CYCLICALLY 5-EDGE CONNECTED NON-BICRITICAL CRITICAL SNARKS

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Abstract

Snarks are bridgeless cubic graphs with chromatic index $\chi' = 4$. A snark $G$ is called critical if $\chi'(G - \{v, w\}) = 3$, for any two adjacent vertices $v$ and $w$.

For any $k \geq 2$ we construct cyclically 5-edge connected critical snarks $G$ having an independent set $I$ of at least $k$ vertices such that $\chi'(G - I) = 4$.

For $k = 2$ this solves a problem of Nedela and Škoviera [6].

Keywords: cubic graphs, snarks, edge colorings.

1991 Mathematics Subject Classification: 05C15, 05C70.

1. Introduction

A snark is a bridgeless cubic graph with chromatic index $\chi' = 4$. The study of the reduction of snarks is as old as the study of these graphs itself.

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For a detailed introduction to this topic we refer the reader to one of [3, 5, 6, 7, 10, 12, 13]. This note deals with a reduction of snarks introduced by Nedela and Škoviera in [6].

Let \( G \) be a snark and let \( F \subset E(G) \) be a \( k \)-edge cut \((k \geq 0)\) whose removal divides \( G \) into two components \( H_1 \) and \( H_2 \). If the chromatic index of one of the components is 4, say \( \chi'(H_1) = 4 \), then \( H_1 \) can be extended to a snark \( H \) with \( |V(H)| \leq |V(G)| \) by adding edges and probably vertices. Graph \( H \) is called a \( k \)-reduction of \( G \). If \( |V(H)| < |V(G)| \), then \( H \) is called a proper \( k \)-reduction of \( G \).

A snark is called \( k \)-irreducible if it has no proper \( m \)-reduction such that \( m < k \), and it is called irreducible if it is \( k \)-irreducible for each \( k \geq 1 \).

A snark \( G \) is called critical if \( \chi'(G - \{v, w\}) = 3 \) for any two adjacent vertices \( v, w \in V(G) \), it is called cocritical if \( \chi'(G - \{v, w\}) = 3 \) for any two non-adjacent vertices \( v, w \in V(G) \), and it is called bicritical if it is critical and cocritical.

Nedela and Škoviera proved the following characterizations.

**Theorem 1.1** [6]. Let \( G \) be a snark. Then the following statements hold true.

1. If \( 5 \leq k \leq 6 \), then \( G \) is \( k \)-irreducible if and only if it is critical.
2. If \( k \geq 7 \), then \( G \) is \( k \)-irreducible if and only if it is bicritical.

Finally, it turns out that

**Theorem 1.2** [6]. A snark is irreducible if and only if it is bicritical.

In [2] it is shown that there are cocritical snarks which are not critical, and that there are snarks which are neither critical nor cocritical. Further, each critical snark on less than 30 vertices is bicritical and henceforth it is irreducible. Nedela and Škoviera [6] state the following problem.

**Problem 1.3** [6]. Does there exist a snark that is critical but not bicritical? Equivalently, does there exist a 6-irreducible snark that is not irreducible?

The answer is “yes”. In [9] the latter author constructed infinite families of cyclically 4-edge connected critical snarks which are not bicritical. The smallest one has 32 vertices. M. Škoviera [8] found another infinite family of cyclically 4-edge connected snarks with these properties by using a different method.
In this note, we improve these results in two directions. We construct cyclically 5-edge connected critical snarks with the property that they have a large independent set of vertices whose removal does not yield an edge 3-colorable graph. Clearly, these graphs are not bicritical.

2. The Main Theorem

We will use the following lemma, due to Blanuša [1].

**Lemma 2.1.** (Parity Lemma) Let $M$ be a multigraph whose edges are colored with colors $1, \ldots, k$, and let $a_i$ be the number of vertices $v$ in $M$ such that no edge incident to $v$ is colored $i$. Then for all $i = 1, \ldots, k : a_i \equiv |V(M)| \pmod{2}$.

**Proof.** For $i = 1, \ldots k$ let $E_i$ be the set of edges colored $i$. Then $a_i = |V(M)| - 2|E_i|$, and hence $a_i \equiv |V(M)| \pmod{2}$. $\blacksquare$

**Theorem 2.2.** For each $k \geq 1$ there is a cyclically 5-edge connected critical snark that has an independent set $I$ of $2k+1$ vertices such that $\chi(G - I) = 4$.

**Proof.** The idea of the construction is as follows. Let $k \geq 1$ be fixed. We construct a multigraph $M_k$ and specify three closed walks in this multigraph. We replace vertices of $M_k$ by well specified graphs to obtain a cubic graph $G_k$. We show that $G_k$ is cyclically 5-edge connected and that it is a snark. Furthermore, each of the walks in $M_k$ can be extended to circuits in $G_k$ to obtain a 2-factor of $G_k$ with precisely two odd circuits. We then show that for each edge $e = vw$ in $G_k$ there is a 2-factor of $G_k$ with precisely two odd circuits which are connected by $e$. If this is true, then $G_k$ has an edge 4-coloring with a color class consisting of precisely two edges, one of them incident to $v$ and the other incident to $w$. Thus $G_k - \{v, w\}$ is edge 3-colorable. Hence $G_k$ is critical.

