multisubdivision number $msd_{\text{pr}}(G)$ of any graph $G$ is at most four. For brevity we refer to a graph $G$ with $msd_{\text{pr}}(G) = 4$ as an msd-4 graph. Msd-4 trees were characterized in [2].

We discuss methods of combining msd-4 graphs to yield new msd-4 graphs and use our results, combined with results from [2], to characterize msd-4 block graphs. Definitions and previous results are given in Section 2. We state the characterization of msd-4 block graphs in Section 3, but defer its proof to Section 6 to allow us to prove a number of results used in the proof; results that apply to general msd-4 graphs are given in Section 4, while results specific to block graphs can be found in Section 5.

2. Definitions and Previous Results

We refer the reader to [8] for domination parameters not defined here. A set $S$ of vertices of a graph $G = (V, E)$ without isolated vertices is a paired dominating set of $G$ if every vertex of $G$ is adjacent to a vertex in $S$, and the subgraph $G[S]$ of $G$ induced by $S$ has a perfect matching. If $u, v \in S$ and there exists a perfect matching $M$ of $G[S]$ such that $uv \in M$, we say that $u$ and $v$ are paired in $S$. The smallest cardinality of a paired dominating set of $G$ is the paired domination number of $G$, denoted by $\gamma_{\text{pr}}(G)$. If $S$ is a paired dominating set of $G$ such that $|S| = \gamma_{\text{pr}}(G)$, we call $S$ a $\gamma_{\text{pr}}(G)$-set, or simply a $\gamma_{\text{pr}}$-set if the graph is clear from the context. If $u$ is a vertex of $G$ such that $G - u$ has no isolated vertices and $\gamma_{\text{pr}}(G - u) < \gamma_{\text{pr}}(G)$ (in which case $\gamma_{\text{pr}}(G - u) = \gamma_{\text{pr}}(G) - 2$), we say that $u$ is a $\gamma_{\text{pr}}(G)$-critical vertex, or simply a $\gamma_{\text{pr}}$-critical vertex, and define $\text{Cr}(G) = \{u \in V(G) : u$ is a $\gamma_{\text{pr}}$-critical vertex}. A neighbour of a vertex $u \in V(G)$ is a vertex adjacent to $u$. The (open) neighbourhood $N(u)$ of a vertex $u$ is the set of all vertices adjacent to $u$, and its closed neighbourhood is $N[u] = N(u) \cup \{u\}$. For a set $S \subseteq V(G)$, the (open) neighbourhood of $S$ is $N(S) = \bigcup_{u \in S} N(u)$, and its closed neighbourhood is $N[S] = N(S) \cup S$. For a vertex $u \in S$, the private neighbourhood of $u$ with respect to $S$ is the set $\text{PN}(u, S) = N[u] \setminus N[S \setminus \{u\}]$. It is possible that $u \in \text{PN}(u, S)$, but if $S$ is a paired dominating set, then $u$ is adjacent to the vertex it is paired with,
so \( u \notin PN(u, S) \) in this case.

An edge \( uv \) of a graph \( G \) is subdivided if it is replaced by a path \( (u, x, v) \), where \( x \) is a new vertex, and multisubdivided if it is replaced by a path \( (u, x_1, \ldots, x_k, v) \), \( k \geq 2 \), where \( x_1, \ldots, x_k \) are new vertices; we also say that \( uv \) is subdivided \( k \) times. Let \( G_{uv,k} \) denote the graph obtained from \( G \) by subdividing the edge \( uv \) \( k \) times.

The paired domination multisubdivision number \( msd_{pr}(G) \) of a graph \( G \) without isolated vertices is the smallest positive integer \( k \) such that there exists an edge \( uv \) which must be subdivided \( k \) times for \( \gamma_{pr}(G_{uv,k}) \) to exceed \( \gamma_{pr}(G) \). As mentioned above, \( msd_{pr}(G) \leq 4 \) for all graphs. The three graphs in Figure 1 are all msd-4 graphs; the red vertices form \( \gamma_{pr} \)-sets.

![Figure 1](image)

Figure 1. (a) The spider \( S(2, 2, 6) \) (b) the corona \( K_3 \circ K_1 \) (c) a flared corona \( K_4 \circ^2 K_1 \).

A leaf of a graph is a vertex of degree one, and its neighbour is called a stem.

The following properties of msd-4 graphs were proved in [2].

**Theorem 1** [2]. Let \( G \) be an msd-4 graph. Then

(i) each edge of \( G \) belongs to a matching of a minimum paired dominating set of \( G \);

(ii) any leaf of \( G \) is a \( \gamma_{pr} \)-critical vertex;

(iii) each stem is adjacent to exactly one leaf.

The complete bipartite graph \( K_{1,k} \), \( k \geq 2 \), is called a star. Let \( K_{1,k} \) have partite sets \( \{u\} \) and \( \{v_1, \ldots, v_k\} \). The spider \( S(\ell_1, \ldots, \ell_k), \ell_i \geq 1, k \geq 2 \), is a tree obtained from \( K_{1,k} \) by subdividing the edge \( uv_i \) \( \ell_i - 1 \) times, \( i = 1, \ldots, k \). Note that \( S(2, 2) \cong P_5 \). See Figure 1(a) for \( S(2, 2, 6) \). The characterization of msd-4 trees in [2] immediately gives the following result.
Proposition 2 [2]. The spider $T = S(2, \ldots, 2)$ satisfies $\text{msd}_{\text{pr}}(T) = 4$, and $\text{Cr}(T)$ consists of the leaves of $T$.

The corona $G \circ K_1$ of a graph $G$ is the graph obtained by joining each vertex of $G$ to a new leaf; $K_3 \circ K_1$ is illustrated in Figure 1(b). A flared corona $G \circ^{\ast t} K_1$ of $G$ is a graph obtained by joining each vertex of $G$, except one vertex $w$, to a new leaf, while $w$ is joined to a single vertex of each of $t \geq 1$ copies of $K_2$. The flared corona $K_4 \circ^{\ast 2} K_1$ is depicted in Figure 1(c). The following facts can be verified easily and are stated without proof.

Remark 3.
(i) A corona $K_n \circ K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is odd.
(ii) A flared corona $K_n \circ^{\ast t} K_1$, $n \geq 2$, is an msd-4 graph if and only if $n$ is even.
(iii) A vertex of $K_{2n+1} \circ K_1$ or $K_{2n} \circ^{\ast t} K_1$ is $\gamma_{\text{pr}}$-critical if and only if it is a leaf (see Theorem 1).

A block of a graph is a maximal connected subgraph with no cut-vertex, and a block graph is a graph, each of whose blocks is a complete graph. Thus, trees are block graphs since each block of a nontrivial tree is a $K_2$. Evidently, coronas and flared coronas are also block graphs. To characterize msd-4 block graphs, we use spiders $S(2, \ldots, 2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^{\ast t} K_1$, combining them by identifying vertices and edges in a prescribed way.

We begin by describing two operations, collectively known as $\oplus$-operations, for joining disjoint graphs; since the operations can be performed on any graphs, we state them in their most general form. (The operations are well known but we need to define our notation.)

$G_1 \oplus^{u_1 u_2} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $u_i \in V(G_i)$ for $i \in \{1, 2\}$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$ by $G_1 \oplus^{u_1 u_2} G_2$ (or by $G_1 \oplus^{u_1 u_2} G_2$ if the label $u$ is unimportant).

$G_1 \oplus^{e_1 e_2} G_2$: Let $G_1$ and $G_2$ be vertex disjoint graphs and $e_i = u_i v_i \in E(G_i)$. We denote the graph obtained from $G_1$ and $G_2$ by identifying $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, $v_1$ and $v_2$ into one vertex $v = v_1 = v_2$, and $e_1$ and $e_2$ into one edge $e = uv$ by $G_1 \oplus^{e_1 e_2} G_2$ (or by $G_1 \oplus^{e_1 e_2} G_2$ if the label $e$ is unimportant).

