SOME NEWS ABOUT THE INDEPENDENCE NUMBER OF A GRAPH

Jochen Harant

Department of Mathematics, Technical University of Ilmenau
D-98684 Ilmenau, Germany

Abstract

For a finite undirected graph $G$ on $n$ vertices some continuous optimization problems taken over the $n$-dimensional cube are presented and it is proved that their optimum values equal the independence number of $G$.

Keywords: graph, independence.

1991 Mathematical Subject Classification: 05C35.

1 Introduction and Results

Let $G$ be a finite simple and undirected graph on $V(G) = \{1, 2, \ldots, n\}$ with its edge set $E(G)$. A subset $I$ of $V(G)$, such that the subgraph of $G$ induced by $I$ is edgeless, is called an independent set of $G$, and the maximum cardinality of an independent set of $G$ is named the independence number $\alpha(G)$ of $G$. $N(i)$ and $d_i$ denote the set and the number of neighbours of $i \in V(G)$ in $G$, respectively, and let $\Delta(G) = \max\{d_i \mid i \in V(G)\}$ and $C^n = \{(x_1, x_2, ..., x_n) \mid 0 \leq x_i \leq 1, \ i = 1, 2, ..., n\}$. For events $A$ and $B$ and for a random variable $Z$ of an arbitrary random space, $P(A)$, $P(A|B)$, and $\mathcal{E}(Z)$ denote the probability of $A$, the conditional probability of $A$ given $B$, and the expectation of $Z$, respectively. Since the computation of $\alpha(G)$ is difficult (INDEPENDENT SET is an NP-complete problem; see [6]), much work was done to establish bounds on $\alpha(G)$ (e.g., see [1, 3, 4, 5, 8, 10, 12, 13, 14, 15, 16, 17]), to find efficient algorithms forming a large independent set of $G$ (e.g., see [2, 7, 8, 9, 10, 12]), or to replace the combinatorial optimization problem to determine $\alpha(G)$ by a continuous one.
The last approach leads to bounds on $\alpha(G)$ as well as to efficient algorithms (e.g., see [8, 9]). In the present paper some new continuous optimization problems taken over $C^n$ are presented and it is proved that their optimum values equal $\alpha(G)$. Theorem 1 gives a remarkable result of T.S. Motzkin and E.G. Straus [11] and Theorem 2 is proved in [9].

**Theorem 1.**

\[
\alpha(G) = \max_{(0,0, \ldots, 0) \neq (x_1, x_2, \ldots, x_n) \in C^n} \left( \frac{\sum_{i \in V(G)} x_i}{\sum_{i \in V(G)} x_i^2 + \sum_{ij \in E(G)} x_ix_j} \right)^2.
\]

**Theorem 2.** $\alpha(G) = \max_{(x_1, x_2, \ldots, x_n) \in C^n} \sum_{i \in V(G)} (x_i \prod_{j \in N(i)} (1 - x_j))$.

A classical lower bound on $\alpha(G)$ due to Y. Caro and V.K. Wei [3, 17] is given by the following theorem.

**Theorem 3.** $\alpha(G) \geq \sum_{i \in V(G)} \frac{1}{1 + d_i}$.

The next Theorems 4, 5, 6, and 7 are the main results of the present paper.

**Theorem 4.** $\alpha(G) = \max_{(x_1, x_2, \ldots, x_n) \in C^n} e_G(x_1, x_2, \ldots, x_n)$, where

\[
e_G(x_1, x_2, \ldots, x_n) = \sum_{i \in V(G)} \left( \frac{x_i}{1 + \sum_{j \in N(i)} x_j} + \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{1 + \sum_{j \in N(i) \setminus (N(i) \cup \{i\})} (1-x_j)} \right).
\]

**Theorem 5.** $\alpha(G) = \max_{(x_1, x_2, \ldots, x_n) \in C^n} f_G(x_1, x_2, \ldots, x_n)$, where

\[
f_G(x_1, x_2, \ldots, x_n) = \sum_{i \in V(G)} \left( x_i + \frac{1-x_i}{1+\sum_{j \in N(i) \setminus (N(i) \cup \{i\})} (1-x_j)} \right) - \sum_{ij \in E(G)} x_ix_j.
\]

The following Theorem 6 looks more "complicate", but it is "stronger" than Theorem 4 and Theorem 5 (see Remark 1).

