MINIMAL REDUCIBLE BOUNDS FOR HOM-PROPERTIES OF GRAPHS

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Abstract

Let $H$ be a fixed finite graph and let $\rightarrow H$ be a hom-property, i.e. the set of all graphs admitting a homomorphism into $H$. We extend the definition of $\rightarrow H$ to include certain infinite graphs $H$ and then describe the minimal reducible bounds for $\rightarrow H$ in the lattice of additive hereditary properties and in the lattice of hereditary properties.

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1. Definitions

In general we follow the notation and terminology of [1]. Denote by $I$ the set of all finite undirected simple graphs. Any isomorphism-closed subset $P$ of $I$ is called a property of graphs. A property $P$ is hereditary if whenever a graph $G$ is in $P$, then all subgraphs of $G$ are also in $P$. A property $P$ is additive if whenever graphs $G$ and $H$ are in $P$, then their disjoint union, denoted by $G \cup H$, is in $P$ too. When partially ordered under set inclusion, the poset of all additive hereditary properties forms a complete distributive lattice, which we will denote by $L^a$. We use $L$ to denote the lattice of hereditary properties. A property is called non-trivial if it contains at least one non-null graph and it is not equal to $I$.

Let $P_1, P_2, \ldots, P_n$ be any properties of graphs. A vertex $(P_1, P_2, \ldots, P_n)$-partition of a graph $G$ is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for
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each \( i = 1, 2, \ldots, n \), the induced subgraph \( G[V_i] \) has the property \( P_i \). Any of the \( V_i \) may be empty. The property \( P_1 \circ P_2 \circ \ldots \circ P_n \) is defined as the set of all graphs having a vertex \((P_1, P_2, \ldots, P_n)\)-partition. If \( P_1, P_2, \ldots, P_n \) are all (additive) hereditary properties, then \( P_1 \circ P_2 \circ \ldots \circ P_n \) is an (additive) hereditary property too. For convenience, we will write \( P_1 \circ P_2 \circ \ldots \circ P_n \) as \( P_1 P_2 \ldots P_n \), omitting the binary operation symbol.

An additive hereditary property \( R \) is called reducible in \( L^a \) if there exist non-trivial properties \( P \) and \( Q \) in \( L^a \) such that \( R = PQ \). Otherwise \( R \) is called irreducible. A reducible property \( R \in L^a \) is called a minimal reducible bound for property \( P \in L^a \) if \( P \subseteq R \) and there is no reducible property \( R_1 \) satisfying \( P \subseteq R_1 \subseteq R \). From this definition, each reducible property is the unique minimal reducible bound for itself. We use the symbol \( B(P) \) to denote the class of all minimal reducible bounds for property \( P \). We do not know whether a minimal reducible bound exists for every property \( P \), and \( B(P) \) is known for only a few properties \( P \). Similar definitions hold in \( L \).

Given any \( P \in L^a \) (or in \( L \)), we define the class of all \( P \)-maximal graphs by \( M(P) = \{G \in P : G + e \notin P \text{ for any } e \in E(G)\} \). \( M(P) \) determines \( P \) in the sense that \( H \in P \) iff there exists some \( P \)-maximal graph \( G \) such that \( H \leq G \).

A homomorphism from a graph \( G \) to a graph \( H \) is a mapping \( f \) of the vertex set \( V(G) \) to the vertex set \( V(H) \) which preserves edges, i.e. if \( \{u, v\} \in E(G) \), then \( \{f(u), f(v)\} \in E(H) \). We say that \( G \) is homomorphic to \( H \) if there exists a homomorphism from \( G \) to \( H \), and we write \( G \rightarrow H \). If \( G \rightarrow H \), then \( \chi(G) \leq \chi(H) \). If \( H \) is a finite graph, then the hom-property generated by \( H \) is the set \( \rightarrow H = \{G \in I : G \rightarrow H\} \). Note that \( \rightarrow H \) is an additive hereditary property for any \( H \in I \).

In Section 2 we summarise some fundamental properties of hom-properties. In Section 3 we extend the definition of hom-properties to include \( \rightarrow H \) where \( H \) may be an infinite union of finite graphs. We then describe \( B(\rightarrow H) \) in the lattice \( L^a \) in Section 4 and consider some applications of these results in Section 5. Section 6 describes \( B(\rightarrow H) \) in the lattice \( L \).

2. Fundamental Properties of Hom-Properties

Given a graph \( G \), a core of \( G \) is any subgraph \( G' \) of \( G \) such that \( G \rightarrow G' \), and such that \( G \) is not homomorphic to any proper subgraph of \( G' \). Every graph \( G \) has a unique core up to isomorphism (see [2]) which is denoted by \( C(G) \). If \( G = C(G) \), i.e. if \( G \) is not homomorphic to any of its proper subgraphs, then we call \( G \) a core. Since any graph homomorphic to \( G \) is
also homomorphic to $C(G)$, and any element of $\rightarrow C(G)$ is in $\rightarrow G$, we have that $\rightarrow G = \rightarrow C(G)$. Hence, given any hom-property, we can assume it is of the form $\rightarrow H$ where $H$ is a core.

The $\rightarrow H$-maximal graphs are known and described in [4]:

Given any $G \in \mathcal{I}$, with $V(G) = \{v_1, v_2, ..., v_n\}$, its multiplications $G^i$ are defined as follows:

1. $V(G^i) = W_1 \cup W_2 \cup ... \cup W_n$,
2. for each $1 \leq i \leq n$, $|W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$, $W_i \cap W_j = \emptyset$,
4. The only edges of $G^i$ are all the edges of the form $\{u, v\}$ where $u \in W_i$, $v \in W_j$ and $\{v_i, v_j\} \in E(G)$.

Thus each vertex $v_i$ of $G$ is replaced by a non-empty set of vertices $W_i$ (also denoted by $v_i^i$) and if $u \in W_i, v \in W_j$, then $u$ and $v$ are adjacent in $G^i$ iff $v_i$ and $v_j$ are adjacent in $G$. $W_1, W_2, ..., W_n$ are independent sets called the multivertices of $G^i$. We also write $G^i$ as $G^i(W_1, W_2, ..., W_n)$ to emphasize its structure, and $G^i(k)$ for $G^i(W_1, W_2, ..., W_n)$ if $|W_i| = k$ for each $i = 1, 2, ..., n$. By mapping all the vertices in $W_i$ to $v_i$ for each $i = 1, 2, ..., n$, it is readily seen that $G^i \rightarrow G$, i.e. $G^i \in \rightarrow G$ and that $C(G^i) = G$ if $G$ is a core.

