K P-D I G R A P H S A N D C K I - D I G R A P H S SATISFYING THE \( k \)-MEYNIEL’S CONDITION

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Abstract

A digraph \( D \) is said to satisfy the \( k \)-Meyniel’s condition if each odd directed cycle of \( D \) has at least \( k \) diagonals.

The study of the \( k \)-Meyniel’s condition has been a source of many interesting problems, questions and results in the development of Kernel Theory.

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the \( k \)-Meyniel’s condition.

Primary keywords: digraph, kernel, independent set of vertices, absorbing set of vertices, kernel-perfect digraph, critical-kernel-imperfect digraph, \( \tau \)-system, \( \tau_1 \)-system.

Secondary keywords: independent kernel modulo \( Q \), co-rooted tree, \( \tau \)-construction, \( \tau_1 \)-construction.

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1. Introduction

For general concepts we refer the reader to [1]. If \( D \) is a digraph, then \( V(D) \) and \( F(D) \) or \( F(D) \) will denote the sets of vertices and arcs of \( D \) respectively.

We write \( D_0 \subseteq D \) (resp: \( D_0 \subseteq \) \( \) \( D \)) whenever \( D_0 \) is a subdigraph (resp: induced subdigraph) of \( D \). For \( S_1, S_2 \subseteq V(D) \), the arc \( u \), \( u \) of \( D \) will be called an \( S_1 S_2 \)-arc provided that \( u \in S_1 \) and \( u \in S_2 \); \( D[S_1] \) will denote the subdigraph of \( D \) induced by \( S_1 \) and \( D[S_1, S_2] \) the subdigraph of \( D \) with vertex set \( S_1 \cup S_2 \) whose arcs are the \( S_1 S_2 \)-arcs of \( D \). The asymmetrical part of \( D \) (resp: symmetrical part of \( D \)), which is denoted by \( \text{Asym} \) \( D \) (resp:
Sym $D$) is the spanning subdigraph of $D$ whose arcs are the asymmetrical (resp. symmetrical) arcs of $D$.

The set $I \subseteq V(D)$ is independent if $FD[I] = \emptyset$. A kernel $N$ of $D$ is an independent set of vertices such that for every $z \in (V(D) - N)$ there exists a $zN$-arc in $D$. A semikernel $S$ of $D$ is an independent set of vertices such that for every $z \in (V(D) - S)$ for which there exists an $Sz$-arc, there also exists a $zS$-arc.

A digraph $D$ is called

(i) *quasi KP-digraph* if every proper induced subdigraph of $D$ has a kernel,

(ii) *kernel-perfect digraph* or *KP-digraph* if every induced subdigraph of $D$ has a kernel,

(iii) *critical kernel-imperfect* or *CKI-digraph* if $D$ is a quasi KP-digraph and has no kernel.

It was proved by Neumann-Lara in [9] that $D$ is a KP-digraph iff every induced subdigraph of $D$ has a non empty semikernel. We will say that a digraph $A$ is a *co-rooted tree* if $A$ is an asymmetrical digraph whose underlying graph is a tree and there exists one and only one vertex $v \in F(A)$ (the *co-root* of $A$) such that there is no arc in $A$ whose initial endvertex is $v$.

Let $C = (1, 2, \ldots, m, 1)$ be a directed cycle of $D$, we denote by $\ell(C)$ its length, for $i \neq j$, $i, j \in V(C)$ we denote by $(i, C, j)$ the $ij$-directed path contained in $C$ and we denote by $\ell(i, C, j)$ its length; an arc $f = ij \in (FD - FC)$ is a *diagonal* of $C$ iff $i \neq j$, $i, j \in V(C)$ and $\ell(i, C, j) < \ell(C) - 1$ and $f$ is a *pseudodiagonal* when $\ell(i, C, j) \leq \ell(C) - 1$.

A digraph $D$ is said to satisfy the $k$-Meyniel’s condition if each odd directed cycle of $D$ has at least $k$ diagonals.

The study of the $k$-Meyniel’s condition has been a source of many interesting problems, questions and results in the development of Kernel Theory (see by example [2], [3], [4], [5], [6]).

