EQUIVALENT CLASSES FOR $K_3$-GLUINGS OF WHEELS

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Abstract

In this paper, the chromaticity of $K_3$-gluings of two wheels is studied. For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq \lfloor n/2 \rfloor$ all $K_3$-gluings of wheels $W_{q+2}$ and $W_{n-q+2}$ create an $\chi$-equivalent class.

Keywords: chromatically equivalent graphs, chromatic polynomial, chromatically unique graphs, wheels.

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INTRODUCTION

The graphs which we consider here are finite, undirected, simple and loopless. Let $G$ be a graph, $V(G)$ be its vertex set, $E(G)$ be its edge set, $\chi(G)$ be its chromatic number and $P(G, \lambda)$ be its chromatic polynomial. Two graphs $G$ and $H$ are said to be \textit{chromatically equivalent}, or in short $\chi$-equivalent, written $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. A graph $G$ is said to be \textit{chromatically unique}, or in short $\chi$-unique, if for any graph $H$ satisfying $H \sim G$, we have $H \cong G$, i.e. $H$ is isomorphic to $G$. A \textit{wheel} $W_n$ is a graph of order $n$, $n \geq 4$, obtained by the join of $K_1$ and a cycle $C_{n-1}$ of order $n-1$. Let for a vertex $x$ of $G$ the symbol $N(x)$ denote a subgraph of $G$ induced by the set of vertices adjacent to $x$.

A $H$-gluing of two graphs $G$ and $F$ is a graph obtained by identifying an induced subgraph of $G$ isomorphic to $H$ with such a subgraph of $F$ in the disjoint union of $G$ and $F$. Koh and Teo [5] gave a survey on several results on chromaticity of $K_r$-gluings of graphs for $r \geq 1$. One of more interesting results has been discovered by Koh and Goh [4]. They completely characterized $\chi$-unique $K_3$-gluings of complete graphs of order $\geq 3$ and a $K_4$-homeomorph.
In this paper, the $\chi$-equivalent classes for $K_3$-gluings of two wheels are studied. In computing chromatic polynomials, we make use of Whitney’s reduction formula given in [8]. The formula is

\[(1) \quad P(G, \lambda) = P(G-e, \lambda) - P(G/e, \lambda)\]
or equivalently

\[(2) \quad P(G_e, \lambda) = P(G, \lambda) + P(G/e, \lambda)\]

where $G-e$ is the graph obtained from $G$ by deleting an edge $e$ and $G/e$ is the graph obtained from $G$ by contracting the edge $e$.

We also make use of the overlapping formula given in [8]. The formula is

\[(3) \quad P(G, \lambda) = P(H, \lambda)P(F, \lambda)/P(K_p, \lambda)\]

where $G$ is a $K_p$-gluing of two disjoint graphs $H$ and $F$, for $p \geq 1$.

**Preliminary Results**

We shall use the known results for $\chi$-equivalent graphs presented in Lemma 1, where $I_G(F)$ denotes the number of induced subgraphs of $G$ which are isomorphic to $F$.

**Lemma 1** [6]. Let $G$ and $H$ be two $\chi$-equivalent graphs. Then

(i) $|V(G)| = |V(H)|$;
(ii) $|E(G)| = |E(H)|$;
(iii) $\chi(G) = \chi(H)$;
(iv) $I_G(C_3) = I_H(C_3)$;
(v) $I_G(C_4) - 2I_G(K_4) = I_H(C_4) - 2I_H(K_4)$;
(vi) $G$ is connected iff $H$ is connected;
(vii) $G$ is 2-connected iff $H$ is 2-connected.

The following simple immediate observation plays an important role in proving that graphs with triangles are $\chi$-unique or $\chi$-equivalent.

**Lemma 2.** Let $T$ be a tree with $n$ vertices. Then there are $n - 1$ triangles in the join $T + K_1$. 
Lemma 3. Let $T$ be a tree with $n$ vertices and let $v \notin V(T)$. Let $H$ denote a graph obtained from $T$ by adding the vertex $v$ and $m$ edges between $v$ and vertices of $T$, $(m \leq n)$. Then the number of triangles of $H$ is $\leq m - 1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to $v$ is a tree.

Lemma 4. Let $F$ be a unicyclic $K_3$-free graph with $n$ vertices and let $v \notin V(F)$. Let $H$ denote a graph obtained from $F$ by adding the vertex $v$ and $m$ edges between $v$ and vertices of $F$, $(m \leq n)$. Then the number of triangles of $H$ is $m$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to $v$ is connected and it contains the cycle of $F$.

