A NOTE ON STRONG AND CO-STRONG PERFECTNESS OF THE X-JOIN OF GRAPHS

ALINA SZELECKA

Institute of Mathematics, Technical University of Zielona Góra
Podgórna 50, 65–246 Zielona Góra, Poland
e-mail: A.Szelecka@im.pz.zgora.pl

AND

ANDRZEJ WŁOCH

Department of Mathematics, Technical University of Rzeszów
W. Pola 2, 35–359 Rzeszów, Poland
e-mail: awloch@ewa.prz.rzeszow.pl

Abstract

Strongly perfect graphs were introduced by C. Berge and P. Duchet in [1]. In [4], [3] the following was studied: the problem of strong perfection for the Cartesian product, the tensor product, the symmetrical difference of \( n, n \geq 2 \), graphs and for the generalized Cartesian product of graphs. Co-strong perfection was first studied by G. Ravindra and D. Basavayya [5]. In this paper we discuss strong perfection and co-strong perfection for the generalized composition (the lexicographic product) of graphs named as the \( X \)-join of graphs.

Keywords: strongly perfect graphs, co-strongly perfect graphs, the \( X \)-join of graphs.

1991 Mathematics Subject Classification: 05C75, 05C60.

1. Introduction

Let \( G \) be a finite undirected connected simple graph. By \( V(G) \) and \( E(G) \) we denote its vertex set and edge set, respectively. The notation \( H = < V_0 >_G, V_0 \subseteq V(G) \) means that \( H \) is the subgraph of \( G \) induced by \( V_0 \). A subset \( S \subseteq V(G) \) is said to be stable in \( G \) if no two distinct vertices of \( S \) are adjacent in \( G \). A subset \( Q \subseteq V(G) \) is a clique of \( G \) if \( < Q >_G \) is a complete subgraph of \( G \). If the stable set \( S \) meets every maximal (with respect to the set inclusion) clique \( Q \), then we will call it a stable
transversal of $G$. A graph $G$ is called strongly perfect ([1]) if its every induced subgraph (including $G$ itself) has a stable transversal. We call $G$ co-strongly perfect ([5]) if $G$ and the complementary graph $\overline{G}$ to $G$ are strongly perfect. Let $G_1, \ldots, G_n$, $n \geq 2$, be graphs of the same order $m \geq 2$ with the vertex sets $V(G_i) = V = \{y_1, \ldots, y_m\}$ for $i = 1, \ldots, n$ and $X$ be a graph such that $V(X) = \{x_1, \ldots, x_n\}$. The $X$-join ([2]) of the sequence of graphs $G_1, \ldots, G_n$ and the graph $X$ is the graph $X[G_1, \ldots, G_n]$ with the vertex set $V(X) \times V$ and the edge set $\{(x_j, y_p), (x_k, y_q) : j = k$ and $[y_p, y_q] \in E(G_i)$ or $[x_j, x_k] \in E(X)\}$.

Observe that if $G_1 = G_2 = \ldots = G_n = Y$, then we obtain the composition (the lexicographic product) of graphs $Y$ and $X$ denoted by $X[Y]$.

Let $V_0 \subseteq V(X) \times V$. By the projection $Pr_X V_0$ of the subset $V_0$ on the graph $X$ we mean the set $Pr_X V_0 = \{x \in V(X) :$ there exists $y \in V(G_i)$, $1 \leq i \leq n$, that $(x, y) \in V_0\}$.

2. Results

Put $G = X[G_1, \ldots, G_n]$, for convenience. Let $H$ be a connected induced subgraph of $G$ such that it is not isomorphic to any induced subgraph $H'$ of the graph $X$ or $G_i$, for $i = 1, \ldots, n$.

Let $Pr_X V(H) = \{x_{i1}, \ldots, x_{ip}\}$, $2 \leq p \leq n$.

We partition the set $V(H)$ on $p$-disjoint sets $V_{ij}(H)$ such that $Pr_X V_{ij}(H) = \{x_{ij}\}$ for $j = 1, \ldots, p$. For an arbitrary subset $R \subseteq V(H)$, in a natural way we can write $R = \bigcup_{j=1}^{t} R \cap V_{ij}(H)$, where $1 \leq t \leq p$.

For $G$ and $H$ given above it follows immediately.

Lemma 1. If $Q$ is a maximal clique of $H$, then $Pr_X Q$ is a maximal clique of $< Pr_X V(H) >$.

Lemma 2. A subset $Q \subseteq V(H)$ is a maximal clique of $H$ if and only if

1. $Q \cap V_{ij}(H)$ is a maximal clique of $< V_{ij}(H) >$ for $j = 1, \ldots, p$ or $Q \cap V_{ij}(H) = \emptyset$ and

2. $Pr_X Q$ is a maximal clique of $< Pr_X V(H) >$.

Proof. I. Assume that $Q$ is a maximal clique of $H$. We can write $Q = \bigcup_{j=1}^{t} Q \cap V_{ij}(H)$ where $1 \leq t \leq p$ with $Q \cap V_{ij}(H) \neq \emptyset$ for each $j = 1, \ldots, t$. Moreover, each of the sets $Q \cap V_{ij}(H)$ must be a clique of $< V_{ij}(H) >$. Suppose there exists $j$, $1 \leq j \leq t$ such that $Q \cap V_{ij}(H)$, is not maximal. In consequence, there exists a vertex $(x_{ij}, y_r) \in V_{ij}(H) \setminus Q \cap V_{ij}(H), 1 \leq j \leq t$
(of course \((x_{ij}, y_r) \notin Q\)) which is adjacent to each vertex from \(Q \cap V_{ij}(H)\). Moreover, by the definition of \(G\) and from the fact that \(Q \cap V_{ij}(H) \subset Q\) it follows that \((x_{ij}, y_r)\) must be adjacent to each vertex from \(Q \setminus Q \cap V_{ij}(H)\). In consequence, \((x_{ij}, y_r)\) is adjacent to all vertices from \(Q\) and \((x_{ij}, y_r) \notin Q\), a contradiction with the assumption that \(Q\) is a maximal clique of \(H\). This shows that the condition in (1) holds.