**Theorem 6.** $\alpha(G) = \max_{(x_1, x_2, \ldots, x_n) \in C^n} g_G(x_1, x_2, \ldots, x_n)$, where

\[
g_G(x_1, x_2, \ldots, x_n) = \sum_{i \in V(G)} \left( x_i + \frac{1-x_i}{1+\sum_{j \in N(i) \setminus (N(i) \cup \{i\})} (1-x_j)} \right) \prod_{j \in N(i)} (1-x_j).
\]
Some News about the Independence Number ...

\[ + \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j} \text{ and } V' = \left\{ i \in V(G) \mid \sum_{j \in N(i)} x_j > 0 \right\}. \]

A "weaker" (see Remark 1), but a more "transparent" and (see Remark 2) an "algorithmically realizable" version of Theorem 5 is the following one.

**Theorem 7.** \( \alpha(G) = \max_{(x_1, x_2, \ldots, x_n) \in C^n} h_G(x_1, x_2, \ldots, x_n), \) where

\[ h_G(x_1, x_2, \ldots, x_n) = \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j. \]

2 Proofs

Throughout the proofs we will use the well-known fact that for a random subset \( M \) of a given finite set \( N, \)

\[ \mathcal{E}(|M|) = \sum_{y \in N} P(y \in M) = \sum_{k=0}^{|N|} kP(|M| = k). \]

Let \( I \) be a maximum independent set of \( G \) and let \( x_i^* = 1 \) if \( i \in I \) and \( x_i^* = 0 \) if \( i \notin I. \) Since \( (1 - x_i^*) \prod_{j \in N(i)} (1 - x_j^*) = 0 \) for \( i \in V(G) \) and \( \sum_{ij \in E(G)} x_i^* x_j^* = 0, \) we obtain

**Lemma 1.** \( \alpha(G) = e_G(x_1^*, x_2^*, \ldots, x_n^*) = f_G(x_1^*, x_2^*, \ldots, x_n^*) = g_G(x_1^*, x_2^*, \ldots, x_n^*) = h_G(x_1^*, x_2^*, \ldots, x_n^*). \)

With Lemma 1, it is clear that Theorem 7 follows from Theorem 5.

Now, let \( (x_1, x_2, \ldots, x_n) \) be an arbitrary member of \( C^n. \) We form a set \( X \subseteq V(G) \) by random and independent choice of \( i \in V(G), \) where \( P(i \in X) = x_i. \)

Let \( H_1, H_2, \) and \( H_3 \) be the subgraph of \( G \) induced by the vertices of \( X, \) by the vertices \( i \in X \) with \( N(i) \cap X \neq \emptyset, \) and by the vertices \( i \notin X \) with \( N(i) \cap X = \emptyset, \) respectively. Furthermore, let \( Y \) be a smallest subset of \( V(H_2) \) covering all edges of \( H_2, \) i.e., the graph induced by \( V(H_2) - Y \) is edgeless, and let \( I_1 \) and \( I_3 \) be a maximum independent set of \( H_1 \) and \( H_3, \) respectively. It can be seen easily that \( |Y| = |V(H_2)| - \alpha(H_2), \) \( |Y| \leq |E(H_2)| \) and that \( (X - Y) \cup I_3 \) and \( I_1 \cup I_3 \) are independent sets of \( G. \) Because of these remarks and the property of the expectation to be an average value, we have Lemma 2 as follows.
Lemma 2. \( \alpha(G) \geq \mathcal{E}(|X - Y|) + \mathcal{E}(\alpha(H_3)), \) \( \alpha(G) \geq \mathcal{E}(\alpha(H_1)) + \mathcal{E}(\alpha(H_3)), \) \( \mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|Y|) \geq \mathcal{E}(|X|) - \mathcal{E}(|E(H_2)|), \) and \( \mathcal{E}(|X - Y|) = \mathcal{E}(|X|) - \mathcal{E}(|V(H_2)|) + \mathcal{E}(\alpha(H_2)). \)

Lower bounds on \( \mathcal{E}(\alpha(H_1)), \mathcal{E}(\alpha(H_2)), \) and \( \mathcal{E}(\alpha(H_3)) \) are given in Lemma 3.

Lemma 3. \( \mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} \frac{x_i}{x_j}, \)
\( \mathcal{E}(\alpha(H_2)) \geq \sum_{i \in V'} \frac{x_i(1- \prod_{j \in N(i)} (1-x_j))^2}{\prod_{j \in N(i)} (1-x_j)}, \) where \( V' = \{ i \in V(G) \mid \sum_{j \in N(i)} x_j > 0 \}, \)
and \( \mathcal{E}(\alpha(H_3)) \geq \sum_{i \in V(G)} \frac{(1-x_i) \prod_{j \in N(i)} (1-x_j)}{\prod_{j \in N(i)\cap (N(i)-i))} (1-x_i). \)