Lemma 5. Let $F$ be a connected $K_3$-free graph with $n$ vertices and with only two fundamental cycles, and let $v \notin V(F)$. Let $H$ be a graph obtained from $F$ by adding the vertex $v$ and $m$ edges between $v$ and $m$ vertices of $F$. Then the number of triangles of $H$ is $\leq m + 1$. Moreover, the equality holds if and only if the subgraph induced by the vertices adjacent to $v$ is connected and contains two fundamental cycles.

Let us assume that $n \geq 6$ is an integer number. For an integer number $q$, $\frac{n}{2} \geq q \geq 3$, the graph $W^q_{n+1}$ is obtained from $W_{n+1}$ by adding exactly one new edge joining two vertices at distance $q$ in the subgraph $C_n$ of $W_{n+1}$. In other words, $W^q_{n+1}$ is a $K_3$-gluing of $W_{n-q+2}$ and $W_{q+2}$ identifying their central vertices.

Lemma 6. $(\lambda - 2)^2 \not\mid P(W^q_{n+1}, \lambda)$. Moreover $W^q_{n+1}$ is uniquely 3-colourable if $n$ is even and $q$ is odd, $\frac{n}{2} \geq q \geq 3$.

Proof. By using Whitney’s reduction formula we have:

\begin{equation}
P(W^q_{n+1}, \lambda) = P(W_{n+1}, \lambda) - \frac{P(W_{n-q+1}, \lambda) \cdot P(W_{q+1}, \lambda)}{P(K_2, \lambda)}.
\end{equation}

Evidently according to the known result for $P(C_n, \lambda)$ (see [1]), we get that

\begin{equation}
P(W_{n+1}, \lambda) = \lambda((\lambda - 2)^n + (-1)^n(\lambda - 2)) = \lambda(\lambda - 1)(\lambda - 2) \cdot P_s(W_{n+1}, \lambda),
\end{equation}

where
\[ P_s(W_{n+1}, \lambda) = \begin{cases} 
(\lambda - 3) \sum_{i=0}^{(n-3)/2} (\lambda - 2)^{2i}, & \text{if } n \text{ is odd}, \\
\sum_{i=0}^{n-2} (-1)^i (\lambda - 2)^i, & \text{if } n \text{ is even}. 
\end{cases} \]

Note that
\[ P_s(W_{n+1}, 2) = \begin{cases} 
-1, & \text{if } n \text{ is odd}, \\
1, & \text{if } n \text{ is even}, 
\end{cases} \]

and
\[ P_s(W_{n+1}, 3) = \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
1, & \text{if } n \text{ is even}. 
\end{cases} \]

From (4) and (5) we get
\[ P(W_n^q, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \cdot [P_s(W_{n+1}, \lambda) - (\lambda - 2) \cdot P_s(W_{q+1}, \lambda)]. \]

Note that \((\lambda - 2) | P(W_n^q, \lambda)\). Let \(P(W_n^q, \lambda) = (\lambda - 2)R(W_{n+1}^q, \lambda)\). Then \(R(W_n^q, 2) = \pm 2\) and \(P(W_n^q, \lambda)\) is not divisible by \((\lambda - 2)^2\). Since for an even \(n\) and an odd \(q\) we have \(P(W_n^q, 3) = 6\), then \(W_n^q\) is uniquely 3-colourable.

**Lemma 7** [2]. Let \(G\) be a graph containing at least two triangles. If there is a vertex of a triangle having degree two in \(G\), then \((\lambda - 2)^2 | P(G, \lambda)\).

**Lemma 8.** Let \(G\) be a graph obtained by \(K_2\)-gluing of two graphs such that each of them has a triangle. Then \((\lambda - 2)^2 | P(G, \lambda)\).

**Proof.** Directly from (3).

**Lemma 9.** Let \(H\) and \(F\) be non-isomorphic \(\chi\)-unique graphs. Then \(K_1 + H \not\sim K_1 + F\).

**Proof.** Evidently \(P(G + K_1, \lambda) = \lambda \cdot P(G, \lambda - 1)\) for any graph \(G\). Let \(H\) and \(F\) be non-isomorphic \(\chi\)-unique graphs. Suppose that \(P(H + K_1, \lambda) = P(F + K_1, \lambda)\) then \(P(H, \lambda - 1) = P(F, \lambda - 1)\) and we get a contradiction.
Main Results

We prove that each of $\chi$-equivalent classes for some cases of $W_{n+1}$ consists of two graphs.