The graph $G_1 \oplus^{e_1 e_2} G_2$, where $G_1 = S(2, 2, 6)$, $G_2 = K_3 \circ K_1$, and $e_i = u_i v_i$ for $i = 1, 2$, is illustrated in Figure 2. Note that $u_i$ is $\gamma_{\text{pr}}(G_i)$-critical for $i = 1, 2$, and $u_1 = u_2$ is $\gamma_{\text{pr}}$-critical in $G_1 \oplus^{e_1 e_2} G_2$. The spider $S(2, 2, 6)$, in turn, is obtained as $H_1 \oplus^{u_1 u_2} H_2$, where $H_1 = S(2, 2, 2)$, $H_2 = P_5 = S(2, 2)$, and $u_i$ is a leaf of $H_i$, $i = 1, 2$. 
3. Characterization of msd-4 Block Graphs

We now state our main result — the characterization of msd-4 block graphs. The proof is deferred to Section 6.

Let $U$ be the collection of all spiders $S(2, \ldots, 2)$, coronas $K_{2n+1} \circ K_1$ and flared coronas $K_{2n} \circ^* K_1$, $n \geq 1$. Define $B$ to be the family of all block graphs $G$ that can be obtained as a graph $G_j$, $j \geq 1$, from a sequence $G_1, \ldots, G_j$ of graphs, where $H_1 = G_1 \in U$, and, if $j > 1$, $G_{i+1}$ can be constructed recursively from $G_i$ by

- adding a graph $H_{i+1} \in U$,
- choosing vertices $u_1 \in \text{Cr}(G_i)$, $u_2 \in \text{Cr}(H_{i+1})$, and if necessary, $v_1 \in \text{N}(u_1)$, $v_2 \in \text{N}(u_2)$,
- performing the operation $G_i \oplus^{u_1u_2} H_{i+1}$ or $G_i \oplus^{u_1v_1} v_2v_2 H_{i+1}$.

**Theorem 4.** Let $G$ be a connected block graph. Then $G$ is an msd-4 graph if and only if $G \in B$. Moreover, if $G$ is an msd-4 graph constructed from the graphs $H_1, \ldots, H_j \in U$, then $\text{Cr}(G) = \bigcup_{i=1}^j \text{Cr}(H_i)$.

The second statement of Theorem 4 implies that any $\gamma_{pr}$-critical vertex $v$ of an msd-4 block graph remains $\gamma_{pr}$-critical after the $\oplus$-operations have been performed any number of times, whether $v$ was identified with another vertex or not. The following corollary of Theorem 4 was proved in [2].

**Corollary 5.** A tree $T$ is an msd-4 graph if and only if $T \in B$, that is, if and only if $T$ can be constructed as described, using only spiders $S(2, \ldots, 2)$. 
4. General Results

In this section we discuss ways of constructing larger msd-4 graphs from smaller ones. We first prove a useful lemma.

**Lemma 6.** Let $G$ be a graph with $\text{msd}_{pr}(G) = 4$. For any edge $uv$ of $G$, subdivide $uv$ by replacing it with the path $(u,x_1,x_2,x_3,v)$. If $D$ is any $\gamma_{pr}(G_{uv,3})$-set, then $D \cap \{u,x_1,x_2,x_3,v\} =$

(i) $\{x_1,x_2\}$ or $\{x_2,x_3\}$, or
(ii) $\{u,x_1,v\}$ or $\{u,x_3,v\}$.

If the first part of (i) holds, then $u$ is $\gamma_{pr}$-critical, and if the second part of (i) holds, then $v$ is $\gamma_{pr}$-critical.

**Proof.** Let $X = \{x_1,x_2,x_3\}$. To dominate $x_2$, $X \cap D \neq \emptyset$. We consider three cases.

**Case 1.** $X \cap D = X$. Without loss of generality assume that $x_1$ is paired with $u \in D$, and $x_2$ and $x_3$ are paired. Then $v \notin D$, otherwise $D \backslash \{x_2,x_3\}$ is also a paired dominating set of $G_{uv,3}$, contradicting the minimality of $D$. But now $D' = (D \backslash X) \cup \{v\}$ is a paired dominating set of $G$, which is impossible because $\text{msd}_{pr}(G) = 4$.

**Case 2.** $|X \cap D| = 2$. If $X \cap D = \{x_1,x_3\}$, then $\{u,v\} \subseteq D$ with $u$ paired with $x_1$, and $v$ with $x_3$. However, then $D \backslash \{x_1,x_3\}$ is a paired dominating set of $G$, contradicting $\text{msd}_{pr}(G) = 4$. Suppose $X \cap D = \{x_1,x_2\}$. Then $x_1$ and $x_2$ are paired in $D$. If $\{u,v\} \cap D \neq \emptyset$, then $D \backslash \{x_1,x_2\}$ is a paired dominating set of $G$, which is a contradiction. Hence $D \cap \{u,x_1,x_2,x_3,v\} = \{x_1,x_2\}$. Now $D \backslash \{x_1,x_2\}$ is a paired dominating set of $G - u$, so $\gamma_{pr}(G - u) < \gamma_{pr}(G_{uv,3}) = \gamma_{pr}(G)$. We conclude that $u$ is $\gamma_{pr}$-critical. Arguing similarly if $X \cap D = \{x_2,x_3\}$, we conclude that (i) and the last part of the statement of the lemma hold.

**Case 3.** $|X \cap D| = 1$. Then $x_2 \notin D$. If $x_1 \in D$, then $x_1$ is paired with $u \in D$, while $v \in D$ to dominate $x_3$. Consequently, $D \cap \{u,x_1,x_2,x_3,v\} = \{u,x_1,v\}$. Similarly, if $x_3 \in D$, then $D \cap \{u,x_1,x_2,x_3,v\} = \{u,x_3,v\}$. ■

Our first result regarding the construction of msd-4 graphs from smaller graphs shows that subdividing any edge of an msd-4 graph four times produces another msd-4 graph. Repeatedly subdividing edges of an msd-4 graph thus yields, for example, msd-4 graphs of arbitrary large girth. In fact, we prove a stronger result: subdividing any edge of any graph $G$ without isolated vertices four times produces a graph that has the same multisubdivision number as $G$.

**Proposition 7.** For any graph $G$ and any edge $e$ of $G$, $\text{msd}_{pr}(G_{e,4}) = \text{msd}_{pr}(G)$.
Proof. Say \( \text{msd}_{pr}(G) = t \leq 4 \) and \( e = uv \) has been subdivided by replacing it with the path \((u, x_1, x_2, x_3, x_4, v)\). Then \( \gamma_{pr}(G_{e,4}) = \gamma_{pr}(G) + 2 \) and there exists an edge \( e' \) of \( G \) such that \( \gamma_{pr}(G_{e',4}) = \gamma_{pr}(G) + 2 \). If \( e \neq e' \), then subdividing \( e \in E(G_{e',4}) \) four times yields the graph \((G_{e',4})_{e,4}\). Since \( \text{msd}_{pr}(G_{e',4}) \leq 4 \), \( \gamma_{pr}((G_{e',4})_{e,4}) = \gamma_{pr}(G_{e',4}) + 2 = \gamma_{pr}(G) + 4 \). But \((G_{e',4})_{e,4} = (G_{e,4})_{e',t} \), hence \( \gamma_{pr}((G_{e,4})_{e',4}) = \gamma_{pr}(G) + 4 = \gamma_{pr}(G_{e,4}) + 2 \). If \( e = e' \), say \( uv \) has been subdivided, in \( G \), by replacing it with \((u, x_1, \ldots, x_t, v)\). Subdividing (without loss of generality) the edge \( x_1v \) four times by replacing it with \((x_t, x_{t+1}, \ldots, x_{t+4}, v)\), we obtain the graph \((G_{e,t})_{x_4e,t} = (G_{e,4})_{x_4e,t} \) with \( \gamma_{pr}((G_{e,4})_{x_4e,t}) = \gamma_{pr}(G_{e,4}) + 2 \). It follows that \( \text{msd}_{pr}(G_{e,4}) \leq t \).

We show that \( \text{msd}_{pr}(G_{e,4}) \geq t \). If \( t = 1 \), this is obvious, hence assume \( t \geq 2 \). Consider any \( e' \in E(G) \). Suppose first that \( e' \neq e \). Since \( \text{msd}_{pr}(G) = t \), \( \gamma_{pr}(G_{e',t-1}) = \gamma_{pr}(G) \). If \( D' \) is any \( \gamma_{pr}(G_{e',t-1}) \)-set, then \( D = D' \cup \{x_1, x_2\} \) (if \( u \) and \( v \) are paired in \( D' \)) or \( D = D' \cup \{x_2, x_3\} \) (otherwise) is a paired dominating set of \((G_{e,4})_{e',t-1}\) of cardinality \( |D| = \gamma_{pr}(G_{e',t-1}) + 2 = \gamma_{pr}(G) + 2 = \gamma_{pr}(G_{e,4}) \).