**Construction**

Let $k \geq 1$ be fixed and $I = \{w_0, w_1, \ldots, w_{2k}\}$. Define $M_k$ to be the multigraph with vertex set $\mathbb{Z}_{3(2k+1)} \cup I$, and for each $i \in \mathbb{Z}_{3(2k+1)}$ vertex $i$ is joined to vertex $i + 1$ by two parallel edges, $e(i, i + 1)$ and $f(i, i + 1)$, and by one edge with $w_m \in I$ if $i \equiv m \pmod{2k+1}$. We call the elements of $\mathbb{Z}_{3(2k+1)}$ the outer vertices of $M_k$. Multigraph $M_1$ is shown in Figure 1.
Figure 1. Multigraph $M_1$

For $k \geq 1$ the circuits $P_e$, $P_f$ and the walk $Q$ in $M_k$ are defined as follows:

Let $2k + 1 = K$,

$P_e = e(0,1), e(1, w_1), e(w_1, K + 1), e(K + 1, K + 2), e(K + 2, w_2), e(w_2, 2), e(2,3), \ldots, e(2K - 1, w_{K-1}), e(w_{K-1}, K - 1), e(K, w_0), e(w_0, 0)$,

$P_f = f(0,1), e(1, w_1), e(w_1, K + 1), f(K + 1, K + 2), e(K + 2, w_2), e(w_2, 2), f(2,3), \ldots, e(2K - 1, w_{K-1}), e(w_{K-1}, K - 1), f(K - 1, K), e(K, w_0), e(w_0, 0)$,

$Q = e(2K - 1, 2K), e(2K, 2K + 1), e(2K + 1, 2K + 2), \ldots, e(3K - 1, 0), f(0, 3K - 1), f(3K - 1, 3K - 2), \ldots, f(2K, 2K - 1)$.

The 7-block $B^i$ ($i \in \mathbb{Z}_{3(2k+1)}$) is the graph obtained from a cycle $C_6$ on the vertices $v_0^i, v_1^i, \ldots, v_5^i$ by adding a vertex $v_6^i$ and edges $v_1^i v_0^i$ and $v_4^i v_5^i$.

For each $k \geq 1$ construct the cubic graph $G_k$ as follows:

For each $i \in \mathbb{Z}_{3(2k+1)}$ take block $B^i$ and vertices $w_0, \ldots, w_{2k}$, add edges $v_0^i v_5^{i+1}, v_2^i v_2^{i+1}$ and for $0 \leq j \leq 2k$ add edges $w_j v_0^i$ if $n \equiv j \pmod{2k + 1}$.

If we contract each subgraph $B^i$ to a single vertex $i$, we obtain multigraph $M_k$. For the following we may assume that the edges $v_0^j v_5^{i+1}$ and $v_2^j v_2^{i+1}$ of $G_k$ are denoted by $e(i,i+1)$ and $f(i,i+1)$ in the contracted graph, respectively. Vice versa $G_k$ can be obtained from $M_k$ by successively replacing the outer vertices by 7-blocks. A fact which we will use in the following.
Claim 2.2.1. \( G_k \) is a cyclically 5-edge connected snark, that has an independent set \( I \) of \( 2k + 1 \) vertices such that \( G_k - I \) is not edge 3-colorable.

**Proof.** A 7-block \( B^i \) is obtained from the Petersen graph by removing three vertices of a path of length 2. Therefore it follows from Lemma 2.1 that for each 3-coloring of \( B^i \) the same colors are missing at \( v_0^i \) and \( v_3^i \) and two different colors are missing at \( v_2^i \) and \( v_5^i \) or vice versa. Thus \( G_k - I \) cannot be edge 3-colorable.

Since between any two outer vertices of \( M_k \) there are five edge disjoint paths and since no 5-circuit of \( B^i \) can be separated by removing less than five edges from \( G_k \), it follows that \( G_k \) is cyclically 5-edge connected. \( \blacksquare \)

Claim 2.2.2. \( G_k \) is critical.

**Proof.** We have to show that for each edge \( vw = e \in E(G_k) \) there is a 2-factor \( \mathcal{F}_e \) of \( G_k \) with precisely two odd circuits which are connected by \( e \).

We reconstruct \( G_k \) from \( M_k \) by successively replacing outer vertices of \( M_k \) by the 7-blocks. We show that the circuits \( P_e, P_f \) and the walk \( Q \) can be extended to circuits of the new graphs. Eventually (after replacing all outer vertices of \( M_k \)), we obtain a spanning 2-factor of \( G_k \) with precisely two odd circuits.

In \( G_k \) we have basically the following types of edges:

\[ v_0^i v_1^i, v_1^i v_2^i, v_2^i v_3^i, v_3^i v_4^i, v_4^i v_5^i, v_5^i v_0^i, v_1^i v_6^i, v_4^i v_7^i, v_0^i v_5^i+1, v_3^i v_2^i+1, \text{ and } v_6^i w_1 \] (where \( i \equiv l \pmod{2k+1} \)).