Kratovil, Mihók and Semanišin proved in [4] that every $\rightarrow H$-maximal graph is a multiplication of a subgraph of $H$ that is itself a core. Thus for every $\rightarrow H$-maximal graph $G$, there exists an integer $k \geq 1$ such that $G$ is contained in $H^i(k)$.

The following lemma describes properties of hom-properties that will be used often in what follows. We use the notation $H + G$ for the join of two graphs $H$ and $G$, i.e. for the graph obtained from $H \cup G$ by adding all edges joining vertices of $H$ to vertices of $G$. A graph that is the join of two non-null graphs is called decomposable, while a graph that is not decomposable is called indecomposable.

**Lemma 1.**

1. $\rightarrow K_1$ is the set of all edgeless graphs, also denoted by $\mathcal{O}$.
2. $\rightarrow K_2$ is the set of all bipartite graphs and $\rightarrow K_2 = \rightarrow H$ for any graph $H$ with chromatic number 2, since $C(H) = K_2$.
3. For any graphs $H$ and $G$, $\rightarrow (H + G) = (\rightarrow H)(\rightarrow G)$ (see [3]).
4. $\rightarrow H$ is irreducible in $\mathcal{L}^a$ iff $H$ is indecomposable (see [3]).
5. For any graphs $H$ and $G$, $\rightarrow H \subseteq \rightarrow G$ iff $H \rightarrow G$ iff $H \in \rightarrow G$ (see [2]).
3. The Hom-Property $\rightarrow H$ for Infinite $H$

Although each hom-property is an additive hereditary property and is thus an element of the complete lattice $\mathbb{L}^a$, the hom-properties do not form a complete sublattice of $\mathbb{L}^a$. For example $\forall \{ \rightarrow R : R \text{ is a triangle-free core} \}$ cannot be a hom-property: If $\forall \{ \rightarrow R : R \text{ is a triangle-free core} \} =\rightarrow H$ for some graph $H$, then $\rightarrow R \subseteq \rightarrow H$ for each triangle-free core $R$. This would imply that $\chi(R) \leq \chi(H)$ for each triangle-free core $R$, which is not true, since triangle-free graphs of arbitrarily high chromatic number can be constructed.

To enable the supremum and infimum (intersection) of an arbitrary set of hom-properties to again be a hom-property, we extend the definition of hom-properties by including $\forall \{ \rightarrow H : \} = \forall \{ \rightarrow H : \}$. For such a graph $H$ we define $\rightarrow H$ by $\rightarrow H = \{ G \in \mathcal{I} : G \rightarrow H \}$, i.e. $\rightarrow H$ is the set of all finite graphs admitting a homomorphism into $H$.

Since the set of all finite graphs is countable, and since only one copy of each connected component of $H$ is sufficient, we can always assume that $H$ is a countable union of finite cores and that these cores are pairwise non-isomorphic. Unlike in the case where $H$ is finite, $H$ itself need no longer have a core e.g. $K_1 \cup K_2 \cup K_3 \cup \ldots$ has no core, and $H$ need not have a finite chromatic number.

Extending the definition of hom-properties to allow $\rightarrow H$ where $H$ is either finite or a countable union of finite graphs makes the hom-properties a complete sublattice of $\mathbb{L}^a$, i.e. the supremum and infimum of any set of hom-properties is again a hom-property, as the following two results show.

**Theorem 2.** Let $\{ H_\alpha : \alpha \in A \}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\forall \{ \rightarrow H_\alpha : \alpha \in A \} =\rightarrow (\cup \{ H_\alpha : \alpha \in A \})$.

**Proof.** In the lattice $\mathbb{L}^a$, $\forall \{ \rightarrow H_\alpha : \alpha \in A \}$ is the least additive hereditary property which contains each $\rightarrow H_\alpha$, $\alpha \in A$. We show that $\rightarrow (\cup \{ H_\alpha : \alpha \in A \})$ satisfies this.

Clearly, if $G \in \rightarrow H_\alpha$ for any $\alpha \in A$, then $G \in \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$. Therefore $\rightarrow H_\alpha \subseteq \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$ for each $\alpha \in A$.

Now suppose that $\rightarrow H_\alpha \subseteq \mathcal{P}$ for each $\alpha \in A$, for some property $\mathcal{P} \in \mathbb{L}^a$. We show that $\rightarrow (\cup \{ H_\alpha : \alpha \in A \}) \subseteq \mathcal{P}$: Let $G \in \rightarrow (\cup \{ H_\alpha : \alpha \in A \})$. By definition, $G$ is finite, and hence there is a homomorphism from $G$ to a finite union of $H_\alpha$’s, say $G \in \rightarrow H_1 \cup H_2 \cup \ldots \cup H_n$. Since each connected component...
of $G$ is homomorphically mapped to exactly one $H_i$, $G$ has a decomposition $G = G_1 \cup G_2 \cup \ldots \cup G_n$, such that $G_i \rightarrow H_i$, for $i = 1, 2, \ldots, n$. But then we have $G_i \in \rightarrow H_i \in \mathcal{P}$ for $i = 1, 2, \ldots, n$. As each $G_i$ is in $\mathcal{P}$, by the additivity of $\mathcal{P}$, $G$ is in $\mathcal{P}$ too. $lacksquare$

**Theorem 3.** Let $\{H_\alpha : \alpha \in A\}$ be a set of graphs, each of which is finite or a countable union of finite graphs. Then $\bigwedge \{\rightarrow H_\alpha : \alpha \in A\} = \rightarrow (\bigcup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\})$.

**Proof.** Suppose $G \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$. Then $G \rightarrow C(G)$ and $C(G) \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$. Then for each $\alpha \in A, C(G) \in \rightarrow H_\alpha$ and so $C(G)$ is contained in a multiplication of a finite subgraph of $H_\alpha$. So we have $G \in \rightarrow C(G) \subseteq \rightarrow (\bigcup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\})$.

Conversely, suppose $G \in \rightarrow (\bigcup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\})$. Then there exists a homomorphism $f : G \rightarrow (\bigcup \{R : R$ is a core contained in a multiplication of a finite subgraph of $H_\alpha$ for each $\alpha \in A\})$. Consider any connected component $K$ of $G$; it is mapped by $f$ to one of these cores, say $R$. By the definition of $R$, $R \in \cap \{\rightarrow H_\alpha : \alpha \in A\}$ and so $K \in \rightarrow R \subseteq \cap \{\rightarrow H_\alpha : \alpha \in A\}$. But then $\cap \{\rightarrow H_\alpha : \alpha \in A\}$ is an additive property containing each connected component of $G$ and we conclude that $G$ itself is in $\cap \{\rightarrow H_\alpha : \alpha \in A\}$. $lacksquare$

4. **Minimal Reducible Bounds for $\rightarrow H$ in $\mathbb{L}^a$**

In this section we describe the set of all minimal reducible bounds for $\rightarrow H$ in the lattice $\mathbb{L}^a$, first dealing with the case where $H$ is finite, and then with the infinite case. The following lemma and its corollary are useful for both cases.