Assume \( e' = e \). Without loss of generality, subdivide the edge \( x_1v \) of \( G_{e,4} \) \( t-1 \) times by replacing it with the path \((x_4, \ldots, x_{t+3}, v)\) and denote the resulting graph \((G_{e,4})_{x_4v,t-1}\) by \( G_{e,3+t} \) for simplicity. Also consider the graph \( G_{e,t-1} \) obtained from \( G \) by subdividing \( e = uv \) by replacing it with \((u, x_1, \ldots, x_{t-1}, v)\). Since \( \text{msd}_{pr}(G) = t \), \( \gamma_{pr}(G_{e,t-1}) = \gamma_{pr}(G) \). Let \( S' \) be any \( \gamma_{pr}(G_{e,t-1}) \)-set. We consider three cases. In each case we construct a paired dominating set \( S \) of \( G_{e,3+t} \) such that \( |S| = |S'| + 2 = \gamma_{pr}(G_{e,4}) \); this shows that \( \text{msd}_{pr}(G_{e,4}) \geq t \).

Case 1. \( t = 2 \). If \( x_1 \notin S' \), then without loss of generality \( u \in S' \) to dominate \( x_1 \), and \( S' \setminus \{u\} \) dominates \( v \). Let \( S = S' \cup \{x_3, x_4\} \). If \( x_1 \in S' \), then again without loss of generality \( x_1 \) is paired with \( u \). Let \( S = S' \cup \{x_4, x_5\} \).

Case 2. \( t = 3 \). If \( S' \cap \{x_1, x_2\} = \emptyset \), then \( u \) dominates \( x_1 \) while \( v \) dominates \( x_2 \); let \( S = S' \cup \{x_3, x_4\} \) (so \( v \) dominates \( x_6 \)). If \( (\text{without loss of generality}) S' \cap \{x_1, x_2\} = \{x_1\} \), then \( u \) and \( x_1 \) are paired, and \( S' \setminus \{u, x_1\} \) dominates \( v \). Let \( S = S' \cup \{x_4, x_5\} \). If \( \{x_1, x_2\} \subseteq S' \), then \( x_1 \) and \( x_2 \) are paired (otherwise \( S' \setminus \{x_1, x_2\} \) is a paired dominating set of \( G \), which is not the case). Let \( S = S' \cup \{x_5, x_6\} \).

Case 3. \( t = 4 \). By Lemma 6, without loss of generality \( S' \cap \{u, x_1, x_2, x_3, v\} = \{x_1, x_2\} \) or \( \{u, x_1, v\} \). In the former case, let \( S = S' \cup \{x_5, x_6\} \), and in the latter case, let \( S = S' \cup \{x_4, x_5\} \).

In all cases, \( S \) is a paired dominating set of \( G_{e,3+t} \) of cardinality \( \gamma_{pr}(G) + 2 = \gamma_{pr}(G_{e,4}) \), and \( \text{msd}_{pr}(G_{e,4}) \geq t \). It follows that \( \text{msd}_{pr}(G_{e,4}) = t \), as required.

We next prove results pertaining to the \( \oplus \)-operations defined above that hold for general \( \text{msd}-4 \) graphs, not only block graphs. We show that the \( \oplus \)-operations can be used to construct new connected \( \text{msd}-4 \) graphs from smaller ones.
Our next result shows that performing the operation $G_1 \oplus_{u}^{n_{1}u_{2}} G_2$ on msd-4 graphs $G_1$ and $G_2$ with $\gamma_{pr}(G_i)$-critical vertices $u_1$ and $u_2$, respectively, results in an msd-4 graph in which each $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.

**Proposition 8.** Let $G_1$ and $G_2$ be disjoint msd-4 graphs with $\gamma_{pr}(G_i)$-critical vertices $u_i$, $i = 1, 2$. Then for the graph $G = G_1 \oplus_{u}^{n_{1}u_{2}} G_2$, $\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, any $\gamma_{pr}(G_i)$-critical vertex (including $u$) is $\gamma_{pr}(G)$-critical and

$$\text{msd}_{pr}(G) = 4.$$ 

**Proof.** Since $u_i \in V(G_i)$ is $\gamma_{pr}(G_i)$-critical, $\gamma_{pr}(G_1 - u_1) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4$, and at most two more vertices are needed to pairwise dominate $G$. Therefore $\gamma_{pr}(G) \leq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$.

Suppose there exists a paired dominating set $S$ of $G$ such that $|S| < \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$ and let $S_1 = S \cap V(G_1)$. First suppose that $u \notin S$. Assume without loss of generality that $S_1$ dominates $u$. Then $S_1$ is a paired dominating set of $G_1$ and $S_2$ is a paired dominating set of $G_2 - u_2$. Hence $|S_1| \geq \gamma_{pr}(G_1)$ and $|S_2| \geq \gamma_{pr}(G_2) - 2$. But then $|S| = |S_1| + |S_2| \geq \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2$, which is not the case. Therefore we may assume that $u \in S$ (in this case $u_i \in S_i$, $i = 1, 2$) and $|S_1| + |S_2| = |S| + 1$. Without loss of generality, $u$ is paired with $v \in V(G_1)$, hence $S_1$ is a paired dominating set of $G_1$. Therefore $|S_1| \geq \gamma_{pr}(G_1)$ so that $|S_2| \leq \gamma_{pr}(G_2) - 3$. If $N_{G_2}(u_2) \subseteq S_2$, then $S_2 \setminus \{u_2\}$ is a paired dominating set of $G_2$, and if there exists $w \in N_{G_2}(u_2) \setminus S_2$, then $S_2 \cup \{w\}$ is a paired dominating set of $G_2$. This is impossible because $|S_2 \cup \{w\}| \leq \gamma_{pr}(G_2) - 2$. Hence

$$\gamma_{pr}(G) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2.$$ 

If $w_i$ is $\gamma_{pr}(G_i)$-critical, then, for $j \neq i$, the union of any $\gamma_{pr}(G_i - w_i)$-set and any $\gamma_{pr}(G_j - u_j)$-set is a paired dominating set of $G - w_j$ (this holds for $w_i = u_i = u$ also), so

$$\gamma_{pr}(G - w_i) \leq \gamma_{pr}(G_i - w_i) + \gamma_{pr}(G_j - u_j) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 4 < \gamma_{pr}(G).$$ 

Therefore $w_i$ is $\gamma_{pr}(G)$-critical.

Without loss of generality consider $e \in E(G_1)$ and subdivide $e$ three times. Then, since $\text{msd}_{pr}(G_1) = 4$ and $u_2$ is $\gamma_{pr}(G_2)$-critical, we obtain

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1e,3}) + \gamma_{pr}(G_2 - u_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) - 2 = \gamma_{pr}(G).$$ 

Therefore $\text{msd}_{pr}(G) = 4$. 

We show next that performing the operation $G_1 \oplus_{e_1e_2} G_2$ on msd-4 graphs $G_i$, $i = 1, 2$, with edges $e_i = x_iy_i$, where $x_i$ is a $\gamma_{pr}(G_i)$-critical vertex, results in an msd-4 graph in which each $\gamma_{pr}(G_i)$-critical vertex is $\gamma_{pr}(G)$-critical.
**Proposition 9.** Let $G_i, i = 1, 2,$ be disjoint msd-$4$ graphs with $e_i = x_iy_i ∈ E(G_i),$ where $x_i ∈ Cr(G_i).$ Then for the graph $G = G_1 ⊕ e_1 ⊕ e_2$ $γ_{pr}(G) = γ_{pr}(G_1) + γ_{pr}(G_2) − 2,$ any $γ_{pr}(G_i)$-critical vertex (including $x = x_1 = x_2$) is $γ_{pr}(G)$-critical and $msd_{pr}(G) = 4.$

**Proof.** By Theorem 1, there exists a $γ_{pr}(G_i)$-set in which $x_i$ and $y_i$ are matched. Therefore

$$\gamma_{pr}(G) ≤ \gamma_{pr}(G_1) + \gamma_{pr}(G_2) − 2. \tag{1}$$

On the other hand, it suffices to add two vertices to a $γ_{pr}(G)$-set when splitting it into paired dominating sets of $G_1$ and $G_2.$ Hence we have equality in (1).