Condition (2) follows from Lemma 1.

II. Suppose that conditions (1) and (2) hold. We can write \(Q = \bigcup_{j=1}^s Q \cap V_{ij}(H), 1 \leq t \leq p\). Note that \(|Q| > 1\), by the assumption about \(H\). Firstly, we shall show that \(Q\) is a clique of \(H\). Let \((x_{ij}, y_r), (x_{ik}, y_s)\) be two distinct vertices from \(Q\). If \(j = k\), then they belong to \(Q \cap V_{ij}(H)\) and are adjacent by (1). If \(j \neq k\), then \(x_{ij}, x_{ik} \in Pr_X Q\) and by (2) they are adjacent in \(X\). Thus, by the definition of \(G\) the vertices \((x_{ij}, y_r), (x_{ik}, y_s)\) are adjacent in \(G\). This proves that \(Q\) is a clique of \(H\).

Assume that \(Q\) is not maximal. This means that there exists \((x_{il}, y_r) \notin Q\) but it is adjacent to each vertex from \(Q\). Moreover, by the definition of \(G\), the vertex \(x_{il}\) is adjacent to all vertices from \(Pr_X Q\). This implies that \(x_{il} \in Pr_X Q\) by (2). In consequence, it must be that \((x_{il}, y_r) \in V_{il}(H) \setminus Q \cap V_{il}(H)\) (evidently \((x_{il}, y_r) \notin Q \cap V_{il}(H)\)). Since \(Q \cap V_{il}(H) \subset Q\) and \((x_{il}, y_r)\) is adjacent to each vertex from \(Q\), then it is adjacent to each vertex from \(Q \cap V_{il}(H)\). Hence by (1) it must be that \((x_{il}, y_r) \in Q \cap V_{il}(H)\), a contradiction. Hence, \(Q\) is a maximal clique of \(H\) and this completes the proof of the lemma.

Using the same method as in the proof of Lemma 2 we prove.

**Lemma 3.** A subset \(S \subset V(H)\) is a maximal stable set of \(H\) if and only if

1. \(S \cap V_{ij}(H)\) is a maximal stable set of \(< V_{ij}(H) >\) for \(j = 1, \ldots, s\) or \(S \cap V_{ij}(H) = \emptyset\) and

2. \(Pr_X S\) is a maximal stable set of \(< Pr_X V(H) >\).

Lemma 4 follows directly from the definition of the graph \(X[G_1, \ldots, G_n]\).

**Lemma 4.** \(X[G_1, \ldots, G_n] = X[G_1, \ldots, G_n]\).

**Theorem 1.** \(X[G_1, \ldots, G_n]\) is strongly perfect if and only if the graphs \(X, G_1, \ldots, G_n\) are strongly perfect.

**Proof.** I. Let \(X[G_1, \ldots, G_n]\) be strongly perfect. Then \(X, G_1, \ldots, G_n\) are strongly perfect since they are isomorphic to some induced subgraphs of \(G\).
II. Suppose that the graphs $X, G_1, \ldots, G_n$ are strongly perfect. We shall show that $G$ is strongly perfect. Let $H$ be a connected induced subgraph of $G$. We shall prove that $H$ has a stable transversal.

If $H$ is an induced subgraph of $X$ or $G_i, 1 \leq i \leq n$, then $H$ has a stable transversal, by the assumption that $X, G_1, \ldots, G_n$ are strongly perfect.

Let $H$ be not induced subgraph of $X, G_i, i = 1, \ldots, n$. Assume that $H$ does not have a stable transversal, i.e., for every maximal stable set $S \subseteq V(H)$ there exists a maximal clique $Q \subseteq V(H)$ such that $S \cap Q = \emptyset$. Moreover, by the definition of $G$ and Lemmas 2, 3 we have that for every maximal stable set $Pr_X S$ of $< Pr_X V(H) >$ there exists a maximal clique $Pr_X Q$ of $< Pr_X V(H) >$ such that $Pr_X S \cap Pr_X Q = \emptyset$. This is a contradiction, since $< Pr_X V(H) >$ has a stable transversal.

This proves that $X[G_1, \ldots, G_n]$ is strongly perfect and the proof is complete. ■

For $G_1 = G_2 = \ldots = G_n = Y$ we obtain

**Corollary 1.** The composition $X[Y]$ of graphs $X$ and $Y$ is strongly perfect if and only if both $X$ and $Y$ are strongly perfect.

Using Lemma 4 and Theorem 1 we obtain

**Corollary 2.** $X[G_1, \ldots, G_n]$ is strongly perfect if and only if the graphs $X, G_1, \ldots, G_n$ are strongly perfect.

In consequence, it follows immediately

**Theorem 2.** $X[G_1, \ldots, G_n]$ is co-strongly perfect if and only if the graphs $X, G_1, \ldots, G_n$ are co-strongly perfect.

**References**


Received 26 April 1996
Revised 8 October 1996