A NOTE ON \((k, l)\)-KERNELS
IN \(B\)-PRODUCTS OF GRAPHS

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Abstract

\(B\)-products of graphs and their generalizations were introduced in [4]. We determined the parameters \(k, l\) of \((k, l)\)-kernels in generalized \(B\)-products of graphs. These results are generalizations of theorems from [2].

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1. Definitions and Notation

By \(G\) we mean a finite connected graph without loops and multiple edges with the vertex set \(V(G)\) and the edge set \(E(G)\). The number \(d_G(x, y)\) denotes the length of the shortest path connecting \(x\) and \(y\) in \(G\). Note that \(d_G(x, y)\) is finite and \(d_G(x, y) \geq 1\) if \(x \neq y\).

Let \(k, l\) be integers, \(k \geq 2\) and \(l \geq 1\). \(J \subset V(G)\) is called a \((k, l)\)-kernel of \(G\) if and only if

1. for distinct \(x, y \in J\), \(d_G(x, y) \geq k\) and
2. for each \(x \notin J\) there exists \(y \in J\) such that \(d_G(x, y) \leq l\).

For \(k = 2, l = 1\) we obtain a kernel in Berge’s sense.

The Cartesian product of two graphs \(G_1, G_2\) is the graph \(G_1 \times G_2\) with the vertex set \(V(G_1) \times V(G_2)\) and the edge set \(E(G_1 \times G_2)\), such that \([(x', y'), (x, y)] \in E(G_1 \times G_2)\) if and only if \([x', x] \in E(G_1)\) and \(y = y'\) or \([y, y'] \in E(G_2)\) and \(x = x'\).

The normal product of two graphs \(G_1, G_2\) is the graph \(G_1 \cdot G_2\), such that \(V(G_1 \cdot G_2) = V(G_1) \times V(G_2)\) and \([(x', y'), (x, y)] \in E(G_1 \cdot G_2)\) if and only if \([x', x] \in E(G_1)\) and \(y = y'\) or \([y', y] \in E(G_2)\) and \(x = x'\) or \([x', x] \in E(G_1)\) and \([y', y] \in E(G_2)\).
So-called $B$-products of graphs were defined in [4] as follows. Let $B \subset N \times N - \{(0,0)\}$, where $N$ is the set of non-negative integers. Then the $B$-product of the graphs $G_1, G_2$ is the graph $B(G_1, G_2)$ with $V(B(G_1, G_2)) = V(G_1) \times V(G_2)$ and $E(B(G_1, G_2)) = \{[(i,j),(i',j')] : (d_{G_1}(i,i'), d_{G_2}(j,j')) \in B\}$. The set $B$ is called the basic set of the $B$-product.

The generalized Cartesian product $B^m_n(G_1, G_2)$ and the generalized normal product $B_{mn}(G_1, G_2)$ are defined by the basic sets $B^m_n = \{(i,0) : 1 \leq i \leq m\} \cup \{(0,j) : 1 \leq j \leq n\}$, $B_{mn} = \{(i,j) : 0 \leq i \leq m \text{ and } 1 \leq j \leq n \text{ or } 1 \leq i \leq m \text{ and } 0 \leq j \leq m\}$, respectively.

If $m = 1$ and $n = 1$, then $B^1_1(G_1, G_2) = G_1 \times G_2$ and $B_{11}(G_1, G_2) = G_1 \cdot G_2$. For $r \geq 1$ the $r$-th power $G^r$ of a graph $G$ is defined as follows: $V(G^r) = V(G)$ and $E(G^r) = \{[x,y] : x,y \in V(G) \text{ and } 1 \leq d_G(x,y) \leq r\}$.

In [4] the following dependences between the well-known products and their generalizations were proved.

**Theorem 1** [4]. $B^m_n(G_1, G_2) = G^m_1 \times G^n_2$, $B_{mn}(G_1, G_2) = G^m_1 \cdot G^n_2$, $B_{nn}(G_1, G_2) = (G_1 \cdot G_2)^n$, for $n, m \geq 1$.

For undefined terms, see [1].

2. **Main Results**

**Theorem 2.** If $J$ is a $(k,l)$-kernel of $G$, then $J$ is a $(k_0,l_0)$-kernel of $G^r$, for $k, k_0 \geq 2$, and $l, l_0 \geq 1$, $r \leq k - 1$ where

$$k_0 = \begin{cases} \frac{k}{r}, & \text{if } \frac{k}{r} \text{ is an integer,} \\ \left\lfloor \frac{k}{r} \right\rfloor + 1, & \text{otherwise,} \end{cases}$$

$$l_0 = \begin{cases} \frac{l}{r}, & \text{if } \frac{l}{r} \text{ is an integer,} \\ \left\lfloor \frac{l}{r} \right\rfloor + 1, & \text{otherwise,} \end{cases}$$

where $\lfloor p \rfloor$ denotes the largest integer less than or equal to $p$.