**Lemma 4.** Let $H$ be a finite core or a countable union of finite cores. If $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ with $\mathcal{O} \subseteq \mathcal{P}$ and $\mathcal{O} \subseteq \mathcal{Q}$ such that $\rightarrow H \subseteq \mathcal{P} \mathcal{Q}$ then there exists a partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq \mathcal{P} \mathcal{Q}$ and $\rightarrow H[V_1] \subseteq \mathcal{P}$ and $\rightarrow H[V_2] \subseteq \mathcal{Q}$.

**Proof.** First suppose that $H$ is finite and let $V(H) = \{v_1, v_2, \ldots, v_n\}$. We will show that there exists a partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and
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For each partition $(V_1^i, V_2^i)$ of $(2k - 1)$, we can ensure that the remaining $v_i^j(k)$ is completely in the $P$ part or completely in the $Q$ part. We can also ensure that neither the $P$ nor the $Q$ part is empty: One of the $v_i^j(k)$ can be moved to the empty part if necessary.

We now have disjoint sets $I_1$ and $I_2$ such that $I_1 \cup I_2 = \{1, 2, ..., n\}$ and $(\{v : v \in v_i^j(k), i \in I_1\}, \{v : v \in v_i^j(k), i \in I_2\})$ forms a $(P, Q)$ partition of $H^i_{V}(k)$.

Since $P$ and $Q$ are hereditary properties, each such pair $(I_1, I_2)$ induces a $(P, Q)$-partition of $H^i_{V}(r)$ for each $r \leq k$, with each $v_i^j(r)$ entirely in the $P$ part or entirely in the $Q$ part. Since there are only finitely many partitions $(I_1, I_2)$ of $\{1, 2, ..., n\}$, there exists a pair $(I_1^*, I_2^*)$ which serves for infinitely many values of $k$, and hence for every value of $k$. Let $V_1 = \{v_i \in V(H) : i \in I_1^*\}$ and $V_2 = \{v_i \in V(H) : i \in I_2^*\}$. Then $H[V_1] \subseteq P$ for all $k \geq 1$ and $H[V_2] \subseteq Q$ for all $k \geq 1$.

Suppose now that $H$ is a countable union of finite graphs, $H = H_1 \cup H_2 \cup ...$. Denote by $G_n$ the graph $H_1 \cup H_2 \cup ... \cup H_n$, $n \geq 1$, and let $G$ be the set of all $G_n$, i.e. $G = \{ G_n : n \geq 1 \}$.

For each $n \geq 1$, $G_n \subseteq \mathcal{P} \mathcal{Q}$ and so by the finite case above, there exists a partition $(W_1^n, W_2^n)$ of $V(G_n)$ with neither part empty such that $G_n-W_1^n \subseteq \mathcal{P}$ and $G_n-W_2^n \subseteq \mathcal{Q}$. Restricted to $V(H_1)$, each $(W_1^n, W_2^n)$ induces a partition of $V(H_1)$ such that $H_1-W_1^n \subseteq \mathcal{P}$ and $H_1-W_2^n \subseteq \mathcal{Q}$. Since $V(H_1)$ has only finitely many partitions, there exists a partition of $V(H_1)$ with these properties induced by infinitely many $(W_1^n, W_2^n)$. Call this partition $(V_1^1, V_2^1)$ and note that $H_1-V_1^1 \subseteq \mathcal{P}$ and $H_1-V_2^1 \subseteq \mathcal{Q}$.

Now delete from $G$ all those $G_n$ whose corresponding $(W_1^n, W_2^n)$ do not induce $(V_1^1, V_2^1)$ and call the resulting set $G'$. Suppose that $i \geq 2$ is the least integer such that $G_i$ is in $G'$. For each $n \geq i$ for which $G_n \in G'$, the partition $(W_1^n, W_2^n)$ of $V(G_n)$ restricted to $V(G_i)$ induces a partition of $V(G_i)$. Since $V(G_1)$ has only finitely many partitions, there exists a partition of $V(G_i)$ induced by infinitely many $(W_1^n, W_2^n)$. This partition of $V(G_i)$ induces $(V_1^i, V_2^i)$ in $V(H_1)$. Label the partitions induced by this partition of $V(G_i)$ in $V(H_2), V(H_3), ..., V(H_i)$ by $(V_1^2, V_2^2)(V_1^3, V_2^3), ..., (V_1^i, V_2^i)$, respectively. For each $k = 1, 2, ..., i$ we have $H_k[V_1^k] \subseteq \mathcal{P}$ and $H_k[V_2^k] \subseteq \mathcal{Q}$. 
We now repeat the procedure: delete from \( G' \) all those \( G_n \) whose corresponding \((W^n_1, W^n_2)\) do not induce \((V^1_1, V^1_2), (V^2_1, V^2_2), \ldots, (V^n_1, V^n_2)\) and call the resulting set \( G'' \). If \( j \geq i + 1 \) is the least integer such that \( G_j \in G'' \), choose a partition of \( V(G_j) \) that is induced by infinitely many of the \((W^n_1, W^n_2)\) which satisfy \( G_n \in G'' \), etc.

Following this procedure, we obtain for each \( n \geq 1 \) a partition \((V^n_1, V^n_2)\) of \( V(H_n) \) which satisfies \( \rightarrow H_n[V^n_1] \subseteq \mathcal{P} \) and \( \rightarrow H_n[V^n_2] \subseteq \mathcal{Q} \). With \( V_1 = \bigcup_{n \geq 1} V^n_1 \) and \( V_2 = \bigcup_{n \geq 1} V^n_2 \), we have a partition of \( V(H) \). If either \( V_1 \) or \( V_2 \) is empty, move an arbitrary vertex into this set. By the construction of \( V_1 \) and \( V_2 \), \( \rightarrow H[V_1] \subseteq \mathcal{P} \) and \( \rightarrow H[V_2] \subseteq \mathcal{Q} \).

**Corollary 5.** Let \( H \) be a finite core or a countable union of finite cores. If \( \mathcal{P} \) and \( \mathcal{Q} \) are non-trivial properties in \( \mathbb{L}^a \) such that \( \rightarrow H \subseteq \mathcal{P} \mathcal{Q} \), then there exists a partition \((V_1, V_2)\) of \( V(H) \) with \( V_1 \neq \emptyset \) and \( V_2 \neq \emptyset \) such that \( \rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq \mathcal{P} \mathcal{Q} \) and \( \rightarrow H[V_1] \subseteq \mathcal{P} \) and \( \rightarrow H[V_2] \subseteq \mathcal{Q} \).