In this paper we present a method to construct a large variety of kernel-perfect (resp. critical kernel-imperfect) digraphs which satisfy the $k$-Meyniel’s condition. This method is also the basis in the study of extensions of kernel-perfect digraphs to critical kernel-imperfect digraphs (see [8]).

**Theorem 1.1** [8]. Let $D_1$, $D_2$ and $D$ be digraphs such that $V(D_1) \cap V(D_2) = \{v\}$ and $D = D_1 \cup D_2$. Then $D$ is a KP-digraph iff $D_1$ and $D_2$ are KP-digraphs.
Theorem 1.2. Let \( G \) be a connected graph without cycles and for each \( e = w_1, w_2 \in E(G) \) let \( \gamma_e \) be a digraph such that \( \{w_1, w_2\} \subseteq V(\gamma_e), V(\gamma_e) \cap V(G) = \{w_1, w_2\} \). Suppose that the digraphs \( \gamma_e - V(G) \) are mutually disjoint. The digraph \( D = \bigcup_{e \in E(G)} \gamma_e \) is a KP-digraph iff \( \gamma_e \) is a KP-digraph for each \( e \in E(G) \).

Proof. Theorem 1.2 follows directly from Theorem 1.1 proceeding by induction on \( |V(G)| \).

Theorem 1.3 [6]. Suppose that \( V(D) \) has a partition \( \{V_1, V_2\} \) such that every \( V_1, V_2 \)-arc in \( D \) is symmetric and \( D[V_1] \) and \( D[V_2] \) are KP-digraphs. Then \( D \) is a KP-digraph.

Theorem 1.4 [6]. If \( D \) is a CKI-digraph, there is no a partition \( \{V_1, V_2\} \) of \( V(D) \) such that \( D[V_1, V_2] \subseteq \text{Sym } D \); in other words \( \text{Asym } D \) is strongly connected.

2. \( \tau_1 \)-System and \( \tau_1 \)-Construction

Definition 2.1. Let \( D \) be a multidigraph and \( u \in V(D) \); a partition \( \pi_u = \{u_0^-, u_1^-, \ldots, u_{m(u)-1}^-, u_+\} \) of \( F_u(D) = F_u^+(D) \cup F_u^-(D) \) will be called a \( \tau \)-partition in \( u \) if it satisfies the following two properties:

1. \( u_i^- \subseteq F_u^-(D) \) for each \( i \in \{0, 1, \ldots, m(u)-1\} \).
2. \( u_+ = F_u^+(D) \).

\( F_u^+(D) \) (resp: \( F_u^-(D) \)) denotes the set of arcs of \( D \) whose initial (resp: terminal) endvertex is \( u \).

When \( \pi_u \) is a \( \tau \)-partition in \( u \) we denote by \( \pi_u \) the set

\[
\pi_u = \{(u, u_+), (u, u_i^-) \mid i \in \{0, 1, \ldots, m(u)-1\}\}.
\]

Definition 2.2. A triple \( t_0 = (D_0, U, A) \) will be called a \( \tau_0 \)-system if it satisfies the following two properties:

1. \( D_0 \) is a multidigraph, \( U \subseteq V(D_0) \).
2. \( A = (A_u)_{u \in U} \) is a family of co-rooted trees with \( V(A_u) = \pi_u \) where \( \pi_u \) is a \( \tau \)-partition in \( u \), \( (u, u_+) \) is the co-root of \( A_u \) and \( |\pi_u| \geq 2 \).