**Theorem 1.** For each even integer $n \geq 6$ and each odd integer $3 \leq q \leq [n/2]$ all $K_3$-gluings of wheels $W_{q+2}$ and $W_{n-q+2}$ create a $\chi$-equivalent class.

**Proof.** Let $n$ be even, $(n \geq 6)$ and let $G \sim W_{n+1}^q$. Then $P(G, \lambda) = P(W_{n+1}^q, \lambda)$ and therefore, by Lemmas 1, 6 and 7 any candidate for $G$ has the following properties: $|V(G)| = n + 1$, $|E(G)| = 2n + 1$, $I_G(C_3) = n + 1$. $G$ is a 2-connected unique 3-colourable graph and no vertex of any triangle of $G$ has degree two in $G$.

Let $V_1, V_2$ and $V_3$ be colour classes of the uniquely 3-colouring of $G$ and let $|V_i| = n_i$, $i = 1, 2, 3$. Evidently $n_1 + n_2 + n_3 = n + 1$.

Let $G_i$ be the subgraph of $G$ induced by $V(G) - V_i$, where $i = 1, 2, 3$. Evidently, each of $G_i$, $i = 1, 2, 3$, is connected (see Theorem 12.16 in [3]). Therefore

$$2n - 1 = (n_1 + n_2 - 1) + (n_1 + n_3 - 1) + (n_2 + n_3 - 1)$$

$$\leq |E(G_3)| + |E(G_2)| + |E(G_1)| = 2n + 1.$$ (6)

Without loss of generality, we have two cases:

**Case 1.** Let $G_3$ and $G_2$ be trees and let $G_1$ be a connected graph with two fundamental cycles, say $C, C'$. Note that $|V(G_1)| = n_2 + n_3 = n + 1 - n_1$ and $|E(G_1)| = n + 2 - n_1$. Consequently, the number $m(V_1, V(G_1))$ of edges from $V_1$ to $V(G_1)$ satisfies the following equality

$$m(V_1, V(G_1)) = 2n + 1 - (n + 2 - n_1) = n + n_1 - 1.$$ (7)

Suppose that no vertex of $V_1$ is adjacent to all vertices of any cycle of $G_1$.

Then by Lemma 3 and formula (7)

$$n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\text{deg}(v_i) - 1) = \sum_{i=1}^{n_1} \text{deg}(v_i) - n_1 = n + n_1 - 1 - n_1 = n - 1,$$

and we get a contradiction. Therefore we can assume that some vertex $v \in V_1$ is adjacent to all vertices of a fundamental cycle of $G_1$, say $C$, and since $G_2$ and $G_3$ are trees, then $v$ is unique. Now if there exists no vertex of
$V_1$ adjacent to all vertices of the cycle $C'$ of $G_1$, where $C' \neq C$ then similarly, by Lemmas 3 and 4 we get that

\[(8) \quad n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i) - 1) + 1 = n,\]

and it leads to a contradiction. Therefore according to the above argument there is exactly one vertex $v' \in V_1$ which is adjacent to all vertices of $C'$. Suppose that a subgraph of $G_1$ induced by the set of all vertices adjacent to a vertex of $V_1$ is disconnected. Looking at the tree structure of $G_2$ and $G_3$ and Lemmas 3-5 we obtain the inequality presented in formula (8), and it leads to a contradiction.

From the above it follows that

**Lemma 10.** One of the vertices of $V_1$, say $v$, is adjacent to all vertices of a connected subgraph of $G_1$ which contains $C$, and one of the vertices of $V_1$, say $v'$, is adjacent to all vertices of a connected subgraph of $G_1$ which contains $C'$, and each of the other vertices of $V_1$ is adjacent to the vertices of a subtree of $G_1$.

Let us consider degrees of the vertices of $G$. Immediately by 2-connectivity of $G$ and Lemmas 6, 7 and 10 we get that each vertex of $V_1$ has degree at least 3 in $G$. Similarly, each 1-degree vertex of $G_1$ has at least two neighbours in $V_1$. Suppose that a 2-degree vertex $x$ of $G_1$ has degree 2 in $G$. Then by Lemma 10 the vertex $x$ does not belong to any cycle of $G_1$ and it is a cut vertex of $G$. It leads to a contradiction to 2-connectivity of $G$. It follows that