We can now describe the minimal reducible bounds for the hom-properties in \( \mathbb{L}^a \).

### 4.1. Finite \( H \)

Let \( H \) be a finite core such that \( \rightarrow H \) is irreducible in \( \mathbb{L}^a \) (i.e. \( H \) is indecomposable). Let \( \mathbf{H} \) be the set of all hom-properties \( \rightarrow C_1 + C_2 = (\rightarrow C_1)(\rightarrow C_2) \) formed as follows:

For each partition \((V_1, V_2)\) of \( V(H) \) with \( V_1 \neq \emptyset \), \( V_2 \neq \emptyset \), let \( C_1 = C(H[V_1]) \) and \( C_2 = C(H[V_2]) \).

**Lemma 6.** \( \rightarrow H \subsetneq \rightarrow C_1 + C_2 \) for each \( \rightarrow C_1 + C_2 \in \mathbf{H} \).

**Proof.** This will follow if we can show that there is a homomorphism from \( H \) to \( C_1 + C_2 \). By the definition of \( C_1 \) and \( C_2 \), there exist homomorphisms \( f_1 : V_1 \rightarrow V(C_1) \) and \( f_2 : V_2 \rightarrow V(C_2) \). Define \( f : V(H) \rightarrow V(C_1 + C_2) \) by \( f(x) = f_i(x) \) if \( x \in V_i \), \( i = 1, 2 \).

Since \( H \) is a finite graph, the set \( \mathbf{H} \) is finite and thus minimal elements (under inclusion of properties) exist. These minimal elements of \( \mathbf{H} \) are precisely all the minimal reducible bounds of \( \rightarrow H \), i.e. they form \( \mathbf{B}(\rightarrow H) \).

**Theorem 7.** \( \mathbf{B}(\rightarrow H) = \text{Min}_{\subset} \mathbf{H} \).
**Proof.** We must show that if there are non-trivial properties $P$ and $Q$ in $L^a$ such that $\rightarrow H \subset P \cup Q$, then there exists a $\rightarrow C_1 + C_2 \subseteq H$ such that $\rightarrow H \subset \rightarrow C_1 + C_2 \subseteq P \cup Q$. This follows immediately by Corollary 5: there exists a $(P, Q)$ partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$, $V_2 \neq \emptyset$ such that $\rightarrow H \subseteq \rightarrow (V_1) \rightarrow (V_2) \subseteq \rightarrow P \cup Q$, and so $\rightarrow H \subseteq (\rightarrow C(H[V_1]))(\rightarrow C(H[V_2])) \subseteq \rightarrow P \cup Q$.

All the minimal reducible bounds in $L^a$ for a hom-property $\rightarrow H$, where $H$ is finite, can thus be found by forming the finite set $H$ (by considering all partitions $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, and then forming the hom-properties $(\rightarrow (C(H[V_1]) + C(H[V_2])))$ and then determining which of these reducible properties are minimal under inclusion.

4.2. Infinite $H$

We now consider minimal reducible bounds in $L^a$ for an irreducible $\rightarrow H$, where $H$ is an infinite union of finite cores. By Corollary 5, if a minimal reducible bounds exists for such a $\rightarrow H$, it is of the same form as in the finite case, i.e. it has the form $(\rightarrow H[V_1])(\rightarrow H[V_2])$ for some partition $(V_1, V_2)$ of $V(H)$ with $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$. We can again form the set $H$ for an infinite graph $H$, $H = \{(\rightarrow H[V_1])(\rightarrow H[V_2]) : (V_1, V_2) \text{ is a partition of } V(H) \text{ and } V_1 \neq \emptyset, V_2 \neq \emptyset\}$ and clearly $\rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2])$ for each $(\rightarrow H[V_1])(\rightarrow H[V_2])$ in $H$. However $H$ will now be an infinite set and the existence of minimal elements is no longer trivial. In the following theorem we show that $H$ has minimal elements and that every element of $H$ contains a minimal element. These minimal elements thus form $B(\rightarrow H)$, the set of all minimal reducible bounds for $\rightarrow H$.

**Theorem 8.** Let $H$ be an countable union of finite cores. Then the set $H$ contains minimal elements, and each element of $H$ contains a minimal element of $H$.

**Proof.** We will first use Zorn’s lemma to show that $H = \{(\rightarrow H[V_1])(\rightarrow H[V_2]) : (V_1, V_2) \text{ is a partition of } V(H), V_1 \neq \emptyset, V_2 \neq \emptyset\}$ has minimal elements. This will follow if we can show that every chain in $H$ has a lower bound in $H$.

Suppose to the contrary that $C = \{(\rightarrow H[V_1])(\rightarrow H[V_2]) : \alpha \in A\}$ is an infinite chain in $H$ that does not have a lower bound in $H$. Then given any element of the chain, there exists an infinite chain of elements of $C$ below it.
Suppose \( H = H_1 \cup H_2 \cup \ldots \). For each \( \alpha \in A \), the partition \((V_1^\alpha, V_2^\alpha)\) of \( V(H) \) induces a partition of \( V(H_1) \). Since \( V(H_1) \) has only finitely many partitions, there exists a partition \((V_{1,1}, V_{2,1})\) of \( V(H_1) \) that is induced infinitely many times and that satisfies: given any \( \alpha \in A \), there exists \( \alpha' \in A \) such that \((\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha]) \subset (\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_2^{\alpha'}])\) and \((V_1^\alpha, V_2^\alpha)\) induces \((V_{1,1}, V_{2,1})\) in \( V(H_1) \). If for each induced partition of \( V(H_1) \) occurring infinitely many times, there exists an \( \alpha \) such that every \( \alpha' \in A \) satisfying \((\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha]) \subset (\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_2^{\alpha'}])\) induces some different partition of \( V(H_1) \), then, since these \( \alpha \) are finite, we can choose the one among them corresponding to the least element of \( \mathcal{C} \). This element of \( \mathcal{C} \) contains only finitely many other elements of \( \mathcal{C} \) below it, contradicting our hypothesis. We have \( H_1[V_{1,1}] \rightarrow H[V_1^{\alpha'}] \) and \( H_2[V_{1,2}] \rightarrow H[V_2^{\alpha'\alpha'}] \).