For each \( u \in U \) and \( f \in F_u(D) \) we denote by \( \pi_u(f) \) the element of \( \pi_u \) containing \( f \).
If \( t_0 = (D_0, U, A) \) is a \( \tau_0 \)-system, then \( \tau_0(t_0) \) denotes the digraph defined as follows:

\[
V(\tau_0(t_0)) = V(D_0 - U) \cup \bigcup_{u \in U} V(A_u),
\]

\[
F(\tau_0(t_0)) = \{ f^* \mid f \in FD_0 \}
\]

for each \( f = wz \in FD_0 \), \( f^* \) is defined by

\[
f^* = \begin{cases} 
  f, & \text{when } \{ w, z \} \subseteq (V(D_0) - U), \\
  w(z, \pi_z(f)), & \text{when } w \in (V(D_0) - U) \text{ and } z \in U, \\
  (w, w_+)z, & \text{when } w \in U \text{ and } z \in (V(D_0) - U), \\
  (w, w_+)(z, \pi_z(f)), & \text{when } \{ w, z \} \subseteq U.
\end{cases}
\]

**Definition 2.3.** A pair \( t_1 = (t_0, \gamma) \) will be called a \( \tau_1 \)-system if \( t_0 = (D_0, U, A) \) is a \( \tau_0 \)-system and \( \gamma = (\gamma_u)_{u \in U} \) is a family, where \( \gamma_u = (\gamma^f_u)_{f \in F(A_u)} \) is a family of internally disjoint directed paths. Moreover, if \( f = w_1w_2 \), then \( \gamma^f_u \) is a \( w_1w_2 \)-directed path of positive even length and \( V(\gamma^f_u) \cap V(A_u) = \{ w_1, w_2 \} \). Also we denote \( t_1 = (D_0, U, A, \gamma) \).

Note that \( V(\gamma^f_{u_1}) \cap V(\gamma^f_{u_2}) = \emptyset \) for any \( f_1 \in F_{A_{u_1}}, f_2 \in F_{A_{u_2}} \) and \( u_1 \neq u_2 \).

If \( t_1 = (t_0, \gamma) \) is a \( \tau_1 \)-system, then we denote \( \tau_1(t_1) = \tau_0(t_0) \cup \bigcup_{u \in U} \bigcup_{f \in F_{A_u}} \gamma^f_u \).

**Definition 2.4** [5]. If \( D \) is a digraph and \( N, Q \subset V(D), N^c = V(D) - N, Q^c = V(D) - Q, \) \( N \) is said to be an independent kernel modulo \( Q \) (i.k. mod \( Q \)) of \( D \) iff

(i) \( N \) is independent,

(ii) For every \( w \in N^c \cap Q^c \) there exists a \( w \)-\( N \)-arc.

**Observation 2.1.** If \( D \) is a directed path of positive even length say \( D = (u_0, u_1, \ldots, u_{2n}), n \geq 1 \), then \( D \) satisfies the following properties:

(i) If \( N \) is an i.k. mod \( \{u_{2n}\} \) of \( D \), then \( u_0 \in N \) iff \( u_{2n} \in N \).

(ii) \( \{ u_i \mid i = 2k, 0 \leq k \leq n \} \) is an i.k. mod \( \{u_{2n}\} \), in fact it is a kernel of \( D \) which contains \( \{ u_0, u_{2n} \} \).

(iii) \( N = \{ u_{2i+1} \mid 0 \leq i \leq n - 1 \} \) is an i.k. mod \( \{u_{2n}\} \) of \( D \) such that \( \{ u_0, u_{2m} \} \subseteq N^c \).
**Definition 2.5.** Let $D$ be a multidigraph, $R, T \subseteq V(D)$; $T$ will be called $R$-homogeneous whenever $T \subseteq R$ or $T \subseteq (V(D) - R)$.

**Lemma 2.1.** Let $A$ be a co-rooted tree with co-root $a_0$, $|V(A)| \geq 2$ and $(\gamma^f)_{f=(u_f,v_f) \in FA}$ a family of internally disjoint directed paths of positive even length such that $\gamma^f$ is a $u_f v_f$-directed path and $V(\gamma^f) \cap V(A) = \{u_f, v_f\}$. If $N$ is an i.k. mod $\{a_0\}$ of $\bigcup_{f \in FA} \gamma^f$, then $V(A)$ is $N$-homogeneous. Moreover, when $V(A) \subseteq N^c$ there is no $a_0 N$-arc in $\bigcup_{f \in FA} \gamma^f$.