**Lemma 11.** $\deg(x) \geq 3$ for each $x \in V(G)$.

Suppose now that $V(N(x)) = V(G_1)$ for some $x \in V_1$. Then by Lemma 5 the vertex $x$ belongs to $n_2 + n_3 + 1$ triangles of $G$, and each of $n + 1 - (n_2 + n_3 + 1) = n_1 - 1$ other triangles contains a vertex of $V_1 - \{x\}$. By formula (7) the number of edges from the set $V_1 - \{x\}$ to $V(G_1)$ is equal to $n + n_1 - 1 - (n_2 + n_3) = 2(n_1 - 1)$. So this fact and 2-connectivity of $G$ imply that $\deg(y) = 2$ for each $y \in V_1 - \{x\}$. Therefore from Lemma 7, the set $V_1$ consists of exactly one vertex $x$ and $G_1$ has not any vertex of degree one. Thus $\deg(x) = n$ and $G$ is isomorphic to the join of $K_1$ and one of the three graphs presented in Figure 1.
Equivalent Classes for \(K_3\)-Gluings of Wheels

If \(G_1\) is isomorphic to a graph of the structure \((C)\) or \((B)\), then Lemma 8 implies \((\lambda - 2)^2|P(G, \lambda)|\) and we get a contradiction to Lemma 6.

Therefore \(G_1\) is isomorphic to a graph of the structure \((A)\). Note that each of the three paths from the vertex \(a\) to \(b\) is odd length, since \(n\) is even and \(C, C'\) have even length. Since each generalized \(\theta\)-graph is \(\chi\)-unique [7], from Lemma 9 we get \(G = \mathbb{W}_{n+1}^q\).

We have to consider the case : \(V(N(x)) \neq V(G_1)\) for each \(x \in V_1\).

First suppose that the vertex \(v \in V_1\) is adjacent to all vertices of \(C\) and \(C'\), i.e., \(v = v'\). The assumption of the case and Lemma 10 imply \(V(G_1) - V(C \cup C') \neq \emptyset\). So there exists a vertex \(u \in V(G_1) - V(N(v))\) such that \(\deg_{G_1}(u) = 1\). Thus

\[
(9) \quad n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v_i) - 1) + 2 = n + 1.
\]

Lemma 5 and \(V(N(v)) \neq V(G_1)\) imply that \(v\) belongs to at most \(n_2 + n_3\) triangles of \(G\), and vertices of \(V_1 - \{v\}\) belong to at least \(n_1\) triangles. Moreover, the number of edges from \(V_1 - \{v\}\) to \(V(G_1)\) is at least \(2(n_1 - 1) + 1\). Therefore \(|V_1| \geq 2\).

Lemma 11 implies that the vertex \(u\) is adjacent to two different vertices \(v_1, v_2 \in V_1 - \{v\}\). Let \(w\) be a neighbour of \(u\) in \(G_1\). From Lemmas 10, 11 we have that \(w\) is adjacent to \(v_1\) and \(v_2\). Therefore we get either a cycle in the subgraph \(N(w)\) or that \(G\) is a \(K_2\)-gluing of two graphs with triangles. The first case contradicts acyclicity of \(G_2\) and \(G_3\). By Lemma 8 the other case gives \((\lambda - 2)^2|P(G, \lambda)|\) and it contradicts Lemma 6.

Therefore suppose now that the vertex \(v \in V_1\) is not adjacent to a vertex of \(C'\). Thus \(v \neq v'\). Applying the same arguments as before we get that
$G_1$ does not have any vertex of degree 1. Hence we can consider only the following three subcases: $G_1$ is a $K_2$-gluing of two cycles of even order, a $K_1$-gluing of two cycles of even order, or it consists of two cycles of even order and exactly one path connecting them.

Since $n$ is even, then for the first case we get that $V_1 - \{v, v'\} \neq \emptyset$ and 2-connectivity of $G$, Lemma 10 and acyclicity of $G_2$ and $G_3$ imply $N(v_1) \cong K_2$ for each $v_1 \in V_1 - \{v, v'\}$ and this gives a contradiction to Lemma 11.

For two other cases Lemma 10 and acyclicity of $G_2$ and $G_3$ imply $|V(N(v_1)) \cap V(N(v_2))| \leq 2$, for each pair of different vertices $v_1, v_2 \in V_1$. Therefore by 2-connectivity of $G$ we get that $G$ is a $K_2$-gluing of two graphs with triangles. Hence we get a contradiction to the Lemma 6.