Let $e ∈ E(G)$ be any edge. If $e ∈ E(G_1) \setminus \{e_1\},$ then

$$\gamma_{pr}(G_{e, 3}) ≤ \gamma_{pr}(G_{e, 3}) + \gamma_{pr}(G_2 − x_2) = \gamma_{pr}(G_1) + \gamma_{pr}(G_2) − 2 = \gamma_{pr}(G).$$

The case when $e ∈ E(G_2) \setminus \{e_2\}$ is analogous. Thus assume $e = xy$ and subdivide $e$ by replacing it with the path $(x, u, v, w, y).$ Let $S$ be any $γ_{pr}(G)$-set. As shown above, $|S| = \gamma_{pr}(G) − 2.$ Now $S \cup \{u, v\}$ is a paired dominating set of $G_{e, 3}$ of cardinality $γ_{pr}(G).$ It follows that $G$ is an msd-$4$ graph. 

We now describe a type of “reverse” operation, called a **split operation**, for each of the $⊕$-operations.

$G ⊕ u.$ Let $G$ be a connected graph with a cut-vertex $u.$ Denote the components of $G − u$ by $F_1, F_2, \ldots, F_k.$ For each $i,$ let $G_i$ be the graph obtained from $F_i$ by adding a new vertex $u_i,$ joining $u_i$ to $v_i ∈ V(F_i)$ if and only if $uv_i ∈ E(G).$ Denote the disjoint union $G_1 + \cdots + G_k$ by $G ⊕ u.$

$G ⊕ xy.$ Let $G$ be a connected graph containing a vertex-cut $\{x, y\},$ where $xy ∈ E(G).$ Denote the components of $G − \{x, y\}$ by $F_1, F_2, \ldots, F_k.$ For each $i,$ let $G_i$ be the graph obtained from $F_i$ by adding the edge $x_iy_i,$ joining $x_i (y_i, \text{respectively})$ to $v_i ∈ V(F_i)$ if and only if $xv_i ∈ E(G)$ $(yv_i ∈ E(G),$ respectively). Denote the disjoint union $G_1 + \cdots + G_k$ by $G ⊕ xy.$

The next proposition shows that if an msd-$4$ graph $G$ is split at a $γ_{pr}$-critical cut-vertex $u,$ the components of $G ⊕ u$ are msd-$4$ graphs having the copies of $u$ as $γ_{pr}$-critical vertices.

**Proposition 10.** Let $G$ be an msd-$4$ graph with a $γ_{pr}$-critical cut-vertex $u.$ Denote the components of $G ⊕ u$ by $G_1, \ldots, G_k.$ Then for each $i = 1, \ldots, k,$ $u_i$ is a $γ_{pr}(G_i)$-critical vertex and $msd_{pr}(G_i) = 4.$
Proof. Since $u$ is $\gamma_{pr}(G)$-critical and $G - u$ is the disjoint union of $G_i - u_i$, $i = 1, \ldots, k$,

$$\gamma_{pr}(G) - 2 = \gamma_{pr}(G - u) = \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i).$$

Suppose $\gamma_{pr}(G_1 - u_1) \geq \gamma_{pr}(G_1)$. Let $R_i$ be a $\gamma_{pr}(G_1)$-set and, for $i \geq 2$, let $R_i$ be a $\gamma_{pr}(G_i - u_i)$-set. Since $R_1$ dominates $u_1$, $R = \bigcup_{i=1}^{k} R_i$ is a paired dominating set of $G$. But then

$$\gamma_{pr}(G) \leq |R| \leq \gamma_{pr}(G_1) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq \sum_{i=1}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - 2,$$

which is impossible. Thus $u_1$ is $\gamma_{pr}(G_1)$-critical. The same argument works for each $i \in \{2, \ldots, k\}$.

Consider an arbitrary edge $e \in E(G_1)$ and subdivide $e$ three times. Then

$$\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i). \tag{2}$$

We show that equality holds in (2). Let $S$ be any $\gamma_{pr}(G_{e,3})$-set and define $S_1 = S \cap V(G_{1,e,3})$ and $S_i = S \cap V(G_i)$ for $i = 2, \ldots, k$ (if $u \in S$, then $u_i \in S_i$ for each $i$). First suppose that $u \notin S$. If $S_1$ dominates $u$, then $S_1$ is a paired dominating set of $G_{1,e,3}$ and $S_i$, $i \geq 2$, is a paired dominating set of $G_i - u_i$. Hence $|S_1| \geq \gamma_{pr}(G_{1,e,3})$ and $|S_i| \geq \gamma_{pr}(G_i - u_i)$, so that $\gamma_{pr}(G_{e,3}) = |S| = \sum_{i=1}^{k} |S_i| \geq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i)$ as required. On the other hand, if $S_1$ does not dominate $u$, then $S_j$ is a paired dominating set of $G_j$ for some $j \geq 2$, so that $|S_j| \geq \gamma_{pr}(G_j) = \gamma_{pr}(G_j - u_j) + 2$ (since $u_j$ is $\gamma_{pr}(G_j)$-critical). Let $S'_j$ be a $\gamma_{pr}(G_j - u_j)$-set, $S'_1 = S_1 \cup \{u, u'\}$ for some $u' \in N_{G_1}(u)$, and $S' = (S' \setminus S'_j) \cup S'_j \setminus S'_j$. Then $|S'| = |S|$, $S'_i$ is a paired dominating set of $G_{1,e,3}$ and the result follows as before.

Now suppose that $u \in S$. Then $|S_1| + \sum_{i=2}^{k} |S_i| = |S| + k - 1$ and $u$ is paired with a vertex in exactly one of the graphs $G_{1,e,3}$ or $G_i$, $i \geq 2$. For each of the $k - 1$ other graphs, either $S_i \cup \{u_i\}$, for some neighbour $w_i \notin S_i$ of $u_i$, or $S_i \setminus \{u_i\}$ (if all neighbours of $u_i$ in $G_i$ belong to $S_i$) is a paired dominating set. Hence

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i) \leq |S| + 2(k - 1).$$

Since $u_i$ is $\gamma_{pr}(G_i)$-critical for each $i = 2, 3 \ldots, k$, $\gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_i) - 2$. This gives

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \leq |S| = \gamma_{pr}(G_{e,3}).$$

Therefore we have equality (2). Now
\[ \gamma_{pr}(G_{1,e,3}) = \gamma_{pr}(G_{e,3}) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) \]

\[ = \gamma_{pr}(G_1) + \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) - \sum_{i=2}^{k} \gamma_{pr}(G_i - u_i) = \gamma_{pr}(G_1). \]

Hence, for any edge \( e \in E(G_1) \), \( \gamma_{pr}(G_{1,e,3}) = \gamma_{pr}(G) \). Thus \( \text{msd}_{pr}(G_1) = 4 \).

Similar reasoning may be applied to \( G_i \) for \( i \in \{2, 3, \ldots, k\} \).

5. MSD-4 Block Graphs

The last three results we need for the proof of Theorem 4 concern block graphs. In the first result we prove that every non-leaf vertex of an MSD-4 block graph is a cut-vertex.

**Theorem 11.** Let \( G \) be a graph containing a block \( B \cong K_n \), where \( n \geq 3 \), such that some vertex of \( B \) is not adjacent to any vertex of \( G - B \). Then

\[ \text{msd}_{pr}(G) < 4. \]

**Proof.** Suppose the hypothesis of the theorem holds but \( \text{msd}_{pr}(G) = 4 \). Let \( V(B) = \{v_0, \ldots, v_{n-1}\} \) and say \( u = v_0 \) is not adjacent to any vertex of \( G - B \). Subdivide the edge \( uv_2 \) by replacing it with the path \( (u, x_3, x_2, x_1, v_2) \) (see Figure 3). Denote \( X = \{x_1, x_2, x_3\} \) and let \( D \) be a \( \gamma_{pr} \)-set of \( G_{uv_2,3} \). By Lemma 6 we only have to consider the cases \( D \cap \{u, x_1, x_2, x_3, v_2\} \in \{\{u, x_1, v_2\}, \{u, x_3, v_2\}, \{x_1, x_2\}, \{x_2, x_3\}\} \).