**Proof.** The proof is by induction on $|V(A)|$. If $|V(A)| = 2$ the result is a directed consequence of Observation 2.1 (i). Suppose that $|V(A)| > 2$ and let $g = u w \in F(A)$ be an arc such that $\delta^A(u) = 0$, $N$ be an i.k. mod $\{a_0\}$ of $\bigcup_{f \in FA} \gamma^f$ and $A_0 = A - \{u\}$. Clearly we have:

1. $N \cap \bigcup_{f \in FA_0} V(\gamma^f)$ is an i.k. mod $\{a_0\}$ of $\bigcup_{f \in FA_0} \gamma^f$ (because $\delta^A(u) = 0$) thus by the inductive hypothesis $V(A_0)$ is $N$-homogeneous.
2. $N \cap V(\gamma^g)$ is an i.k. mod $\{u, w\}$ of $\gamma^g$ and Observation 2.1 (i) implies $\{u, w\}$ is $N$-homogeneous.

It follows from (1) and (2) that $V(A)$ is $N$-homogeneous. When $V(A) \subseteq N^c$ it follows from the choice of $a_0$ that there is no $a_0 \bigcup_{f \in FA} \gamma^f$-arc, so there is no $a_0 N$-arc in $\bigcup_{f \in FA} \gamma^f$. ■

**Theorem 2.1.** Let $t_1 = (D_0, U, A, \gamma)$ be a $\tau_1$-system. If $D_0$ has a kernel, then $D = \tau_1(t_1)$ has a kernel.

**Proof.** Let $N_0$ be a kernel of $D_0$, Observation 2.1 implies that for each $u \in U$ and $f = w_1, w_2 \in F(A_u)$ there exist $N^i_{u,f}, \ i \in \{0, 1\}$ independent kernels mod $\{w_i\}$ of $\gamma^i_{u,f}$ such that $\{w_1, w_2\} \subseteq N^0_{u,f}$ and $\{w_1, w_2\} \subseteq (N^1_{u,f})^c$. It is easy to see by using Lemma 2.1 that

$$N = [N_0 \cap (V(D_0) - U)] \cup \left( \bigcup_{u \in N_0 \cap U} \bigcup_{f \in FA_u} N^0_{u,f} \right) \cup \left( \bigcup_{u \in N_0 \cap U^c} \bigcup_{f \in F(A_u)} N^1_{u,f} \right)$$

is a kernel of $\tau_1(t_1)$. ■
Theorem 2.2. Let \( t_1 = (D_0, U, A, \gamma) \) be a \( \tau_1 \)-system. If \( D = \tau_1(t_1) \) has a kernel, then \( D_0 \) has a kernel.

**Proof.** Let \( N \) be a kernel of \( D \); it is easy to see that for each \( u \in U \), \( N \cap ( \bigcup_{f \in FA_u} V(\gamma'_u) ) \) is an i.k. mod \( \{ (u, u_+) \} \) of \( \bigcup_{f \in FA_u} \gamma'_u \) and Lemma 2.1 implies \( V(A_u) \) is \( N \)-homogeneous and when \( V(A_u) \subseteq N^c \) there is no \( (u, u_+) \)-arc in \( D \) and it follows that

\[
N_0 = \left( N \cap \left( \bigcup_{u \in U} \bigcup_{f \in FA_u} V(\gamma'_u) \right) \right) \cup \left\{ u \in U \mid V(A_u) \subseteq N \right\}
\]

is a kernel of \( D_0 \).

Definition 2.6. Let \( A \) be a co-rooted tree, a subset \( S \) of \( V(A) \) will be called an initial section of \( A \) if for each \( w \in A \) such that there exists a \( wS \)-directed path in \( A \), we have \( w \in S \).

Clearly the empty set is an initial section of any co-rooted tree.

Theorem 2.3. Let \( t_1 = (D_0, U, A, \gamma) \) be a \( \tau_1 \)-system. Suppose that for each non trivial family \( S = (S_u)_{u \in U} \), where \( S_u \) is an initial section of \( A_u \), the digraph \( D_0 - \bigcup_{u \in U} \{ f \in FD_0 \mid f^* \text{ incides in } S_u \} \) is a KP-digraph (for each \( f \in FD_0, f^* \) denotes the arc of \( \tau_0(t_0) \) defined as in Definition 2.2). If every proper induced subdigraph of \( D_0 \) has a kernel, then every proper induced subdigraph of \( D = \tau_1(t_1) \) has a kernel.