Now form \( A' \) from \( A \) by deleting all those \( \alpha \) for which \((V_1^\alpha, V_2^\alpha)\) does not induce \((V_{1,1}, V_{2,1})\). For any \( \alpha \in A \), there exists \( \alpha' \in A' \) such that \((\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha])\) and \( H_1[V_{1,1}] \rightarrow H[V_1^{\alpha'}] \) and \( H_1[V_{2,1}] \rightarrow H[V_1^\alpha] \). We now have a new infinite chain, \( \mathcal{C}' = \{(\rightarrow H[V_1^{\alpha'}]) : \alpha \in A' \} \), and we repeat the procedure using \( H_2 \) and \( \mathcal{C}' \), to form \( \mathcal{C}'' \), etc. For each \( H_i \) we obtain a partition \((V_{1,i}, V_{2,i})\) of \( V(H_i) \) and after completing the procedure \( i \) times, we have a chain of \((\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha])\) such that for all \( \alpha \) in the new index set, the partition \((V_1^\alpha, V_2^\alpha)\) of \( V(H) \) induces the partition \((V_{1,j}, V_{2,j})\) of \( V(H_j) \) for all \( j = 1, 2, \ldots, i \). Also, for any \( \alpha \in A \), there exists \( \alpha' \) in the new index set such that \((\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha])\) and \( H_j[V_{1,j}] \rightarrow H[V_1^{\alpha'}] \) and \( H_j[V_{2,j}] \rightarrow H[V_2^\alpha] \) for all \( j = 1, 2, \ldots, i \).

Now let \( V_1 = \bigcup_{i \geq 1} V_{1,i} \) and let \( V_2 = \bigcup_{i \geq 1} V_{2,i} \). There are now two possibilities: either both \( V_1 \) and \( V_2 \) are non-empty, or one of them (say \( V_2 \)) is empty while the other (\( V_1 \)) equals \( V(H) \).

Suppose first that both \( V_1 \) and \( V_2 \) are non-empty. Then \((\rightarrow H[V_1])(\rightarrow H[V_2])\) is itself in \( \mathbf{H} \). We will show that \((\rightarrow H[V_1])(\rightarrow H[V_2])\) is a lower bound for the chain \( \mathcal{C} \).

Let \( \alpha \in A \) and let \( G \in (\rightarrow H[V_1])(\rightarrow H[V_2]) \). Then there exists a partition \((A, B)\) of \( V(G) \) such that \( G[A] \rightarrow H[V_1] \) and \( G[B] \rightarrow H[V_2] \). Since both \( G[A] \) and \( G[B] \) are finite, there exists an integer \( n \) such that \( G[A] \rightarrow \cup \{H_i[V_{1,i}] : i = 1, 2, \ldots, n\} \) and \( G[B] \rightarrow \cup \{H_i[V_{2,i}] : i = 1, 2, \ldots, n\} \). Now by the remark at the end of the previous paragraph, after \( n \) steps of the procedure, there exists an \( \alpha' \) in the modified index set of the chain with \((\rightarrow H[V_1^{\alpha'}]) \subset (\rightarrow H[V_2^{\alpha'}]) \subset (\rightarrow H[V_1^\alpha]) \subset (\rightarrow H[V_2^\alpha])\) and such that \( H_i[V_{1,i}] \rightarrow H[V_1^{\alpha'}] \) and \( H_i[V_{2,i}] \rightarrow H[V_2^\alpha] \) for \( i = 1, 2, \ldots, n \). Hence \( G[A] \rightarrow H[V_1^{\alpha'}] \).
and $G[B] \in \rightarrow H[V^2_2]$, so $G \in \rightarrow H[V^1_2](\rightarrow H[V^2_2]) \subset \rightarrow H[V^2_2]$, i.e. $\rightarrow H[V^1_2](\rightarrow H[V^2_2]) \subseteq \rightarrow H[V^2_2]$.

Now suppose that $V_2$ is empty and that $V_1 = V(H)$. We claim that in this case, any element of $H$ of the form $(\rightarrow H[V^1_1])(\rightarrow H[V^2_2])$ where $W_2$ is independent, is a lower bound for the chain $C$. To prove this, fix such an element of $H$. Suppose it is $(\rightarrow H[V^1_1])(\rightarrow H[W_2])$, with $W_2$ independent. Let $\alpha \in A$ and let $G \in (\rightarrow H[V^1_1])(\rightarrow H[W_2])$. We must show that $G \in (\rightarrow H[V^1_1])(\rightarrow H[W^2])$: Since $G$ is finite, there exists an integer $n$ such that $G \in (\rightarrow (H_1 \cup H_2 \cup \ldots \cup H_n)[W_1])((H_1 \cup H_2 \cup \ldots \cup H_n)[W_2])$. Now there exists an $\alpha' \in A$ such that $(\rightarrow H[V^1_1])(\rightarrow H[V^2_2]) \subset (\rightarrow H[V^1_1])(\rightarrow H[V^2_2])$ and $(V^1_1, V^2_2)$ induces $(V^1_i, V^2_j) = (V(H_i), \emptyset)$ for each $i = 1, 2, \ldots, n$. Then $(H_1 \cup H_2 \cup \ldots \cup H_n)[W_1] \rightarrow H[V^1_1]$ (the inclusion map) and $(H_1 \cup H_2 \cup \ldots \cup H_n)[W_2] \rightarrow H[V^2_2]$ (since $W_2$ is independent and $V^2_2$ is non-empty). Hence $G \in (\rightarrow H[V^1_1])(\rightarrow H[V^2_2]) \subset (\rightarrow H[V^1_1])(\rightarrow H[V^2_2])$.

We can conclude by Zorn’s lemma that the set $H$ has minimal elements. By fixing an element of $H$ and considering only chains of elements of $H$ each of which is contained in that fixed element, the same argument as above shows that each element of $H$ contains at least one of these minimal elements of $H$. Hence, as in the case where $H$ is finite, the minimal elements of $H$ form $B(\rightarrow H)$ when $H$ is an infinite union of finite graphs.

5. Some Applications

In the following applications, we allow the graph $H$ to be either finite or a countable union of finite graphs and we show the existence of minimal reducible bounds of certain types in $\mathbb{L}^\omega$ for $\rightarrow H$. In this section we assume throughout that $\rightarrow H$ is irreducible, while if $H$ is finite it is assumed to be a core.

Proposition 9. If $H$ is a graph with chromatic number 3, then $O^3$ is the unique minimal reducible bound for $\rightarrow H$.

Proof. Since $\chi(H) = 3$, there exists a partition $(V_1, V_2)$ of $V(H)$ such that $H[V_1]$ is an independent set of vertices and $H[V_2]$ has chromatic number 2, i.e. $\rightarrow C(H[V_1]) \rightarrow C(H[V_2]) = \rightarrow K_1 + K_2 = \rightarrow K_3 = O^3$.