![Figure 3. The block B with the edge uv_2 subdivide with vertices x_1, x_2, x_3.](image-url)
Case 1. \(|X \cap D| = 1\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_1, v_2\}\), then \(x_1\) and \(v_2\) are paired in \(D\), while \(u\) is paired with \(v_i\) for some \(i \neq 0, 2\). However, then \(D \setminus \{x_1, u\}\), with \(v_2\) and \(v_1\) paired, is a smaller paired dominating set of \(G\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{u, x_3, v_2\}\), then \(D \setminus \{x_3, u\}\) is a smaller paired dominating set of \(G\). In either case \(\operatorname{msd}_{pr}(G) < 4\), contrary to our assumption.

Case 2. \(|X \cap D| = 2\). If \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_1, x_2\}\), then \(x_1\) and \(x_2\) are paired in \(D\). To pairwise dominate \(u, v_i \in D\) for some \(i \neq 0, 2\). But then \(D \setminus \{x_1, x_2\}\) is a paired dominating set of \(G\) (with \(v_i\) paired as in \(D\)) and \(\operatorname{msd}_{pr}(G) < 4\), contrary to our assumption. Hence assume \(D \cap \{u, x_1, x_2, x_3, v_2\} = \{x_2, x_3\}\). Then \(x_2\) and \(x_3\) are matched in \(D\). If \(v_1 \in D\) for some \(i\), then \(D \setminus \{x_2, x_3\}\) is a paired dominating set of \(G\) (again with \(v_i\) paired as in \(D\)), a contradiction.

We therefore assume henceforth that

(i) \(D\) contains \(x_2\) and \(x_3\), but neither \(x_1\) nor any \(v_0, \ldots, v_{n-1}\).

By Lemma 6, \(u\) is \(\gamma_{pr}\)-critical, that is,

(ii) \(\gamma_{pr}(G - u) = \gamma_{pr}(G) - 2\).

For each \(i = 1, \ldots, n - 1\), let \(G_i\) be the component of \(G - E(B)\) that contains \(v_i\). Since \(B\) is a block of \(G\), the subgraphs \(G_i\) are distinct and pairwise vertex-disjoint. Let \(D_i = D \cap V(G_i)\). Then \(|\bigcup_{i=1}^{n-1} D_i| = |D \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2\). By (i), each \(D_i\) is a \(\gamma_{pr}(G_i)\)-set that does not contain \(v_i\).

We next show that

(iii) no \(\gamma_{pr}(G)\)-set contains \(u = v_0\) and at least two \(v_i, i \geq 1\).

Suppose there exists such a set \(Z\); assume without loss of generality that \(\{u, v_1, v_2, \ldots, v_k\} \subseteq Z, k \geq 2\). Necessarily, \(u\) is paired with some \(v_i, i = 1, \ldots, k,\) in \(Z\). Assume (again without loss of generality) \(u\) is paired with \(v_1\). Let \(Z_1 = Z \cap V(G_1)\) \(\setminus \{v_1\}\) and, for \(i \geq 2\), let \(Z_i = Z \cap V(G_i)\). Then \(|\bigcup_{i=1}^{n-1} Z_i| \subseteq V(G - u)\) and \(|\bigcup_{i=1}^{n-1} Z_i| = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G)\), by (ii). Since \(v_1\) and \(u\) are paired, \(G_1[Z_1]\) contains a perfect matching, as does \(G[\bigcup_{i=2}^{n-1} Z_i]\). Since \(v_1\) is not adjacent to any vertex of \(G_1 - v_1, i \geq 2,\) and \(v_2\) dominates \(B\) in \(G_1, \bigcup_{i=2}^{n-1} Z_i\) is a paired dominating set of \(G - G_1\).

Suppose \(|Z_1| < |D_1|\). Since both \(Z_1\) and \(D_1\) have even cardinality, \(|Z_1| \leq |D_1| - 2\). Then \(Z_1\) does not dominate \(G_1 - v_1\), otherwise \(\bigcup_{i=1}^{n-1} Z_i\) is a paired dominating set of cardinality less than \(\gamma_{pr}(G)\), which is impossible. Since \(Z_1 \cup \{v_1\}\) dominates \(G_1\), there exists a vertex \(w \in N_{G_1}(v_1)\) that is undominated by \(Z_1\). Then \(W_1 = Z_1 \cup \{w, v_1\}\) is a paired dominating set of \(G_1\) of cardinality at most \(|D_1|\) that contains \(v_1\). But now \(W_1 \cup D_2 \cup D_3 \cup \cdots \cup D_{n-1}\) is a paired dominating set of \(G\) of cardinality at most \(|D_1 \setminus \{x_2, x_3\}| = \gamma_{pr}(G) - 2\), which is impossible. We conclude that \(|Z_1| = |D_1|\).

Let \(Z' = D_1 \cup \left(\bigcup_{i=2}^{n-1} Z_i\right)\). Since \(\bigcup_{i=2}^{n-1} Z_i\) is a paired dominating set of \(G - G_1\) and \(D_1\) is a paired dominating set of \(G_1\), \(Z'\) is a paired dominating set of \(G\).
Moreover,

\[ |Z'| = \bigcup_{i=2}^{n-1} Z_i + |D| = \bigcup_{i=1}^{n-1} Z_i = |Z| - 2 = \gamma_{pr}(G - u) < \gamma_{pr}(G), \]

which is impossible. This concludes the proof of (iii).

Subdivide the edge \( v_1v_2 \) with vertices \( y_1, y_2, y_3 \), where \( y_1 \) is adjacent to \( v_1 \) and \( y_3 \) is adjacent to \( v_2 \) (see Figure 4). Denote \( Y = \{y_1, y_2, y_3\} \) and let \( Q \) be a \( \gamma_{pr} \)-set of \( G_{v_1v_2,3} \). Without loss of generality, by Lemma 6 we only have to consider the cases \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} \in \{\{y_1, y_2\}, \{v_1, v_2, y_1\}\}. \)

![Figure 4. The block B with the edge v1v2 subdivided with vertices y1, y2, y3.](image)

**Case 3a.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{y_1, y_2\} \). Then these two vertices are paired in \( Q \). To pairwise dominate \( u, v_i \in Q \) for some \( i \). It follows that \( Q \setminus \{y_1, y_2\} \) is a paired dominating set of \( G \), so \( msd_{pr}(G) < 4 \), contrary to our assumption.

**Case 3b.** \( Q \cap \{v_1, v_2, y_1, y_2, y_3\} = \{v_1, v_2, y_1\} \). Then \( y_1 \) is paired with \( v_1 \). If \( u \notin Q \), then \( Q' = (Q \setminus \{y_1\}) \cup \{u\} \) is a paired dominating set of \( G \) containing \( u, v_1, v_2 \). By (iii), \( Q' \) is not a \( \gamma_{pr} \)-set of \( G \), from which it follows that \( \gamma_{pr}(G) < |Q| \) and \( msd_{pr}(G) < 4 \). Assume therefore that \( u \in Q \). Then \( u \) is paired in \( Q \) with \( v_i \) for some \( i > 1 \). Now \( Q'' = Q \setminus \{y_1, u\} \) is a paired dominating set of \( G \) in which \( v_1 \) and \( v_i \) are paired. In both cases we again have a contradiction and the proof is complete.

The graph in Figure 5 shows that the statement of Theorem 11 is false if the complete subgraph \( B \) is not a block of \( G \).

The next result in this section shows that \( msd \)-4 block graphs have many \( \gamma_{pr} \)-critical vertices.
Suppose there exists an edge \( uv \) in \( G \).