**Proof.** First we recall that if \( G \) and \( H \) are digraphs then \( G \cap H \) denote the digraph whose vertex set is \( V(G) \cap V(H) \) and \( A(G \cap H) = A(G) \cap A(H) \). Now, if Theorem 2.3 were false, \( D \) would contain a proper induced CKI-subdigraph. Let \( H \) be a proper induced CKI-subdigraph of \( D \). First we will prove that for each \( u \in U \),

\[
H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma'_u) \right] = D \left[ \bigcup_{f \in F(A_u-S_u)} V(\gamma'_u) \right],
\]
where $S = (S_u = V(A_u) - V(H))_{u \in U}$ is a family such that $S_u$ is an initial section of $A_u$.

Let $u \in U$, when $H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma^f_u) \right] = D \left[ \bigcup_{f \in FA_u} V(\gamma^f_u) \right]$, then $S_u = \emptyset$ satisfies the required properties. If

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma^f_u) \right] \subseteq D \left[ \bigcup_{f \in FA_u} V(\gamma^f_u) \right]$$

then there exists $f = w_1 w_2 \in FA_u$ such that $H \cap D \left[ V(\gamma^f_u) \right] \subseteq D \left[ V(\gamma^f_u) \right] = \gamma^f_u$. Since $\text{Asym } H$ is strongly connected (see Theorem 1.3) Definition 2.3 implies

$$H \cap D \left[ V(\gamma^f_u) \right] \subseteq D \left[ \{w_1, w_2\} \right]$$

now we consider $A_u^w = A_u \{z \in V(A_u) \mid \text{there exists a } z w_1\text{-directed path contained in } A_u\}$ and

$$H_{w_1} = H \cap D \left[ \bigcup_{f \in FA_u^w} V(\gamma^f_u) \right].$$

So, we have that $H_{w_1} = \emptyset$ since, if $H_{w_1} \neq \emptyset$ then $H_2 = H \left[ V(H) - V(H_{w_1}) \right]$ is a $KP$-digraph such that $H_2 \cap D \left[ V(\gamma^f_u) \right] \subseteq D \left[ \{w_2\} \right]$ (since $H \cap D \left[ V(\gamma^f_u) \right] \subseteq D \left[ \{w_1, w_2\} \right]$). Furthermore, since for each $f \in FA_u$, $\gamma^f_u$ is a $KP$-digraph and, $A_u$ is a co-rooted tree, Theorem 1.2 implies that $\bigcup_{f \in FA_u} \gamma^f_u$ is a $KP$-digraph and clearly $H_{w_1} \subseteq^* \left( \bigcup_{f \in FA_u} \gamma^f_u \right)$ so $H_{w_1}$ is a $KP$-digraph;

Definition 2.3 and $H_2 \cap D \left[ V(\gamma^f_u) \right] \subseteq \{w_2\}$ imply there is no $H_{w_1} H_2$-arcs in $H$ and using Theorem 1.3 we conclude that $H$ is a $KP$-digraph which is impossible. So, we have proved that $H_{w_1} = \emptyset$ and then

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma^f_u) \right] \subseteq^* D \left[ \bigcup_{f \in FA_u^w} V(\gamma^f_u) \right]$$
for each $f \in FA_u$ such that

$$H \cap D \left[ V(\gamma'_u) \right] \subsetneq D \left[ V(\gamma'_u) \right],$$

and this implies

$$H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma'_f) \right] = D \left[ \bigcup_{f \in (A_u - S_u)} V(\gamma'_f) \right],$$

where $S_u = S^1_u \cup S^2_u$, $S^1_u = \{ w \in V(A_u) \mid \text{there exists } f = wz \in FA_u \text{ such that} \}

$$H \cap D \left[ V(\gamma'_u) \right] \subsetneq \gamma'_u$$

and $S^2_u = \bigcup_{w \in S^1_u} V(A^w_u)$. Clearly, $S_u$ is an initial section of $A_u$.

