SOME CONJECTURES ON PERFECT GRAPHS

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The complement of a graph $G$ is denoted by $\overline{G}$. $\chi(G)$ denotes the chromatic number and $\omega(G)$ the clique number of $G$. The cycles of odd length at least five are called odd holes and the complements of odd holes are called odd anti-holes.

1.

A graph $G$ is called perfect if, for each induced subgraph $G'$ of $G$, $\chi(G') = \omega(G')$. Classical examples of perfect graphs consist of bipartite graphs, chordal graphs and comparability graphs. Examples of nonperfect graphs are the odd holes and odd anti-holes. The most important result on perfect graphs is the following one, due to L. Lovász.

The Perfect Graph Theorem ([6]). The complement of a perfect graph is perfect.

The Perfect Graph Theorem used to be the so-called weak perfect graph conjecture posed by C. Berge around 1960s. His stronger conjecture on perfect graphs, which is still open, is as follows.

The Strong Perfect Graph Conjecture. Graphs without induced odd holes and odd anti-holes are perfect.

Clearly, if the Strong Perfect Graph Conjecture is true, it implies the Perfect Graph Theorem. See [1, 2, 3, 4] for more information on perfect graphs.
2.

The edge-version of graph perfection has been considered by L.E. Trotter [8]. It can be formulated in terms of line graphs as follows. The well-known line graph \( L(G) \) of a graph \( G \) has the edge set of \( G \) as its vertex set, and two distinct edges of \( G \) are adjacent in \( L(G) \) if and only if they have an endvertex in common; see Figure 1. A graph \( H \) is a line graph if there exists a graph \( G \) such that \( H \) is (isomorphic to) \( L(G) \). It is well-known that line graphs can be recognized in linear time.

A graph \( G \) is called line perfect if its line graph \( L(G) \) is perfect. It is well-known that \( L(G) \) is perfect if and only if \( L(G) \) contains no induced odd holes. Trotter proved that a graph is line perfect if and only if it has no (not necessary induced) odd holes. It then follows easily that line perfect graphs are perfect.

By definition, \( L(G) \) is perfect if and only if, for all induced subgraphs \( H \) of \( L(G) \), \( \chi(H) = \omega(H) \). Since the induced subgraphs of \( L(G) \) are in one-to-one correspondence with the line graphs of subgraphs of \( G \), \( L(G) \) is perfect if and only if, for all subgraphs \( G' \) of \( G \), \( \chi(L(G')) = \omega(L(G')) \).

Call a graph \( G \) weakly line perfect if, for all induced subgraphs \( G' \) of \( G \), \( \chi(L(G')) = \omega(L(G')) \). Clearly, line perfect graphs are weakly line perfect. The graph \( G \) in Figure 1 is weakly line perfect but not line perfect (\( L(G) \) contains an induced odd hole of length five). It is easy to see that weakly line perfect graphs cannot have induced odd holes and odd anti-holes. Thus, the Strong Perfect Graph Conjecture implies the following

**Conjecture A.** Weakly line perfect graphs are perfect.

![Figure 1. The line graph and the Gallai graph of a graph](image)

3.

The Gallai graph \( \Gamma(G) \) of a graph \( G \) has the edge set of \( G \) as its vertex set, and two distinct edges of \( G \) are adjacent in \( \Gamma(G) \) if, and only if, they have
an endvertex in common and the two other endvertices are nonadjacent in $G$; see Figure 1. Thus, $\Gamma(G)$ is a spanning subgraph of the line graph $L(G)$.

A graph $H$ is a Gallai graph if there exists a graph $G$ such that $H$ is (isomorphic to) $\Gamma(G)$. Note that, in contrast to line graphs, not every induced subgraph of $\Gamma(G)$ is again a Gallai graph.

**Problem B.** Recognize Gallai graphs in polynomial time, or prove that recognizing Gallai graphs is NP-complete.

A nice connection between Gallai graphs and perfect graphs is the following: Call the graph $G$ Gallai perfect if its Gallai graph $\Gamma(G)$ has no induced odd holes. L. Sun proved

**Theorem ([7]).** Gallai perfect graphs are perfect.

4.

It is shown in [5] that if $\Gamma(G)$ contains an induced odd anti-hole, it also contains an induced odd hole. Thus, the Strong Perfect Graph Conjecture implies the following

**Conjecture C ([5]).** Gallai graphs without induced odd holes are perfect.

If Conjecture C is true, it implies Sun’s Theorem (see [5]). Also, it is easy to see that Conjecture C implies a theorem of D. König, saying that line graphs of bipartite graphs are perfect. In [5], Conjecture C is proved for Gallai graphs $\Gamma(G)$ of graphs $G$ with $\chi(G) \leq 4$.

5.

Sun’s Theorem implies, in particular, that $G$ is perfect if $\Gamma(G)$ is perfect. By definition, $\Gamma(G)$ is perfect if and only if, for all induced subgraphs $H$ of $\Gamma(G)$, $\chi(H) = \omega(H)$.

Recall that not every induced subgraph of a Gallai graph is again a Gallai graph. This motivates the following definition: Call the graph $G$ weakly Gallai perfect if, for all induced subgraphs $G'$ of $G$, $\chi(\Gamma(G')) = \omega(\Gamma(G'))$.

The graph $G$ in Figure 1 is weakly Gallai perfect but not Gallai perfect ($\Gamma(G)$ contains an induced odd hole of length seven). It is not clear whether all Gallai perfect graphs are weakly Gallai perfect. However, if Conjecture C is true, they are.
It is easy to see that weakly Gallai perfect graphs cannot contain odd holes and odd anti-holes (indeed, if \( C \) is an odd hole or an odd anti-hole, then \( \chi(\Gamma(C)) = 3 \) while \( \omega(\Gamma(C)) = 2 \)). So, the Strong Perfect Graph Conjecture implies the following

**Conjecture D.** Weakly Gallai perfect graphs are perfect.

6.

It is shown in [5] that, for all graphs \( G \), \( \chi(\Gamma(G)) \leq \chi(\overline{G}) \). Thus, if \( G \) is perfect, then \( \chi(\Gamma(G')) \leq \omega(G') \) and \( \chi(\Gamma(G')) \leq \omega(G') \) hold for all induced subgraphs \( G' \) of \( G \). We conjecture that these properties characterize perfect graphs.

**Conjecture E** ([5, 2]). A graph \( G \) is perfect if and only if, for all induced subgraphs \( G' \) of \( G \), \( \chi(\Gamma(G')) \leq \omega(\overline{G'}) \) and \( \chi(\Gamma(G')) \leq \omega(G') \).

This conjecture has a “semi-strong” property: It implies the Perfect Graph Theorem and it is implied by the Strong Perfect Graph Conjecture.

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**References**


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