**Theorem 12.** If \( G \) is a block graph with \( \text{msd}_{pr}(G) = 4 \), then for any edge \( uv \in E(G) \),

\[
(N_G[u] \cup N_G[v]) \cap \text{Cr}(G) \neq \emptyset.
\]

**Proof.** Suppose there exists an edge \( uv \in E(G) \) such that \((N_G[u] \cup N_G[v]) \cap \text{Cr}(G) = \emptyset\). By Theorem 1, no vertex in \( N_G[u] \cup N_G[v] \) is a leaf. We subdivide the edge \( uv \) by replacing it with the path \((u, x_1, x_2, x_3, v)\) to obtain the graph \( G_{uv,3} \). By Lemma 6, for any \( \gamma_{pr}\)-set \( S \) of \( G_{uv,3} \), \( S \cap \{u, v, x_1, x_2, x_3\} \in \{\{u, v, x_1\}, \{u, v, x_3\}\} \). Without loss of generality assume there exists such a set \( S \) such that \( S \cap \{u, v, x_1, x_2, x_3\} = \{u, v, x_1\} \), and among all such sets \( S \), let \( D \) be one for which \( \text{PN}(u, D) \) is as small as possible. Then \( x_1 \) and \( u \) are paired in \( D \).

Say \( v \) is paired with \( v' \) and let \( B \) be the block of \( G \) that contains \( uv \). If \( v' \in V(G)\setminus V(B) \), let \( G_v \) be the subgraph of \( G - E(B) \) that contains \( v \), and if \( v' \in V(B) \), let \( G_v \) be the subgraph of \( G - (E(B)\setminus\{vv'\}) \) that contains \( v \). In either case, \( v' \in V(G_v) \). Let \( D_v = D \setminus V(G_v) \) and \( D' = D \setminus \{x_1, u\} \). Then \( G[D'] \) has a perfect matching and \( D_v \) is a paired dominating set of \( G_v \) containing \( v \) and \( v' \). In fact, \( D_v \) is a \( \gamma_{pr}(G_v) \)-set, for if not, let \( D'' \) be a smaller paired dominating set of \( G_v \).

Consider \( N_G(u)\setminus V(B) \). If \( B \cong K_2 \), then \( N_G(u)\setminus V(B) = N_G(u)\setminus \{v\} \) is nonempty because \( u \) is not a leaf, and if \( B \cong K_n \) for \( n \geq 3 \), then \( N_G(u)\setminus V(B) \) is nonempty by Theorem 11. If \( N_G(u)\setminus V(B) \subseteq D \), then \( D' \) is a paired dominating set of \( G \), and if there exists \( w \in N_G(u)\setminus V(B) \setminus D \), then \((D\setminus\{x_1\}\setminus D_v) \cup D'' \cup \{w\} \) is a smaller paired dominating set of \( G \) than \( D \). In both cases we have a contradiction to \( \text{msd}_{pr}(G) = 4 \).

Since \( \text{msd}_{pr}(G) = 4 \), \( |D'| = \gamma_{pr}(G_{uv,3}) - 2 = \gamma_{pr}(G) - 2 \). Consequently, \( D' \) does not dominate \( G \). Since \( v \in D' \) dominates \( B \) in \( G \), there exist vertices \( w_1, \ldots, w_k \in N_G(u)\setminus N_G[v] \subseteq N_G(u)\setminus B \) that are undominated by \( D' \), that is,

\[
G
\]

Figure 5. A graph \( G \) with \( \text{msd}_{pr}(G) = 4 \) and a subgraph \( K_3 \) that is not a block of \( G \).
\{w_1, \ldots, w_k\} = \text{PN}(u, D). For i = 1, \ldots, k, let G_i be the component of G − u that contains w_i. Possibly, G_i = G_j for i \neq j; this happens exactly when w_i w_j \in E(G), and then w_i and w_j also belong to the same (complete) block of G. Since no w_i is adjacent to v or v', \indent V(G_i) \cap V(G_v) = \emptyset for each i. Define D_i = D \cap V(G_i). Then \indent G_i[D_i] has a perfect matching, but does not dominate w_i. If it is nevertheless true that \gamma_{pr}(G_i) = |D_i| for some i, let Q_i be a \gamma_{pr}(G_i) set. Then \indent D^*_i = (D \setminus D_i) \cup Q_i is a \gamma_{pr}(G_{uv,3}) set such that PN(u, D^*_i) \subseteq PN(u, D) \setminus \{w_i\}, contrary to the choice of D. Therefore \gamma_{pr}(G_i) \geq |D_i| + 2 for each i.

Since each stem belongs to all paired dominating sets, no w_i is a stem, and by our initial assumption, no w_i is a leaf. Subdivide the edge uw_i by replacing it with the path (u, y_1, y_2, y_3, w_i). Consider a \gamma_{pr}(G_{uw,1,3}) set S. Since u, w_1 \notin \text{Cr}(G), Lemma 6 states that \indent S \cap \{u, y_1, y_2, y_3, w_i\} = \{u, y_1, y_2, y_3, w_1\}.

- In the former case, y_1 is paired with u and S_1 = S \cap V(G_1) is a paired dominating set of G_1; hence |S_1| \geq \gamma_{pr}(G_1) \geq |D_1| + 2. Since w_i is adjacent to all w_i \in V(G_1), D_1 \cup \{w_i\} dominates G_1 (but not pairwise). Now S' = (S \setminus S_1) \cup D_1 \cup \{w_1, y_3\} is a paired dominating set of G_{uw,1,3} such that |S'| \leq |S|, hence S' is a \gamma_{pr}(G_{uw,1,3}) set. Moreover, S' \cap \{u, y_1, y_2, y_3, w_i\} = \{u, y_1, y_3, w_1\}, contrary to Lemma 6.

- In the latter case, y_3 is paired with w_1. Then S_2 = (S \cap V(G_1)) \cup \{y_3\} is a paired dominating set of the graph obtained from G_1 by joining y_3 to w_1. If all neighbours of w_1 in G_1 belong to S_2, then S_2 \setminus \{w_1, y_3\} is a paired dominating set of G_1. But then S'' = S \setminus \{w_1, y_3\} is a paired dominating set of G such that |S''| < |S|, contradicting \text{msd}_{pr}(G) = 4. Assume some neighbour z of w_1 in G does not belong to S_2. Then S_3 = (S_2 \setminus \{y_3\}) \cup \{z\} is a paired dominating set of G_1, so that |S_2| = |S_3| \geq |D_1| + 2. Since u \in S and \{w_1, \ldots, w_k\} \subseteq N(u), S^* = (S \setminus S_2) \cup D_1 is a paired dominating set of G such that |S^*| < |S|, again a contradiction.

This completes the proof of the theorem.

Although the graph G in Figure 5 satisfies \text{msd}_{pr}(G) = 4 without being a block graph, Theorem 12 holds for G as well.

Our final result in this section concerns the reverse operation \indent G \ominus xy for certain msd-4 block graphs.

**Proposition 13.** Let G be a connected msd-4 block graph such that the only \gamma_{pr}(G)-critical vertices are leaves. Let x be a leaf adjacent to the stem y, where \{x, y\} is a vertex-cut, and denote the components of G \ominus xy by G_1, \ldots, G_k. Then \indent for each i = 1, \ldots, k, G_i is an msd-4 graph and \indent x_i \in \text{Cr}(G_i).

**Proof.** If G_i is an msd-4 graph, it will follow from Theorem 1(ii) that x_i \in \text{Cr}(G_i). However, we need the fact that x_i is \gamma_{pr}(G_i)-critical to show that \text{msd}_{pr}(G_i) = 4, hence this is what we prove first.
Since $G$ is a block graph, $N_{G_i - x_i}(y_i)$ induces a clique for each $i = 1, \ldots, k$. 
Since $x$ is a leaf, $y$ belongs to every paired dominating set of $G$, and by Theorem 1(ii), $x \in \text{Cr}(G)$. Hence $y$ belongs to no $\gamma_{pr}(G - x)$-set (for such a set would dominate $x$ and thus $G$, contradicting $x \in \text{Cr}(G)$).

Let $D$ be a $\gamma_{pr}(G - x)$-set such that $|D \cap N(y)|$ is maximum and let $D_i = D \cap V(G_i), i = 1, \ldots, k$. Since $x \in \text{Cr}(G)$ and $y \not\in D$, $|D| = \sum_{i=1}^{k} |D_i| = \gamma_{pr}(G) - 2$. Also, $D_i$ is a paired dominating set of $G_i - \{x_i, y_i\}$ for each $i$, and a paired dominating set of $G_i - x_i$ for at least one $i$. We show that, in fact,

(A) $D_i$ is a paired dominating set of $G_i - x_i$ for each $i$.