Let $H_0$ be a subdigraph (not necessarily induced) of $D_0$ obtained from $H$ by identifying $\bigcup_{f \in FA_u} V(\gamma'_f)$ with $u$, for each $u \in U$ such that $H \cap D \left[ \bigcup_{f \in FA_u} V(\gamma'_f) \right] \neq \emptyset$. So, we have that $H \cong \tau_1(H_0, U_0, (A_u - S_u)_{u \in U_0}, \gamma_0)$

where, $U_0 = U \cap V(H_0)$ and $\gamma_0$ is the restriction of $\gamma$ to $\bigcup_{u \in U} F(A_u - S_u)$.

Now we will prove that $H_0$ has a kernel.

If $S_u = \emptyset$ for each $u \in U$, then there exists

$$z \in \left( V(D) - \bigcup_{u \in U} \bigcup_{f \in FA_u} V(\gamma'_u) \cap (V(D) - V(H)) \right)$$

and hence $H_0$ is a proper induced subdigraph of $D_0$ and the hypothesis implies $H_0$ has a kernel.

If $S_u \neq \emptyset$ for some $u \in U$, then $H_u$ is an induced subdigraph of $D_0 - \bigcup_{u \in U'} \{ f \in FD_0 \mid f^* \text{ incidet in } S_u \}$, where $U' = \{ u \in U \mid S_u \neq \emptyset \}$ ($f^*$ is defined as in Definition 2.2) and the hypothesis implies that $H_0$ has a kernel.

Since $H_0$ has a kernel and $H = \tau_1(H_0, U_0, (A_u - S_u)_{u \in U_0}, \gamma_0)$, it follows from Theorem 3.1 that $H$ has a kernel contradicting that $H$ is a CKI-digraph.

**Theorem 2.4.** Let $t_1 = (D_0, U, A, \gamma)$ be a $\tau_1$-system. If every proper induced subdigraph of $D = \tau_1(t_1)$ has a kernel, then every proper induced subdigraph of $D_0$ has a kernel.
Proof. Let $D'_0$ be a proper induced subdigraph of $D_0$ and

$$D' = 	au_1(D'_0, U', (A_u)_{u \in U'}, \gamma'),$$

where $U' = (U \cap V(D'_0))$ and $\gamma'$ is the restriction of $\gamma$ to $\bigcup_{u \in U'} FA_u$; since $D'$ is a proper induced subdigraph of $D$, we have that $D'$ has a kernel and Theorem 2.2 implies that $D'_0$ has a kernel. \hfill \blacksquare

Theorem 2.5. Let $t_i = (D_0, U, \Lambda, \gamma)$ be a $\tau_1$-system such that for every non trivial family $S = (S_u)_{u \in U}$, where $S_u$ is an initial section of $A_u$, the digraph $D_0 - \bigcup_{u \in U} \{ f \in FD_0 | f \text{ incides in } S_u \}$ is a $KP$-digraph. Then $\tau_1(t_i)$ is a $KP$-digraph (resp: $CKI$-digraph) if and only if $D_0$ is a $KP$-digraph (resp: $CKI$-digraph).

3. $\tau_1$-Constructions

In these section we present a method to realize in a simple way some $\tau_1$-constructions and we obtain a large variety of $KP$ digraphs and $CKI$-digraphs satisfying the $k$-Meyniel’s condition.