Proof. For \( i \in V(G) \) define the random variable \( Z_i^1 \) with \( Z_i^1 = \frac{1}{1+x_i} \) if \( i \in X \) and \( |N(i) \cap X| = k \geq 0, \) and \( Z_i^1 = 0 \) if \( i \notin X \). Using Theorem 3,
\( \mathcal{E}(\alpha(H_1)) \geq \mathcal{E}(\sum_{i \in V(G)} Z_i^1) = \sum_{i \in V(G)} \mathcal{E}(Z_i^1) \)
\( = \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+x_i} P(i \in X \text{ and } |N(i) \cap X| = k) \)
\( = \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+x_i} P(i \in X)P(|N(i) \cap X| = k) \)
\( = \sum_{i \in V(G)} x_i \sum_{k=0}^{d_i} \frac{1}{1+x_i} P(|N(i) \cap X| = k)). \)

For \( i \in V(G) \) we have \( \sum_{k=0}^{d_i} P(|N(i) \cap X| = k) = 1. \) With Jensen’s inequality
\( \sum_{l=1}^{m} \tau_l \phi(y_l) \geq \phi(\sum_{l=1}^{m} \tau_l y_l) \) for any convex function \( \phi \) and any \( \tau_l \geq 0 \) for \( l = 1, 2, \ldots, m \) with \( \sum_{l=1}^{m} \tau_l = 1, \)
\( \mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} \frac{x_i}{x_j}, \)
\( \mathcal{E}(\alpha(H_1)) \geq \sum_{i \in V(G)} \frac{1}{1+ \sum_{k=0}^{d_i} k P(|N(i) \cap X| = k) \sum_{j \in N(i)} x_j} \sum_{i \in V(G)} \frac{x_i}{x_j}. \)

Now, let \( V' = \{ i \in V(G) \mid \sum_{j \in N(i)} x_j > 0 \}. \) For \( i \in V(G) \) let \( Z_i^2 \) be the random variable with \( Z_i^2 = \frac{1}{1+x_i} \) if \( i \in X \) and \( |N(i) \cap X| = k \geq 1, \) and \( Z_i^2 = 0 \) otherwise. Then,
\[ \mathcal{E}(\alpha(H_2)) \geq \mathcal{E}(\sum_{i \in V(G)} Z_i^2) = \sum_{i \in V(G)} \mathcal{E}(Z_i^2) \]

\[ = \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X \text{ and } |N(i) \cap X| = k) \]

\[ = \sum_{i \in V(G)} \sum_{k=1}^{d_i} \frac{1}{1+k} P(i \in X) P(|N(i) \cap X| = k) \]

\[ = \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} P(|N(i) \cap X| = k). \]

\[ P(|N(i) \cap X| = 0) + \sum_{k=1}^{d_i} P(|N(i) \cap X| = k) = 1 \text{ for } i \in V(G) \text{ and with } \]

\[ \mu_i = P(|N(i) \cap X| = 0) \prod_{j \in N(i)} (1 - x_j) \text{ and } \sigma_{ik} = P(|N(i) \cap X| = k), \]

\[ \mathcal{E}(\alpha(H_2)) \geq \sum_{i \in V(G)} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V(G), \mu_i < 1} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} \]

\[ = \sum_{i \in V'} x_i \sum_{k=1}^{d_i} \frac{1}{1+k} \sigma_{ik} = \sum_{i \in V'} x_i (1 - \mu_i) \sum_{k=1}^{d_i} \frac{\sigma_{ik}}{(1+k)(1-\mu_i)}. \]

For \( \lambda_{ik} = \frac{\sigma_{ik}}{\mu_i} \), we have \( \lambda_{ik} \geq 0, \sum_{k=1}^{d_i} \lambda_{ik} = 1 \text{ if } i \in V', \) and again using Jensen’s inequality,

\[ \mathcal{E}(\alpha(H_2)) \geq \sum_{i \in V'} x_i (1 - \mu_i) \frac{1}{1+\sum_{k=1}^{d_i} k \lambda_{ik}} \]

\[ = \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1-x_j))^2}{1 - \prod_{j \in N(i)} (1-x_j) + \sum_{k=1}^{d_i} k P(|N(i) \cap X| = k)} = \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1-x_j))^2}{1 - \prod_{j \in N(i)} (1-x_j) + \sum_{j \in N(i)} x_j}. \]