First suppose $|N_{G_i - x_i}(y_i)| \geq 2$; say $z_1, z_2 \in N_{G_i - x_i}(y_i)$. Since $N_{G_i - x_i}(y_i)$ induces a clique, $z_1z_2 \in E(G)$. By Theorem 12, $(N_{G}z_1 \cup N_{G}[z_2]) \cap \text{Cr}(G) \neq \emptyset$. Since $N_G[z_i] = N_{G_i - x_i}[z_i]$ and $z_i$ is not a leaf (and thus, by the hypothesis, not $\gamma_{pr}(G)$-critical), $z_1$ or $z_2$ is adjacent to a $\gamma_{pr}(G)$-critical vertex, i.e., a leaf. Say $z_1$ is adjacent to a leaf $z'$. Then $z_1$ belongs to any paired dominating set of any subgraph of $G$ containing both $z_1$ and $z'$, so $z_1 \in D$. Therefore $D_i$ dominates $y_i$ and (A) holds.

Assume therefore that $|N_{G_i - x_i}(y_i)| = 1$, say $N_{G_i - x_i}(y_i) = \{z\}$. If $z \in D$, we are done, hence assume $z \not\in D$. By Theorem 1(iii), $z$ is not a leaf, hence there exists a vertex $z' \in N_{G_i - x_i}(z) \{y_i\}$. By Theorem 1(i), $G$ has a $\gamma_{pr}$-set $X$ such that $zz'$ belongs to a matching of $G[X]$. Now $y \in X$, but $y$ is not paired with any vertex of $G_i - x_i$, since $N_{G_i - x_i}(y_i) = \{z\}$. Therefore $X_i = (X \setminus \{x, y\}) \cap V(G_i)$ is a paired dominating set of $G_i - x_i$. Moreover, $|X_i| \leq |D_i|$, otherwise $(X - X_i) \cup D_i$ is a smaller paired dominating set of $G$, which is impossible. However, now $D' = (D \setminus D_i) \cup X_i$ is a paired dominating set of $G - x$, hence a $\gamma_{pr}(G - x)$-set, containing more neighbours of $y$ than $D$, contrary to the choice of $D$. Hence (A) holds in this case as well.

Therefore $\gamma_{pr}(G_i - x_i) \leq |D_i|$ for each $i$, so that

$$\sum_{i=1}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |D_i| = |D| = \gamma_{pr}(G - x).$$

Suppose there exists a $\gamma_{pr}(G_i - x_i)$-set $Y_i$ containing $y_i$. Since no $D_j$ contains $y_j, D' = (D \setminus D_i) \cup Y_i$ is a paired dominating set of $G - x$ such that $|D'| \leq |D| = \gamma_{pr}(G) - 2$ and $D'$ dominates $x$. Then $D'$ is a paired dominating set of $G$, which is impossible. Therefore no $\gamma_{pr}(G_i - x_i)$-set contains $y_i$. Similarly, if $\gamma_{pr}(G_i - x_i) < |D_i|$ for some $i$ and $Z_i$ is a $\gamma_{pr}(G_i - x_i)$-set, then $D'' = (D \setminus D_i) \cup Z_i$ is a paired dominating set of $G - x$ such that $|D''| < |D|$, which is also impossible. From these two facts we deduce that $D_i$ is a $\gamma_{pr}(G_i - x_i)$-set, equality holds in (3) and $\gamma_{pr}(G_i) = \gamma_{pr}(G_i - x_i) + 2$, that is, $x_i$ is $\gamma_{pr}(G_i)$-critical for each $i$.

We show that $\text{msd}_{pr}(G_1) = 4$: it will follow similarly that $\text{msd}_{pr}(G_i) = 4$ for each $i$. Since $D_1$ is a $\gamma_{pr}(G_1 - x_1)$-set, it is easy to see that we can pairwise
dominate $G_{1,y,3}$ by $|D_1| + 2 = \gamma_{pr}(G_1)$ vertices. Hence consider any edge $e \in E(G_1 - x_1)$ and the graphs $G_{e,3}$ and $G_{1,e,3}$. Since combining any $\gamma_{pr}(G_{1,e,3})$-set with the sets $D_j$, $j = 2, \ldots, k$, produces a paired dominating set of $G_{e,3}$,

\begin{equation}
\gamma_{pr}(G_{e,3}) \leq \gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i).
\end{equation}

We show that equality holds in (4). For convenience of notation, define $H_1 = G_{1,e,3}$ and $H_i = G_i$, $i \geq 2$. Let $S$ be a $\gamma_{pr}(G_{e,3})$-set and define $S_i = S \cap V(H_i)$ for $i = 1, \ldots, k$ (since $y \in S$, $y_i \in S_i$ for each $i$, and if $x \in S$, then $x_i \in S_i$ for each $i$). We consider two cases, depending on whether $x \in S$ or not.

**Case 1.** $x \notin S$. Then $\sum_{i=1}^{k} |S_i| = |S| + k - 1$. Note that $y$ is paired with $w \in V(H_1) \setminus \{x_i, y_i\}$ for exactly one $i$. Then $S_i$ is a paired dominating set of $H_i$. For $j \neq i$, $S_j \cup \{x_j\}$ is a paired dominating set of $H_j$. Therefore $\gamma_{pr}(H_1) \leq |S_1|$ and $\gamma_{pr}(H_j) \leq |S_j| + 1$ for $j \neq i$. For $\ell \geq 2$, $x_\ell$ is $\gamma_{pr}(H_\ell)$-critical, hence $\gamma_{pr}(H_\ell - x_\ell) \leq \gamma_{pr}(H_\ell) - 2$. Therefore

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq \sum_{i=1}^{k} |S_i| - 2(k - 1) + (k - 1) = \sum_{i=1}^{k} |S_i| - (k - 1) = |S|$$

and equality holds in (4).

**Case 2.** $\{x, y\} \subseteq S$. Then $x$ and $y$ are paired in $S$, $\{x_i, y_i\} \subseteq S_i$ for each $i$, and $S_i$ is a paired dominating set of $H_i$. Also, $\sum_{i=2}^{k} |S_i| = |S| + 2(k - 1) - |S_1|$. Since $x_i$ is $\gamma_{pr}(G_i)$-critical,

$$\gamma_{pr}(G_{1,e,3}) + \sum_{i=2}^{k} \gamma_{pr}(G_i - x_i) \leq |S_1| + \sum_{i=2}^{k} |S_i| - 2(k - 1) = |S| = \gamma_{pr}(G_{e,3}),$$

giving equality in (4).

It now follows as in the proof of Proposition 10 that $msd(G_1) = 4$. Similarly, $msd(G_i) = 4$ for $i \geq 2$.

6. **Proof of Theorem 4**

We are now ready to prove our main theorem, the characterization of msd-4 block graphs. We restate the theorem here for convenience.

**Theorem 4 (again).** Let $G$ be a connected block graph. Then $G$ is an msd-4 graph if and only if $G \in B$. Moreover, if $G$ is an msd-4 graph constructed from the graphs $H_1, \ldots, H_j \in \mathcal{U}$, then $\text{Cr}(G) = \bigcup_{i=1}^{j} \text{Cr}(H_i)$.
Proof. If $G \in \mathcal{B}$, it follows immediately from Propositions 8 and 9 that $G$ is an msd-4 graph and $\text{Cr}(G) = \bigcup_{i=1}^{\ell} \text{Cr}(H_i)$.

For the converse, let $G$ be an msd-4 block graph. If $G$ is a tree, the result follows from Corollary 5, hence we assume that $B \cong K_n$, $n \geq 3$, is a block of $G$. By (the contrapositive of) Theorem 11, each vertex of $B$ is a cut-vertex, so $\text{deg}(v) \geq n$ for each $v \in V(B)$. Since each non-leaf vertex of a $K_2$-block is a cut-vertex, we deduce that each vertex of $G$ is either a leaf or a cut-vertex.

Suppose $v \in V(B)$ is $\gamma_{\text{pr}}$-critical. Applying Proposition 10 to $v$ we obtain an msd-4 graph $G_1$ with $v_1 = v$ and $N_{G_1}[v_1] = B$, which contradicts Theorem 11. Thus every $\gamma_{\text{pr}}(G)$-critical vertex belongs only to $K_2$-blocks.