Let $D_0$ be a multidigraph, $U \subseteq V(D_0)$, $<^P$ be a total order in $\{ v(f) = \{u_1, u_2\} | f \text{ is an } u_1u_2\text{-arc} \}$, and $<^{u_1u_2}$ be a total order in $\{ f \in FD_0 | f \text{ is an } u_1u_2\text{-arc} \}$. We will denote by $<$ the total order defined in

$$\bigcup_{u \in U} \{(v(f), f) | f \in F(Sym D_0) \cap F_u^-(D_0)\}$$

as follows: $(v(f), f) < (v(g), g)$ if and only if $v(f) <^P v(g)$ or $v(f) = v(g) = \{u_1, u_2\}$ and $f <^{u_1u_2} g$. And for each $u \in U$ we will denote by $u_-(f) = \{f\}$ when $f \in F(Sym D_0) \cap F_u^-(D_0)$; $u^- = F(Asym D_0) \cap F_u^-(D_0)$, $u_+ = F_u^+(D_0)$,

$$\Pi_u = \{u_+, u_-, u_-(f) | f \in F(Sym D_0) \cap F_u^-(D_0)\},$$

(clearly $\Pi_u$ is a $\tau$-partition in $u$), $A_u^<$ the $u_+u_-$-directed path defined as follows

$$A_u^< = \left( u_-, u_-(f_1), u_-(f_2), \ldots, u_-(f_r), u_+ \right),$$

where

$$(v(f_1), f_1) < (v(f_2), f_2) < \ldots < (v(f_r), f_r)$$
and

\[ \{f_1, \ldots, f_r\} = F Sym D_0 \cap F_0 D_0. \]

Finally we denote by \( A^\prec = (A^\prec_u)_{u \in U} \).

**Theorem 3.1.** Let \( D_0 \) be a multidigraph which is a quasi KP-digraph and \( t_o = (D_o, U, A^\prec) \) any \( \tau_o \)-system defined as at the beginning of this section. For any non trivial family \( S = (S_u)_{u \in U} \), where \( S_u \) is an initial section of \( A_u^\prec \), \((D_0 - \bigcup_{u \in U} \{ f \in FD_o \mid f^* \text{ incidis in } S_u \}) = D_0(S) \) is a KP-digraph.

**Proof.** Suppose that there exists a non trivial family \( S = (S_u)_{u \in U} \), where \( S_u \) is an initial section of \( A_u^\prec \); such that \( D_o(S) \) is not a KP-digraph and let \( D_1 \) be a CKI-digraph which is an induced subdigraph of \( D_0(S) \); since \( D_o(S) \) is a proper subdigraph of \( D_0 \), we have that \( D_1 \) is a non induced subdigraph of \( D_0 \) and there exists an \( uv \)-arc in \( FD_o \) \( \cup (D_1) - F(D_1) \) and \( S_0 \) is not empty, so \( v^0 \cap FD_1 = \emptyset \). Since \( D_1 \) is a CKI-digraph, Theorem 1.4 implies that there exists some \( uv \)-arc in \( Asym D_1 \) and \( \emptyset = v^0 \cap FD_1 = F Asym D_0 \cap Fv^{-} (D_0) \cap FD_1 \) implies \( uv \in F(Asym D_0) \), so there exists \( w \in V(D_1) \) such that \( \delta^+_{D_1}(w) \neq 0 \), where \( D_1^* = Asym D_1 \). Furthermore, if \( \emptyset \neq F(D_1) \), then \( S_0 \neq \emptyset \), \( z^0 \cap F(D_1) = \emptyset \) and since \( D_1 \) is a CKI-digraph, Theorem 1.4 implies \( F(Asym D_1) \cap Fz^{-}(D_1^*) \neq \emptyset \) and then there exists \( w \in F(D_1^*) \); hence \( \delta^+_{D_1}(w) \neq 0 \). We have proved:

(a) there exists \( w \in V(D_1^*) \) such that \( \delta^+_{D_1}(w) \neq 0 \).

(b) if \( \delta^+_{D_1}(z) \neq 0 \), then \( \delta^+_{D_1}(z) \neq 0 \).