If $\rightarrow H \subset \rightarrow C_1 \rightarrow C_2$ for any other $\rightarrow C_1 \rightarrow C_2 \in H$, then either $C_1$ or $C_2$ must contain an edge (since $\chi(C_1) + \chi(C_2) \geq 3$) and hence $K_1 + K_2 \in \rightarrow C_1 \rightarrow C_2$, i.e. $\rightarrow H \subset \rightarrow K_1 + K_2 = O^3 \subset \rightarrow C_1 \rightarrow C_2$. ■
Proposition 10. If $H$ is a graph with chromatic number 4, then all minimal reducible bounds of $\rightarrow H$ are of the form $O(\rightarrow X)$ for some graph $X \subset H$.

Proof. Since $\chi(H) = 4$, there exists a partition $(V_1, V_2)$ of $V(H)$ such that $\chi(H[V_1]) = 2$ and $\chi(H[V_2]) = 2$, i.e. $\rightarrow C(H[V_1]) \rightarrow C(H[V_2]) = \rightarrow K_2 + K_2 = \rightarrow K_1 + K_3 = O(\rightarrow K_3)$.

Consider all partitions $(V_1, V_2)$ of $V(H)$. If $H[V_1]$ or $H[V_2]$ is independent, we get a reducible bound for $\rightarrow H$ of the form $O(\rightarrow H[V_1])$ or $O(\rightarrow H[V_2])$. If neither $H[V_1]$ nor $H[V_2]$ is independent, then $K_2 \rightarrow H[V_1]$ and $K_2 \rightarrow H[V_2]$, so $\rightarrow K_2 + K_2 = O(\rightarrow K_3) \subseteq \rightarrow H[V_1] \rightarrow H[V_2]$.

We can now conclude that all the minimal elements of $H$ are of the form $O(\rightarrow X)$ for some graph $X \subset H$. \hfill \blacksquare

Proposition 11. If $H$ is a graph with chromatic number 5, then $\rightarrow H$ has a minimal reducible bound of the form $O(\rightarrow X)$ for some graph $X \subset H$.

Proof. Since $\chi(H) = 5$, there exists a bound of the form $O(\rightarrow X) = (\rightarrow K_1)(\rightarrow X)$ for $\rightarrow H$ with $X \subset H$ and $\chi(X) = 4$. Suppose that $\rightarrow X_1 \rightarrow X_2$ is any other element of $H$ satisfying $\rightarrow H \subseteq \rightarrow X_1 \rightarrow X_2 \subseteq O(\rightarrow X)$. Since $\chi(H) = \chi(K_1) + \chi(X) = 5$, we must have $\chi(X_1) + \chi(X_2) = 5$ and this is only possible if one of $X_1$ or $X_2$ has chromatic number at most 2.

Say $\chi(X_1) \leq 2$. Then we can assume that $X_1 = K_1$ or $X_1 = K_2$. In the first case, $\rightarrow X_1 \rightarrow X_2 = \rightarrow K_1 \rightarrow X_2 = O(\rightarrow X_2)$, while in the second, $\rightarrow X_1 \rightarrow X_2 = (\rightarrow K_1)(\rightarrow K_1 \rightarrow X_2)$. By Corollary 5, there exists a bound for $\rightarrow H$ of the form $O(\rightarrow Y)$ with $Y \subset H$ satisfying $\rightarrow H \subseteq O(\rightarrow Y) \subseteq (\rightarrow K_1)(\rightarrow K_1 \rightarrow X_2)$. In either case there exists a bound for $\rightarrow H$ of the form $O(\rightarrow Y)$ with $Y \subset H$ satisfying $\rightarrow H \subseteq O(\rightarrow Y) \subseteq \rightarrow X_1 \rightarrow X_2$, so we conclude that $H$ has a minimal element of the form $O(\rightarrow Y)$ for some $Y \subset H$. \hfill \blacksquare

Proposition 12. If $H$ is a graph with chromatic number either infinite or finite and greater than or equal to 6, and if $K_4$ is not a subgraph of $H$, then $\rightarrow H$ has a minimal reducible bound of the form $O(\rightarrow X)$ for some $X \subset H$.

Proof. There exists a bound for $\rightarrow H$ of the form $O(\rightarrow X)$ where $X \subset H$, and $\chi(X) \geq 5$, which is minimal of this type.

Suppose $\rightarrow H \subset (\rightarrow X_1)(\rightarrow X_2) \subseteq O(\rightarrow X)$ where $(\rightarrow X_1)(\rightarrow X_2) \in H$ is not of the form $O(\rightarrow Y)$ for any graph $Y$. If the chromatic number of either $X_1$ or $X_2$ is one, say $\chi(X_1) = 1$, then $(\rightarrow X_1)(\rightarrow X_2) = O(\rightarrow X_2)$, contradicting our assumption on the form of $(\rightarrow X_1)(\rightarrow X_2)$. If one of $X_1$ or $X_2$ has chromatic number 2, say $\chi(X_1) = 2$, then $(\rightarrow X_1)(\rightarrow X_2) = O$
(\(\mathcal{O}(\rightarrow X_2)\)) and by Corollary 5 there exists an element of \(H\) of the form \(\mathcal{O}(\rightarrow Y)\) between \(\rightarrow H\) and \(\mathcal{O}(\rightarrow X_2)\), contradicting the minimality of \(\mathcal{O}(\rightarrow X)\).

Thus \(\chi(X_1) \geq 3\) and \(\chi(X_2) \geq 3\) so that both \(X_1\) and \(X_2\) contain an odd cycle, say \(S_1\) and \(S_2\) respectively. But then \(S_1 + S_2 \in (\rightarrow X_1)(\rightarrow X_2) \subseteq \mathcal{O}(\rightarrow X)\), so \(V(S_1 + S_2)\) has an \((\mathcal{O}, (\rightarrow X))\)-partition, say \((V_1, V_2)\). Thus \((S_1 + S_2)[V_1]\) is an independent subgraph of either \(S_1\) or \(S_2\), and (since \(\chi(S_1) = 3\) and \(\chi(S_2) = 3\), \((S_1 + S_2)[V_2]\) must contain \(K_4\) as a subgraph, a contradiction since \((S_1 + S_2)[V_2] \in \rightarrow X\), and any \(K_4\) in \((S_1 + S_2)[V_2]\) would force a \(K_4\) in \(X \subseteq H\).