**Case 2.** Let $G_3$ be a tree, and $G_2$, $G_1$ be unicyclic graphs with even cycles. Note that

$$|E(G_1)| = |V(G_1)| = n + 1 - n_1,$$

$$|E(G_2)| = |V(G_2)| = n_1 + n_3 = n + 1 - n_2.$$ 

The number of edges from $V_1$ to $V(G_1)$ is equal to

$$2n + 1 - (n + 1 - n_1) = n + n_1.$$  (10)

Similarly, the number of edges from $V_2$ to $V(G_2)$ is equal to

$$2n + 1 - (n + 1 - n_2) = n + n_2.$$  (11)

Let $C^1$ be the cycle of $G_1$, and $C^2$ be the cycle of $G_2$.

Suppose that there is no vertex in $V_1$ adjacent to all of the vertices of $C^1$. Then each vertex of $V_1$ is adjacent to a subforest in $G_1$.

By Lemma 3 the number of triangles in $G$ containing a vertex $v^1_1 \in V_1$ is at most $d(v^1_1) - 1$. So the number of triangles in $G$ is at most

$$n + 1 = I_G(C_3) \leq \sum_{i=1}^{n_1} (\deg(v^1_i) - 1)$$

$$= \sum_{i=1}^{n_1} \deg(v^1_i) - n_1 = n + n_1 - n_1 = n,$$  (12)

and we get a contradiction.

Therefore there exists at least one vertex $v^1 \in V_1$ adjacent to all of the vertices of $C^1$. Suppose that there is another such vertex, i.e., let $w^1 \in V_1 - \{v^1\}$ and let $w^1$ be adjacent to all of the vertices of $C^1$. Assume also without loss of generality that $u_1, u_2, \ldots, u_{2m}$ are consecutive vertices of $C^1$, where $u_1, u_3, \ldots, u_{2m-1} \in V_2$ and $u_2, u_4, \ldots, u_{2m} \in V_3$. Note that the subgraph
induced by \( \{u_1, v^1, u_3, w^1\} \) is a cycle in \( G_3 \). This contradicts the fact that \( G_3 \) is a tree. Thus we have proved that there exists exactly one vertex \( v^1 \) in \( V_1 \) adjacent to all vertices in \( C^1 \). Similarly, there exists exactly one vertex \( v^2 \) in \( V_2 \) adjacent to all vertices in \( C^2 \). Suppose that a subgraph of \( G_1 \) induced by all vertices adjacent to a vertex of \( V_1 \) is disconnected. Hence by Lemmas 3-4 we get the formula (12), and it leads to a contradiction.

Thus we have the following observations.

**Lemma 12.** One vertex, \( v^1 \in V_1 \), is adjacent to all of the vertices of a connected subgraph of \( G_1 \) which contains the even cycle. Each other vertex of \( V_1 \) is adjacent to the vertices of a subtree of \( G_1 \).

Similarly, by symmetry, the vertices of \( V_2 \) must satisfy the respective conditions of the following result.

**Lemma 13.** One vertex, \( v^2 \in V_2 \), is adjacent to all of the vertices of a connected subgraph of \( G_2 \) which contains the even cycle. Each other vertex of \( V_2 \) is adjacent to the vertices of a subtree of \( G_2 \).

Lemma 12 and acyclicity of \( G_3 \) give the following lemma.

**Lemma 14.** \(|V(N(v)) \cap V(N(v'))| \leq 3\) for \( v, v' \in V_1, v \neq v' \).

Moreover, Lemma 11 presented in case 1 holds for \( G \).

**Subcase 2.1.** Suppose that \( N(v^1) = V(G_1) \). Then by Lemma 4 the vertex \( v^1 \) belongs to \( n+1-n_1 \) triangles in \( G \), and each of other \( n+1-(n+1-n_1) = n_1 \) triangles contains a vertex of \( V_1 - \{v^1\} \neq \emptyset \). Note that the number of edges from \( V_1 - \{v^1\} \) to \( V(G_1) \) is equal to \( 2n+1-2(n+1-n_1) = 2n_1 - 1 = 2(n_1 - 1) + 1 \). This and Lemma 11 lead to \(|V_1| = 2\). Hence there exists exactly one vertex in \( V_1 \) different from \( v^1 \), say \( w^1 \), and its degree equals 3.