It follows that \( D_1^* \) contains a directed cycle \( C = (w_0, f_0, w_1, f_1, \ldots, w_n, f_n, w_0) \) where \( \{w_0, \ldots, w_n\} \subseteq V(D_1^*) \), \( \{f_0, \ldots, f_n\} \subseteq FD_1^* \). Since \( \prec \) is a total order in \( \{v(f) \mid f \in Sym D_0\} \) and \( C \subseteq D_1^* \), it follows that for some \( i \in \{0, 1, \ldots, n\} \), \( \{w_{i-1}, w_i\} \prec^p \{w_i, w_{i+1}\} \) (the indices are taken mod \( n+1 \)). It follows from the definition of \( t_o = (D_0, U, A^\prec) \) that

\[ A^\prec_w \{\{w_{i-1}(g) \mid g \text{ is a } w_{i+1} w_i \text{-arc}\} \]

is a subpath of the subpath of \( A^\prec_w \) between the vertices \( w_{i-1}(f_{i-1}) \) and \( w_i^+ \) and since \( f_{i-1} \in F(C) \subseteq FD_1^* \) it follows \( w_{i-1}(f_{i-1}) \notin S_{w_i} \) and then \( \{w_{i-1}(g) \mid g \text{ is a } w_{i+1} w_i \text{-arc}\} \cap S_{w_i} = \emptyset \). Since \( f_i \in C \subseteq D_1^* \) and

\[ \{w_{i-1}(g) \mid g \text{ is a } w_{i+1} w_i \text{-arc}\} \cap S_{w_i} = \emptyset , \]
there exists some $w_{i+1}w_i$-arc in $D_0$ which is also in $D_0(S)$ and since $D_1$ is an induced subdigraph of $D_0$, it follows that $f_i \in F(Sym D_1)$ contradicting $f_i \in F(D')$.

A digraph $D$ is said to satisfy the $k$-Meyniel’s condition if each odd directed cycle of $D$ has at least $k$ diagonals and we write $D$ satisfies $M(k)$.

Let $D_0$ be a digraph, we will denote by $D_0^{(k)}$ the multidigraph obtained from $D_0$ by adding to each symmetrical arc the multiplicity $k$.

**Lemma 3.1.** If $D_0$ is a digraph such that every odd directed cycle has a symmetrical arc and $t_1 = (D_0^{(k)}), V(D_0), A^\leq, \gamma)$, then $\tau_1(t_1)$ satisfies $M(k)$.

**Proof.** Let $C$ be an odd directed cycle contained in $\tau_1(t_1)$; since $\bigcup_{f \in FA_u^\leq} \gamma^f_u$ is a directed path of even length, we have that $C'$ is the digraph obtained from $C$ by identifying $\bigcup_{f \in FA_u^\leq} \gamma^f_u$ with $u$ for each $u \in V(D_0)$; $C'$ is an odd directed cycle in $D_0^{k}$ and clearly $C \cong t_1(C', V(C'), A^\leq / V(C'), \gamma')$, where $\gamma'$ is the restriction of $\gamma$ to $\bigcup_{u \in V(C')} \bigcup_{f \in FA_u^\leq} \gamma^f_u$ and Definition 2.3 implies that each pseudodiagonal of $C'$ is a diagonal of $C$.

As a direct consequence of Theorems 3.1, 2.5 and Lemma 3.1 we obtain.

**Theorem 3.2.** If $D_0$ is a KP-digraph (resp. CKI-digraph) such that every odd directed cycle has a symmetrical arc and $t_1 = (D_0^{(k)}), V(D_0), A^\leq, \gamma)$, then $\tau_1(t_1)$ is a KP-digraph (resp. CKI-digraph) which satisfies $M(k)$.

**Corollary 3.1.** For each natural number $k$, there exists some KP-digraph (resp. CKI-digraph) $D_k$ which satisfies the $k$-Meyniel’s condition.

**Proof.** Define the digraph $C = C_n(j_1, \ldots, j_k)$ by $V(C) = \{0, 1, \ldots, n-1\}$, $F(C) = \{uw \mid v-u \equiv j_s \pmod{n} \text{ for } s = 1, \ldots, k\}$ and denote $D_0 = C_n(1, \pm 2, \ldots, \pm r)$ for an even natural number $n \not\equiv 0 \pmod{r+1}$. In [6] it was proved that $D_0$ is a CKI-digraph; so it follows from Theorems 3.1, 2.3 and Lemma 3.1 that $\tau_1(t_1)$ is a CKI-digraph which satisfies $M(k)$.

**References**


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