We say that a vertex $u$ is a type-A vertex if it is a $\gamma_{\text{pr}}(G)$-critical cut-vertex, and an edge $uv$ is a type-A edge if $u$ is a leaf (hence $\gamma_{\text{pr}}(G)$-critical) and $G - \{u, v\}$ is disconnected. Denote the number of type-A elements (vertices and edges together) of $G$ by $a(G)$. First we show that

(B) if $a(G) = 0$, then $G \in \mathcal{U}$.

Suppose $a(G) = 0$. Then every $\gamma_{\text{pr}}(G)$-critical vertex is a leaf. Say $V(B) = \{v_1, \ldots, v_n\}$. Since no vertex of $B$ is $\gamma_{\text{pr}}(G)$-critical, Theorem 12 implies that $v_1$ or $v_n$ is adjacent to a $\gamma_{\text{pr}}(G)$-critical vertex. Without loss of generality we assume that $v_1u_1 \in E(G)$, $u_1 \notin V(B)$, and $u_1$ is $\gamma_{\text{pr}}(G)$-critical. Similarly, without loss of generality, $v_1$ is adjacent to a $\gamma_{\text{pr}}(G)$-critical vertex $u_i \notin V(B)$ for $i = 2, \ldots, n - 1$. Since $a(G) = 0$ and each vertex of $G$ is either a leaf or a cut-vertex, $\text{deg}_G(u_i) = 1$ for each $i = 1, \ldots, n - 1$ and $G - \{v_1, u_i\}$ is connected. Thus, $v_1$ belongs to only the two blocks $B$ and $v_1u_1$, so $\text{deg}_G(v_1) = n$ for each $i = 1, \ldots, n - 1$.

Since $v_n$ is a cut-vertex, $N(v_n) \setminus V(B) \neq \emptyset$. If $v_n$ is adjacent to a $\gamma_{\text{pr}}(G)$-critical vertex, say $u_n$, then, arguing as above, $\text{deg}(u_n) = 1$, $\text{deg}(v_n) = n$ and $G = K_n \circ K_1$. By Remark 3(i), $n$ is odd, hence $G$ belongs to the family $\mathcal{U} \subseteq B$. If no vertex in $N(v_n) \setminus V(B)$ is critical, let $N(v_n) \setminus V(B) = \{w_1, \ldots, w_t\}$ for $t \geq 1$. By Theorem 12, each $w_i$ is adjacent to a critical vertex $w_i' \neq v_n$, and since $a(G) = 0$, $w_i'$ is a leaf. We show that

(C) $\{w_1, \ldots, w_t\}$ is an independent set of $G$.

Suppose (without loss of generality) that $w_1w_2 \in E(G)$ and consider $G_{w_1w_2,3}$. Let $w_1, x_1, x_2, x_3, w_2$ be the $w_1 - w_2$ path in $G_{w_1w_2,3}$ and let $D$ be a $\gamma_{\text{pr}}(G_{w_1w_2,3})$-set. Since $w_1'$ and $w_2'$ are leaves, $w_1, w_2 \in D$. To dominate $x_2$, $\{x_1, x_2, x_3\} \cap D \neq \emptyset$. If $|\{x_1, x_2, x_3\} \cap D| = 2$, then $D \setminus \{x_1, x_2, x_3\}$ is a paired dominating set (with $w_1$ and $w_2$ paired) of $G$ of smaller cardinality than $D$, contrary to $\text{msd}(G) = 4$. Hence assume without loss of generality that $\{x_1, x_2, x_3\} \cap D = \{x_1\}$, so $w_1$ and $x_1$ are paired (and $w_1' \notin D$), while $w_2$ is paired with either $w_2'$ or $v_n$. However, each vertex in $N_G(v_n)$ is adjacent to a leaf and belongs to $D$, thus $D \setminus \{v_n\}$ dominates $G$. Therefore, either $D \setminus \{x_1, w_2\}$ or $D \setminus \{x_1, v_n\}$ is a paired dominating set of $G$ in which $w_1$ and $w_2$ are paired, contrary to $\text{msd}(G) = 4$. It follows that (C) holds.
Since $G$ is a block graph, $w_i$ and $w_j$ belong to different components of $G - v_n$ for all $i \neq j$.

Consequently, if there exists a vertex $z \notin \{v_n, w'_i\}$ adjacent to $w_i$, then $z$ and $v_n$ belong to different components of $G - \{w_i, w'_i\}$. But now $w_i w'_i$ is a type-A edge, which is not the case as $a(G) = 0$. Hence $\deg(w_i) = 2$ and $G \cong K_n \circ t K_1$.

Since $\operatorname{msd}(G) = 4$, $n$ is even, by Remark 3(ii). Therefore $G \in U \subseteq B$. Thus (B) holds.

Now suppose $a(G) \geq 1$. If $G$ has a type-A critical cut-vertex $u$, perform the operation $G \oplus u$; each resulting graph is an msd-4 graph by Proposition 10, and clearly a block graph. Moreover, the copies of $u$ in each graph are $\gamma_{pr}$-critical. Repeat this process until no resulting msd-4 block graph has a type-A critical cut-vertex. Let $G_1, \ldots, G_k$ be the resulting graphs. Then each critical vertex of each $G_i$ is a leaf. If any $G_i$ has a type-A critical edge $uv$, where $u$ is a leaf, perform the operation $G \ominus uv$. Each resulting graph is an msd-4 block graph by Proposition 13. Repeat this process until all resulting graphs $H_j$ satisfy $a(H_j) = 0$. If $H_j$ is a tree, then $H_j \cong S(2, \ldots, 2) \in U$ by Corollary 5, otherwise $H_j \not\in U$ by (B). Now $G$ can be reconstructed by performing the $\oplus$-operations on the $H_j$, hence $G \in B$, as required.

7. Open Problems

We conclude with a short list of open problems for future consideration.

**Question 1.** Does Theorem 12 hold for all msd-4 graphs?

Define another $\oplus$-operation as follows.

$\oplus_{u_i}^{Q_1, u_2} Q_2$. Let $G_1$ and $G_2$ be vertex disjoint graphs containing (not necessarily maximal) cliques $Q_1$ and $Q_2$ of equal size, and vertices $u_i \in V(Q_i)$ for $i \in \{1, 2\}$. We denote a graph obtained from $G_1$ and $G_2$ by identifying $Q_1$ and $Q_2$ into one clique $Q$, and $u_1$ and $u_2$ into one vertex $u = u_1 = u_2$, by $G_1 \oplus_{u_i}^{Q_1, u_2} Q_2$ (or by $G_1 \oplus_{u_i}^{Q_1, u_2} Q_2 G_2$ if $u$ and $Q$ are unimportant).

Note that if the cliques $Q_i$ have order at least three, then identifying the vertices of $Q_i - u_i$ in different ways may yield different graphs. Both operations $\oplus_{u_i}^{u_1} u_2$ and $\oplus_{e}^{u_1} e_2$ are special cases of $\oplus_{u_i}^{Q_1, u_2} Q_2$.

**Question 2.** Let $G_1$ and $G_2$ be disjoint msd-4 graphs containing cliques $Q_1$ and $Q_2$ of equal size and $\gamma_{pr}(G_1)$-critical vertices $u_i \in V(Q_i)$, $i = 1, 2$. Is it true that for any graph $G = G_1 \oplus_{u_i}^{Q_1, u_2} Q_2 G_2$, $u$ is $\gamma_{pr}(G)$-critical and $\operatorname{msd}_{pr}(G) = 4$?

If $G_1$ and $G_2$ are copies of the msd-4 graph in Figure 5, with $u_i = u$, which is $\gamma_{pr}$-critical, and $Q_i$ is the triangle containing $u$, then both graphs obtainable as $G_1 \oplus_{u_i}^{Q_1, u_2} Q_2 G_2$ are msd-4 graphs having $u$ as critical vertex.
Question 3. Let $G$ be a graph with $\text{msd}_{pr}(G) = 4$. What is the largest number of edges of $G$ that can be subdivided three times before the paired domination number increases? If this number can be arbitrarily high, what is its ratio to the number of edges of $G$?

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