Finally, let us consider the random variable \( Z_i^3 \) with \( Z_i^3 = \frac{1}{1+k} \) if \( i \in V(H_3) \) and \( |N(i) \cap V(H_3)| = k \geq 0, \) and \( Z_i^3 = 0 \) if \( i \notin V(H_3) \). Then

\[ \mathcal{E}(\alpha(H_3)) \geq \mathcal{E}(\sum_{i \in V(G)} Z_i^3) = \sum_{i \in V(G)} \mathcal{E}(Z_i^3) \]

\[ = \sum_{i \in V(G)} \sum_{k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3) \text{ and } |N(i) \cap V(H_3)| = k) \]

\[ = \sum_{i \in V(G), k=0}^{d_i} \frac{1}{1+k} P(i \in V(H_3)) P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3)) \]

\[ = \sum_{i \in V(G)} (1-x_i) \prod_{j \in N(i)} (1-x_j) \sum_{k=0}^{d_i} \frac{1}{1+k} P(|N(i) \cap V(H_3)| = k \mid i \in V(H_3))). \]
To prove Using Lemma 4, even we have to show
\[
\sum_{i \in V(G)} ((1 - x_i) \prod_{j \in N(i)} (1 - x_j) \frac{1}{1 + \sum_{k=0}^{d_i} kP(|N(i)\cap V(H_3)|=k \mid i \in V(H_3))})
\]
\[
= \sum_{i \in V(G)} ((1 - x_i) \prod_{j \in N(i)} (1 - x_j) \frac{1}{1 + \sum_{j \in N(i)} \prod_{l \in N(j)\setminus(N(i)\setminus(i))} (1-x_l)})
\]
and Lemma 3 is proved.

Theorem 4, 5, and 6 follow with \(E(|X|) = \sum_{i \in V(G)} x_i, E(|E(H_2)|) = \sum_{ij \in E(G)} x_i x_j, E(|V(H_2)|) = \sum_{i \in V(G)} x_i(1 - \prod_{j \in N(i)} (1 - x_j))\), Lemma 1, 2, and 3.

3 Remarks

For \(\phi, \psi \in \{e, f, g, h\}\) define \(\phi \leq \psi\) if \(\phi_G(x_1, x_2, ..., x_n) \leq \psi_G(x_1, x_2, ..., x_n)\) for every graph \(G\) on \(n\) vertices and for every \((x_1, x_2, ..., x_n) \in C^n\). We write \(\phi \leftrightarrow \psi\) if neither \(\phi \leq \psi\) nor \(\psi \leq \phi\).

**Remark 1.** \(h \leq f \leq g, e \leq g\) and \(e \leftrightarrow f\).

**Proof.** We will use the following Lemma 4, which can be seen easily by induction on \(r\).

**Lemma 4.** For an integer \(r \geq 1\) and \(a_1, a_2, ..., a_r \in [0, 1]\),
\[
\sum_{i=1}^{r} a_i + \prod_{q=1}^{r} (1 - a_q) \geq 1.
\]

The inequality \(h \leq f\) is obvious. To see \(f \leq g\), first notice that
\[
\sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j = \sum_{i \in V(G)} x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j).
\]
If \(\sum_{j \in N(i)} x_j = 0\) for an \(i \in V(G)\) then \(x_i = x_i (\prod_{j \in N(i)} (1 - x_j)) = x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j)\). Hence, with the abbreviation \(\mu_i = \prod_{j \in N(i)} (1 - x_j)\) and \(\rho_i = \sum_{j \in N(i)} x_j\) for \(i \in V(G)\), we have to show
\[
\sum_{i \in V'} (x_i (\mu_i + \frac{(1-\mu_i)^2}{1 - \rho_i + \rho_i})) \geq \sum_{i \in V'} (x_i (1 - \frac{1}{2} \rho_i)),
\]
where again \(V' = \{i \in V(G) \mid \sum_{j \in N(i)} x_j > 0\}\).

Using Lemma 4, even \(\mu_i + \frac{(1-\mu_i)^2}{1 - \rho_i + \rho_i} \geq 1 - \frac{1}{2} \rho_i\) for all \(i \in V'\).

To prove \(e \leq g\), we have to show
\[
\sum_{i \in V(G)} x_i (1 - \frac{1}{2} \sum_{j \in N(i)} x_j) \geq \sum_{i \in V(G)} x_i (1 - \frac{1}{2} \rho_i),
\]
which can be proved by induction on \(r\).
Since \(x_\alpha\) theorems 1, 2, 4, 5, 6, and 7 are of that type that the independence number open, whether optimization problem \(O\). With \(h\), \(X\) vector \((i\). It is obvious that the algorithm in an \(O\) stops.