We conclude that \(H\) has a minimal element of the form \(\mathcal{O}(\rightarrow Y)\) for some \(Y \subseteq H\).

\[\textbf{Proposition 13.} \text{ If } H \text{ is a graph with finite chromatic number satisfying } \chi(H) = n \geq 6, \text{ and } K_{n-1} \subseteq H, \text{ then } \rightarrow H \text{ has a minimal reducible bound of the form } \mathcal{O}(\rightarrow X) \text{ for some } X \subseteq H.\]

\[\textbf{Proof.} \text{ There exists an element } \mathcal{O}(\rightarrow X) \in H \text{ with } \chi(X) = n-1. \text{ Suppose now that } \rightarrow H \subseteq (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq \mathcal{O}(\rightarrow X), \text{ with } (\rightarrow H[V_1])(\rightarrow H[V_2]) \subseteq H. \text{ Then } \chi(H[V_1]) + \chi(H[V_2]) = n. \text{ Since } K_{n-1} \subseteq H, \text{ there exists } K_i \subseteq H[V_1] \text{ and } K_j \subseteq H[V_2] \text{ with } i + j = n - 1.\]

If \(i \geq \chi(H[V_1])\), then \(C(H[V_1]) = K_i\), so \((\rightarrow H[V_1])(\rightarrow H[V_2]) = (\rightarrow K_1)(\rightarrow K_{i-1} \rightarrow H[V_2])\) and by Corollary 5, there exists a bound for \(\rightarrow H\) of the form \(\mathcal{O}(\rightarrow Y)\) for some \(Y \subseteq H\), contained in \((\rightarrow H[V_1])(\rightarrow H[V_2])\).

However if \(i < \chi(H[V_1])\), then \(j \geq \chi(H[V_2])\) and \(C(H[V_2]) = K_j\), and once again \((\rightarrow H[V_1])(\rightarrow H[V_2])\) contains a bound for \(\rightarrow H\) of the form \(\mathcal{O}(\rightarrow Y)\) for some \(Y \subseteq H\).

We conclude that \(H\) has a minimal element of the form \(\mathcal{O}(\rightarrow Y)\) for some \(Y \subseteq H\).

\[\textbf{Proposition 14.} \text{ If } H \text{ is a triangle-free graph with finite chromatic number satisfying } \chi(H) \geq 6, \text{ then } \rightarrow H \text{ has a minimal reducible bound not of the form } \mathcal{O}\mathcal{P} \text{ for any } \mathcal{P} \in \mathbb{L}^a.\]

\[\textbf{Proof.} \text{ Since } \chi(H) \geq 6, \text{ there exists } (\rightarrow X_1)(\rightarrow X_2) \in H \text{ such that } \chi(X_1) \geq 3, \chi(X_2) \geq 3, \chi(X_1) + \chi(X_2) = \chi(H). \text{ Suppose } (\rightarrow X_1)(\rightarrow X_2) = \mathcal{O}(\rightarrow X) \text{ for some } X \subseteq H. X_1 \text{and } X_2 \text{ each contain an odd cycle, say } S_1 \text{ and } S_2 \text{ respectively. We then have that } S_1 + S_2 \in \mathcal{O}(\rightarrow X) \text{ so } V(S_1 + S_2) \text{ has an } (\mathcal{O}, (\rightarrow X))\text{-partition, say } (V_1, V_2). \text{ Since } (S_1 + S_2)[V_1]\text{ is an independent}\]

subset of either \(S_1\) or \(S_2\), \((S_1 + S_2)[V_2]\) must contain a triangle, forcing \(H\) to
contain a triangle, contradicting our hypothesis. So \((\rightarrow X_1)(\rightarrow X_2)\) is not of the form \(O \mathcal{P}\) for any \(\mathcal{P} \in \mathbb{L}^a\).

Suppose now that \(\rightarrow H \subset O(\rightarrow X) \subset (\rightarrow X_1)(\rightarrow X_2)\) for some \(X \subset H\). Since \(\chi(H) = \chi(X_1) + \chi(X_2)\), it must be true that \(\chi(X) = \chi(H) - 1\). Let \(G\) be any finite subgraph of \(X\) with \(\chi(G) = \chi(X)\). The graph \(G + \{v\}\) is in \(O(\rightarrow X)\) and therefore in \((\rightarrow X_1)(\rightarrow X_2)\), and so \(V(G + \{v\})\) has a \((X_1, X_2)\)-partition \((V_1, V_2)\). Suppose that \(v \in V_1\). If \(\{w \in V(G) : w \in V_1\}\) is not an independent set of vertices, then \((G + v)[V_1]\) contains a triangle, and so \(X_1\) contains a triangle, which is not possible. If \(\{w \in V(G) : w \in V_1\}\) is an independent set of vertices, then \(\chi((G + v)[V_2]) \geq \chi(H) - 2\). But \((G + v)[V_2] \in X_2\) and \(\chi(X_2) \leq \chi(H) - 3\), again a contradiction. Hence no bound of the form \(O \mathcal{P}\) with \(\mathcal{P} \in \mathbb{L}^a\) can occur between \(\rightarrow H\) and \((\rightarrow X_1)(\rightarrow X_2)\).

We conclude that \(H\) has a minimal element not of the form \(O \mathcal{Y}\) for any \(\mathcal{Y} \subset H\). ●

The previous result is not true if we allow \(\chi(H)\) to be infinite since the set of all triangle-free graphs, \(I_1\), has the unique minimal reducible bound \(O I_1\) (see [1], [6]). \(I_1\) is the hom-property \(\rightarrow \cup \{R : R\) is a triangle free core\}, with infinite chromatic number.

Corollaries 12 and 14 show that if \(H\) has a finite chromatic number greater than or equal to 6, and \(H\) is triangle-free, then \(\rightarrow H\) has a minimal reducible bound of the form \(O \mathcal{P}\) for some \(\mathcal{P} \in \mathbb{L}^a\) and a minimal reducible bound not of this form.

6. MINIMAL REDUCIBLE BOUNDS FOR \(\rightarrow H\) IN \(\mathbb{L}\)

We now describe the minimal reducible bounds of a hom-property \(\rightarrow H\) in the lattice of hereditary properties, \(\mathbb{L}\). Again, we will describe the case for a finite \(H\) first, and then draw conclusions about an infinite \(H\). The following lemma and its corollary are useful in both the finite and infinite cases.

**Lemma 15.** Let \(H\) be a finite graph or a countable union of finite graphs. If \(\rightarrow H \subseteq \mathcal{P} \mathcal{Q}\), where \(\mathcal{P}\) and \(\mathcal{Q}\) are non-trivial properties in \(\mathbb{L}\) such that \(O \nsubseteq \mathcal{Q}\), then \(\rightarrow H \subseteq \mathcal{P}\).