**Proof.** Suppose, that $J$ is a $(k,l)$-kernel of $G$. We shall show that $J$ is a $(k_0,l_0)$-kernel of $G^r$, for $k_0, l_0$ as described above. By the definition of $G^r$ it follows that if there exists a path of length $\leq r$ connecting $x_i$ to $x_j$ in $G$, then $[x_i,x_j] \in E(G^r)$. It is clear, that for distinct vertices $x_i, x_j \in J$
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holds \(d_G(x_i, x_j) \geq k\). This means that there is the shortest path of length \(\geq k\), say \((x_i, x_{i+1}, x_{i+2}, \ldots, x_j)\), connecting vertices \(x_i, x_j\) in \(G\). Moreover, using the definition of \(G^r\), we obtain that the shortest path between \(x_i, x_j\) in \(G^r\) is of the form: \((x_i, x_{i+r}, x_{i+2r}, \ldots, x_{i+k}, \ldots, x_j)\), if \(\frac{k}{r}\) is an integer, and \((x_i, x_{i+r}, x_{i+2r}, \ldots, x_{i+\lfloor \frac{k}{r} \rfloor r}, \ldots, x_j)\), otherwise. Note, that if \(d_G(x_i, x_j) = k\), then \(i + k = j\), if \(\frac{k}{r}\) is an integer, and \(i + \lfloor \frac{k}{r} \rfloor r + 1 \leq j\), if \(\frac{k}{r}\) is not an integer.

Finally,

\[
d_G^r(x_i, x_j) \geq \begin{cases} \frac{k}{r}, & \text{if } \frac{k}{r} \text{ is an integer}, \\ \lfloor \frac{k}{r} \rfloor + 1, & \text{otherwise}. \end{cases}
\]

Let \(x_i \notin J\). So it is clear that there exists \(x_j \in J\) in \(G\), such that \(d_G(x_i, x_j) \leq l\). Moreover, using the definition of \(G^r\) we have analogously that

\[
d_G^r(x_i, x_j) \leq \begin{cases} \frac{l}{r}, & \text{if } \frac{l}{r} \text{ is an integer}, \\ \lfloor \frac{l}{r} \rfloor + 1, & \text{otherwise}. \end{cases}
\]

Thus, the theorem is proved.

\[\Box\]

For \(r = k - 1\) we obtain the result from [3].

Using Theorems 1, 2 and Theorems 3 and 4 given below we obtain immediately Theorems 5, 6.

**Theorem 3** [2]. If the subset \(J_i\) is a \((k_i, l_i)\)-kernel of \(G_i\), where \(k_i \geq 2\), \(l_i \geq 1\), for \(i = 1, 2\), then the set \(J = J_1 \times J_2\) is a \((k, l)\)-kernel of the graph \(G_1 \times G_2\), where \(k = \min\{k_1, k_2\}\), \(l = l_1 + l_2\).

**Theorem 4** [2]. If the subset \(J_i\) is a \((k_i, l_i)\)-kernel of \(G_i\), \(k_i \geq 2\), \(l_i \geq 1\), for \(i = 1, 2\), then the set \(J = J_1 \times J_2\) is a \((k, l)\)-kernel of the graph \(G_1 \cdot G_2\), where \(k = \min\{k_1, k_2\}\), \(l = \max\{l_1, l_2\}\).

**Theorem 5.** If \(J_i\) is a \((k_i, l_i)\)-kernel of \(G_i\), for \(k_i \geq 2\), \(l_i \geq 1\), \(i = 1, 2\), then the set \(J = J_1 \times J_2\) is a \((k, l)\)-kernel of \(B^m_n(G_1, G_2)\), for \(m \leq k_1 - 1\), \(n \leq k_2 - 1\), where \(k = \min\{\alpha_1, \alpha_2\}\), \(l = \beta_1 + \beta_2\) and

\[
\alpha_1 = \begin{cases} \frac{k_1}{m}, & \text{if } \frac{k_1}{m} \text{ is an integer}, \\ \lfloor \frac{k_1}{m} \rfloor + 1, & \text{otherwise}, \end{cases}
\]

where \(\beta_1 = \lfloor \frac{k_1}{m} \rfloor\) and \(\beta_2 = \lfloor \frac{k_2}{m} \rfloor\).
\( \alpha_2 = \begin{cases} \frac{k_2}{n}, & \text{if } \frac{k_2}{n} \text{ is an integer}, \\ \lfloor \frac{k_2}{n} \rfloor + 1, & \text{otherwise}, \end{cases} \)

\( \beta_1 = \begin{cases} \frac{l_1}{m}, & \text{if } \frac{l_1}{m} \text{ is an integer}, \\ \lfloor \frac{l_1}{m} \rfloor + 1, & \text{otherwise}, \end{cases} \)

\( \beta_2 = \begin{cases} \frac{l_2}{n}, & \text{if } \frac{l_2}{n} \text{ is an integer}, \\ \lfloor \frac{l_2}{n} \rfloor + 1, & \text{otherwise}. \end{cases} \)

**Theorem 6.** If \( J_i \) is a \((k_i, l_i)\)-kernel of \( G_i \), for \( k_i \geq 2, l_i \geq 1, i = 1, 2 \), then the set \( J = J_1 \times J_2 \) is a \((k, l)\)-kernel of \( B_{mn}(G_1, G_2) \), for \( m \leq k_1 - 1, n \leq k_2 - 1 \), where \( k = \min\{\alpha_1, \alpha_2\} \), \( l = \max\{\beta_1, \beta_2\} \) and numbers \( \alpha_i, \beta_i \) are defined as in Theorem 5.

If \( m = 1, n = 1 \), then from Theorem 5 we obtain Theorem 3 and from Theorem 6 it follows Theorem 4.

**References**


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