We have \( V(M_k) = \mathbb{Z}_{3(2k+1)} \cup \{w_0, \ldots, w_{2k}\} \), and we define for \( j = 1, \ldots, 3(2k+1) - 1 \) the function \( f_j : V(M_k) \to V(M_k) \) with \( f_j(i) = i + j \) if \( i \in \mathbb{Z}_{3(2k+1)} \) and \( f_j(w_k) = w_{k+j} \), where the indices are added modulo \( 2k+1 \). This function is an automorphism on \( M_k \).

Thus the aforementioned construction can be applied on \( M_k \) where vertices \( v \) are labeled by \( f_j(v) \), for each \( j = 1, \ldots, 3(2k+1) - 1 \). Hence it suffices to show that for each edge type there is a 2-factor of \( G_k \) containing precisely two odd circuits connected by at least one edge of that type.

It is easy to see that there are hamiltonian paths in \( B^i \) with terminal vertices \( v_0^i, v_6^i \) and \( v_3^i, v_0^i \) and \( v_2^i, v_6^i \) and \( v_3^i, v_6^i \), respectively.

Then, if for \( 1 \leq i \leq 2(2k+1) - 2 \) vertex \( i \) of \( M_k \) is replaced by \( B^i \) circuits, \( P_e \) and \( P_f \) can be extended to circuits of the new graph, respectively.

Paths \( v_2^i, v_5^i, v_6^i, v_4^i, v_3^i \) and \( v_0^i, v_5^i \) span \( B^i \). Then, if for \( 0 \leq l \leq 2k \) vertex \( 2(2k+1) + l \) of \( M_k \) is replaced by \( B^{2(2k+1) + l} \), the walk \( Q \) can be extended to a circuit of the new graph.
Let $G_k^-$ be the graph obtained from $M_k$ by replacing all degree 5 vertices but 0 and $2(2k+1) - 1$. To show that $G_k$ is critical we consider the following cases: Let $s = 2(2k + 1) - 1$.

**Case 1.** We consider the extended circuit $P_e$ and the extended walk $Q$ in $G_k^-$.

Replace vertices 0 and $s$ by $B^0$ and $B^s$, respectively. In $B^0$ extend $P_e$ by the path $v_0^0, v_1^0, v_6^0$, and $Q$ by the path $v_0^0, v_3^0, v_4^0, v_5^0$. In $B^s$ extend $P_e$ by the path $v_3^s, v_4^s, v_6^s$, and $Q$ by the path $v_0^s, v_1^s, v_2^s, v_3^s$ to obtain two disjoint spanning circuits $P^*$ and $Q^*$ of odd length which form a 2-factor of $G_k$.

These two odd circuits are connected by edges $v_0^0v_5^0, v_1^0v_2^0, v_0^0v_6^0, v_4^sv_6^s, v_3^sv_4^s, v_3^sv_2^s$, and by edges $v_6^{s+1+l}w_l$ for $0 \leq l \leq 2k$.

**Case 2.** We consider the extended circuit $P_f$ and the extended walk $Q$ in $G_k^-$.

Let vertices 0 and $s$ be replaced by $B^0$ and $B^s$, respectively. In $B^0$ extend $P_f$ by the path $v_3^0, v_4^0, v_6^0$ and $Q$ by the path $v_0^0, v_1^0, v_2^0, v_3^0$. In $B^s$ extend $P_f$ by the path $v_3^s, v_4^s, v_6^s$, and $Q$ by $v_0^s, v_3^s, v_4^s, v_5^s$. As in case 1 we obtain two disjoint spanning circuits $P^*$ and $Q^*$ of odd length which form a 2-factor of $G_k$. These two circuits are connected by $v_2^sv_3^0, v_1^sv_5^0, v_0^sv_4^s$ and $v_0^{s-1}v_6^s$.

Flower snark $J_{2k+1}$ (cf. [5]) has vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i|i = 1, 2, \ldots, 2k+1\}$ and edge set $E(J_{2k+1}) = \{b_ia_i, b_ic_i, b_id_i; a_ia_{i+1}; c_id_{i+1}; d_id_{i+1}|i = 1, 2, \ldots, 2k+1\}$. The above construction can also be carried out by using copies of $J_7 - \{a_1, a_2, a_7\}$ instead of the $B^i$’s. This yields a cyclically 5-edge connected snark $H_k$ with girth 6. By the same argumentation as above, it follows that $H_k$ is critical and that $\chi'(H_k - I) = 4$, for each $k \geq 1$. Because the proof is long and tedious we omit it here and state:

**Theorem 2.3.** For any $k \geq 1$ there are cyclically 5-edge connected critical snarks with girth 6 having an independent set of $2k+1$ vertices whose removal does not yield an edge 3-colorable graph.

**References**


Received 24 November 1997
Revised 21 December 1998