**Proof.** Suppose first that \(H\) is finite, and suppose that the cardinality of the largest edgeless graph in \(\mathcal{Q}\) is \(k\). For any \(m > k\), \(H^k(m) \in \mathcal{P} \mathcal{Q}\) and by the restriction on \(\mathcal{Q}\), \(H^k(m - k)\) must be in \(\mathcal{P}\). This is true for any \(m > k\) so that \(H^k(r) \in \mathcal{P}\) for all \(r \geq 1\), i.e. \(\rightarrow H \subseteq \mathcal{P}\).
If $H$ is infinite, then since $\rightarrow H' \subseteq \mathcal{P}Q$ for any finite subgraph $H'$ of $H$, by the finite case we can conclude that $\rightarrow H' \subseteq \mathcal{P}$ for every finite subgraph $H'$ of $H$. Since any graph in $\rightarrow H$ is contained in some $\rightarrow H'$ where $H'$ is a finite subgraph of $H$, we can conclude that $\rightarrow H \subseteq \mathcal{P}$.

**Corollary 16.** Let $H$ be a finite graph or a countable union of finite graphs. If $\rightarrow H \subseteq \mathcal{P}Q$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$ such that $O \not\subseteq Q$, then $\rightarrow H \subseteq (\rightarrow H)((\{K_1\}) \subseteq \mathcal{P}Q$.

**Proof.** The proof is immediate as $\rightarrow H \subseteq \mathcal{P}$ and, since $\mathcal{Q}$ is non-trivial, $K_1 \in \mathcal{Q}$.

We now describe the minimal reducible bounds for hom-properties in $\mathbb{L}$.

### 6.1. Finite $H$

**Theorem 17.** If $H$ is a finite indecomposable core then the minimal reducible bounds for $\rightarrow H$ in $\mathbb{L}$ are the minimal elements of $H$ as well as the property $(\rightarrow H)((\{K_1\})$.

**Proof.** By Lemma 4 and Corollary 16 we know that if $\rightarrow H \subseteq \mathcal{P}Q$, where $\mathcal{P}$ and $\mathcal{Q}$ are non-trivial properties in $\mathbb{L}$, then if $O \not\subseteq \mathcal{P}$ and $O \not\subseteq \mathcal{Q}$, we have a minimal element of $H$ between $\rightarrow H$ and $\mathcal{P}Q$, while if $O \not\subseteq \mathcal{Q}$, then $(\rightarrow H)((\{K_1\})$ lies between $\rightarrow H$ and $\mathcal{P}Q$. Note that the case $O \not\subseteq \mathcal{P}$ and $O \not\subseteq \mathcal{Q}$ cannot occur since by Lemma 15, if $O \not\subseteq \mathcal{Q}$, then $\rightarrow H \subseteq \mathcal{P}$, and since $H$ is assumed to have at least one vertex, all multiplications of this vertex must be in $\mathcal{P}$ i.e. $O \subseteq \mathcal{P}$.

To complete the proof of the theorem, we must show that $(\rightarrow H)((\{K_1\})$ is not contained in any minimal element of $H$, and that no minimal element of $H$ is contained in $(\rightarrow H)((\{K_1\})$.

First suppose to the contrary that $\rightarrow H[V_1] + H[V_2]$ is a minimal element of $H$ satisfying $\rightarrow H[V_1] + H[V_2] \subseteq (\rightarrow H)((\{K_1\})$. By Lemma 15 we then have $\rightarrow H[V_1] + H[V_2] \subseteq \rightarrow H$, and so $H[V_1] + H[V_2] \rightarrow H$. If this homomorphism is a surjection, then $H$ is decomposable, a contradiction, while if this homomorphism is not a surjection, then we can use it to map $H$ into a proper subgraph of itself, a contradiction to the fact that $H$ is a core.

Now suppose that $\rightarrow (H[V_1] + H[V_2])$ is a minimal element of $H$ and that $(\rightarrow H)((\{K_1\}) \subseteq (H[V_1] + H[V_2])$. Now $H + K_1 \in (\rightarrow H)((\{K_1\}) \subseteq (H[V_1] + H[V_2])$, so we have the inclusions $\rightarrow H \subseteq (H + K_1) = (\rightarrow H)((O) \subseteq (H[V_1] + H[V_2])$. By Lemma 4 there exists an element
→ (H|W₁ + H|W₂) in H satisfying → H ⊆ (H|W₁ + H|W₂) ⊆ (→ H)
(Ø) ⊆ (H|V₁ + H|V₂), and → H|W₁ ⊆ H and → H|W₂ = Ø. By the minimality of → (H|V₁ + H|V₂) in H, the two elements of H must be equal, and so we have (→ H)(Ø) = (H|V₁ + H|V₂) i.e. (→ H)(Ø) = → H|W₁ → H|W₂. By the unique factorisation theorem [3], and the fact that → H|W₂ = Ø, we can conclude that → H =→ H|W₁ and Ø =→ H|W₂. But then we have a homomorphism from H to H|W₁, a proper subgraph of H, contradicting the fact that H is a core.

6.2. Infinite H

Theorem 18. If H is an infinite union of finite graphs, then the minimal elements of the set H ∪{(→ H)(K₁)} are the minimal reducible bounds for → H in L.

This result immediately follows from Lemma 4 and Corollary 16. The sharper result from the finite case is no longer true since when H is infinite, it may be possible that (→ H)(K₁) is properly contained in a minimal element of H e.g. T₁ has the unique minimal reducible bound in Lα of T₁Ø, the unique minimal element of H. In L however, we have T₁ ⊆ T₁{K₁} ⊆ T₁Ø, so that T₁ has unique minimal reducible bound T₁{K₁}.

It is not true that (→ H)(K₁) is contained in every minimal element of H, since if (→ H)(K₁) ⊆ (→ H|V₁)(→ H|V₂) where (→ H|V₁)(→ H|V₂) is minimal in H, then we have → H ⊆ (→ H)(Ø) ⊆ (→ H|V₁)(→ H|V₂). (The second inclusion follows since any graph G in (→ H)(Ø) is in → (H' + K₁) for some finite subgraph H' of H, and since H' + K₁ ⊆ H|V₁ → H|V₂, we have that → (H' + K₁) ⊆ H|V₁ → H|V₂.) By Lemma 4 there should be another element of H between → H and (→ H)(Ø). By the minimality of → (H|V₁ + H|V₂), we now have that (→ H)(Ø) = (→ H|V₁)(→ H|V₂). However (Corollary 14) if H is infinite and triangle-free with finite chromatic number at least six, H contains at least one minimal element which does not contain the factor Ø.

References


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