Proof. First we give the Algorithm:

\[
\sum_{i \in V(G)} \frac{x_i}{1 + \sum_{j \in N(i)} x_j} \leq \sum_{i \in V(G)} x_i \prod_{j \in N(i)} (1 - x_j) + \sum_{i \in V'} \frac{x_i (1 - \prod_{j \in N(i)} (1 - x_j))^2}{1 - \prod_{j \in N(i)} (1 - x_j) + \sum_{j \in N(i)} x_j}.
\]

Since \(\frac{x_i}{1 + \sum_{j \in N(i)} x_j} = x_i \prod_{j \in N(i)} (1 - x_j)\) if \(\sum_{j \in N(i)} x_j = 0\), it is sufficient to establish \(\frac{1}{1 + \mu_i} \leq \mu_i + \frac{(1 - \mu_i)^2}{1 - \mu_i + \mu_i} \) if \(\sum_{j \in N(i)} x_j > 0\), what is verified easily.

For a cycle \(C_n\) on \(n\) vertices \(e_{C_n}(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}) < f_{C_n}(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}), e_{C_n}(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3}) > f_{C_n}(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3})\) and Remark 1 is proved.

With \(h \leq f\) and \(e \not\simeq f\) it is clear that \(e \leq h\) does not hold. It remains open, whether \(h \not\simeq e\) or \(h \leq e\).

Theorems 1, 2, 4, 5, 6, and 7 are of that type that the independence number \(\alpha(G)\) of a graph \(G\) on \(n\) vertices equals the optimum value of a continuous optimization problem \(O(G)\) to maximize a certain function \(\phi_G\) over \(C^n\). Hence, \(\phi_G(x_1, x_2, \ldots, x_n)\) is a lower bound on \(\alpha(G)\) for every \((x_1, x_2, \ldots, x_n) \in C^n\). Let \((x'_1, x'_2, \ldots, x'_n) \in C^n\) be the solution of an arbitrary approximation algorithm for \(O(G)\). How to find an independent set \(I\) of \(G\) in polynomial time such that \(|I| \geq \phi_G(x'_1, x'_2, \ldots, x'_n)\) ? In \([8]\) and \([9]\) efficient algorithms forming \(I\) with \(|I| \geq \phi_G(x'_1, x'_2, \ldots, x'_n)\) are given if \(O(G)\) is the optimization problem of Theorem 1 or of Theorem 2. Remark 2 shows that this is also possible if we consider the case \(\phi_G = h_G\). In case \(\phi_G = e_G, \phi_G = f_G\) or \(\phi_G = g_G\) the problem remains open, whether such an algorithm exists.

Remark 2. There is an \(O(\Delta(G)n)\)-algorithm with

**INPUT:** \((x_1, x_2, \ldots, x_n) \in C^n\),

**OUTPUT:** an independent set \(I \subseteq V(G)\) with \(|I| \geq \sum_{i \in V(G)} x_i - \sum_{ij \in E(G)} x_i x_j\).

**Proof.** First we give the Algorithm:

1. For \(i = 1\) to \(n\) do if \(\sum_{j \in N(i)} x_j < 1\) then \(x_i := 1\) else \(x_i := 0\).
2. For \(i = 1\) to \(n\) do if \((x_i = 1\) and \(\prod_{j \in N(i)} (1 - x_j) = 0\)) then \(x_i := 0\).
3. \(I := \{i \in V(G) \mid x_i = 1\}\).

STOP

It is obvious that the algorithm in an \(O(\Delta(G)n)\)-algorithm. For the input vector \((x_1, x_2, \ldots, x_n) \in C^n\) set

\[\sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_k x_j = a.\]
After step 1, the current \((x_1, x_2, \ldots, x_n)\) is a 0-1-vector and

\[ \sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_kx_j \geq a \]

because

\[
\frac{\partial}{\partial x_i} \left( \sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_kx_j \right) = 1 - \sum_{j \in N(i)} x_j,
\]
i.e., \(\sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_kx_j\) is multilinear.

In step 2, \(\prod_{j \in N(i)} (1 - x_j) = 0\) if and only if there is at least one \(j \in N(i)\) such that \(x_j = 1\). With \(x_i = 0\) instead of \(x_i = 1\) the sum \(\sum_{k \in V(G)} x_k\) decreases by 1 and the sum \(\sum_{k \in E(G)} x_kx_j\) decreases by at least 1, hence

\[
\sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_kx_j \text{ does not decrease.}
\]

After step 2,

\[ x_kx_j = 0 \text{ for all } k \in E(G), |I| = \sum_{k \in V(G)} x_k = \sum_{k \in V(G)} x_k - \sum_{k \in E(G)} x_kx_j \geq a \]

and Remark 2 is proved. \(\blacksquare\)

References


Some News about the Independence Number ...


Received 8 February 1999