Therefore, from Lemma 7 and from the fact that \( n \) is even, the graph \( G_1 \) consists of \( C^1 \) and exactly one tree \( T \) rooted at a vertex of \( C^1 \). Moreover, for each pair \( x, y \) of leaves of \( T \) we have that \( \text{dist}_{G_1}(x, y) = 2 \) and then \( T \) has only two leaves. Since \( n \) is even, \( T \) has an even number of vertices (including root vertex). Therefore \( T \equiv P_{2t} \) or \( T \) is a \( K_1 \)-gluing of \( P_{2t-1} \) and \( K_2 \), where \( t \geq 1 \), and \( G_1 \) is one of the two graphs presented in Figure 2.

By Lemma 11 each leaf of the rooted tree \( T \) is adjacent to \( w^1 \) and \( v^1 \). Lemmas 6, 8 imply that the graph \( G \) is not any \( K_2 \)-gluing of two graphs with triangles in each of them. Therefore \( G_1 \) is a unicyclic graph with one leaf and a cycle of length \( n - 2 \).
If two of the vertices which are adjacent to $w^1$ have colour 2, then 
$\{x, w^1, y, v^1\}$ induces $C_4$ in $G_3$, and we have a contradiction.

Therefore two of the vertices which are adjacent to $w^1$ have colour 3
and then $\{x, w^1, y, v^1\}$ induces $C_4$ in $G_2$.

Hence $G$ is $K_3$-gluing of $W_{n-1}$ and $W_5$ such that the centers of the wheels are not overlapped. Note that by Lemma 1(v) the graph $G$ is isomorphic to $W_{n+1}^q$ and this is possible only for $q = 3$.

**Subcase 2.2.** We can assume that $N(v^1) \neq V(G_1)$ and by symmetry $N(v^2) \neq V(G_2)$. Then by Lemmas 12, 13 each of the graphs $G_1$, $G_2$ is unicyclic with a vertex of degree one. Evidently by Lemma 11 each leave in $G_1$ is adjacent to at least two vertices of $V_1$. Let $v^1$, $v^2$ be the vertices of Lemmas 12 and 13, respectively. Let $x$ be a leave in $G_1$ which is not adjacent to $v^1$, and let $x^1$ be the neighbour of $x$ in $G_1$.

Let $x^2$ be a neighbour of $x^1$ in $G_1$ such that $x^2 \neq x$ and $\deg(x^2) \geq 2$.

Lemmas 11, 12 imply that the vertex $x$ has at least two neighbours in $V_1$. Let us consider $N(x^1)$. Since $G$ is not any $K_2$-gluing of two graphs with triangles and $G_3$ has not any cycle, then Lemmas 6, 7, 11, 12 and 14 imply that $N(x^1)$ contains a cycle belonging to $G_2$. Evidently, the cycle is unique. The same arguments give $x^1 \in V(C^1)$ and therefore $G_1$ has a unique rooted tree and it is isomorphic to a graph presented in Figure 3. Similarly, $G_2$ is isomorphic to a graph presented in Figure 3.
Let $a, b \in V(N(x^1)) \cap V(C^1)$, $\{w_1, ..., w_t\} = V_1 - \{v^1\}$ and let $x = x_1, x_2, ..., x_m$ be the leaves of $G_1$. If neither $a$ nor $b$ is adjacent to a vertex $w_j, j = 1, ..., t$, then $G$ is a $K_2$-gluing of two graphs with triangles, for $K_2$ induced by $\{v^1, x^1\}$ and we get a contradiction. Thus without loss of generality, we can assume that $a$ is adjacent to $w_1$. Then there exists an alternating sequence passing through all vertices of $V_1$ and all leaves of $V(G_1)$ and having one of the two forms

\begin{align*}
a, w_1, x_1, w_2, x_2, ..., x_m, w_m, b, v^1 \\
a, w_1, x_1, w_2, x_2, ..., x_m, v^1.
\end{align*}

The first case gives an odd cycle in $G_2$ and we get a contradiction. The other one gives a $K_3$-gluing of two wheels which does not identify their central vertices. Since each generalized $\theta$-graph is $\chi$-unique [7], from Lemma 9 we get that these wheels must be isomorphic to $W_{q+2}$ and $W_{n-q+2}$, respectively.

The proof is complete. \hfill \blacksquare

Since the wheels $W_6, W_8$ are not $\chi$-unique graphs [2], [9] the $\chi$-equivalent classes for other cases of $n$ and $q$ can contain more than two graphs. The graphs $G \cong W_{n+1}^n$, for $n$ odd or $q$ even are not uniquely $\chi(G)$-colourable. Thus, the proof techniques used in this paper cannot be used to characterize $\chi$-equivalent classes for these graphs.

References


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