NEIGHBOR SUM DISTINGUISHING TOTAL CHROMATIC NUMBER OF PLANAR GRAPHS WITHOUT 5-CYCLES\textsuperscript{1}

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Abstract

For a given graph \( G = (V(G), E(G)) \), a proper total coloring \( \phi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, k\} \) is neighbor sum distinguishing if \( f(u) \neq f(v) \) for each edge \( uv \in E(G) \), where \( f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v) \), \( v \in V(G) \). The smallest integer \( k \) in such a coloring of \( G \) is the neighbor sum distinguishing total chromatic number, denoted by \( \chi''_{\Sigma}(G) \). Pilśniak and Woźniak first introduced this coloring and conjectured that \( \chi''_{\Sigma}(G) \leq \Delta(G) + 3 \) for any graph with maximum degree \( \Delta(G) \). In this paper, by using the discharging method, we prove that for any planar graph \( G \) without 5-cycles, \( \chi''_{\Sigma}(G) \leq \max\{\Delta(G) + 2, 10\} \). The bound \( \Delta(G) + 2 \) is sharp. Furthermore, we get the exact value of \( \chi''_{\Sigma}(G) \) if \( \Delta(G) \geq 9 \).

Keywords: neighbor sum distinguishing total coloring, discharging method, planar graph.

2010 Mathematics Subject Classification: 05C15.

\textsuperscript{1}This work was supported by NSFC (No.11671232), HNSF(No.A2015202301) and HUSTP (No.ZD2015106, QN2017044).

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1. Introduction

In this paper, all graphs considered are simple, finite and undirected. For the terminology and notation not defined in this paper can be found in [1]. For a graph $G$, we denote its vertex set, edge set and maximum degree by $V(G)$, $E(G)$ and $\Delta(G)$, respectively. If $G$ is a planar graph embedded in the plane, we use $F(G)$ to denote its face set. A vertex $v$ is a $t$-vertex, $t^-$-vertex, $t^+$-vertex if $d_G(v) = t$, $d_G(v) \leq t$, $d_G(v) \geq t$ in $G$, respectively. A $t$-face is defined similarly. An $t$-face $v_1v_2 \cdots v_l$ is a $(b_1, b_2, \ldots, b_l)$-face, where $v_i$ is a $b_i$-vertex, for $i = 1, 2, \ldots, l$. Let $d'_G(v)$ denote the number of $t$-vertices adjacent to $v$ in $G$. Let $n^d_G(v)$ denote the number of $d$-faces incident with $v$ in $G$. A configuration $F$ is reducible to $G$, if it cannot be a configuration of $G$.

Given a graph $G$, set $n_i(G) = |\{v \in V(G) : d_G(v) = i\}|$ for $i = 1, 2, \ldots, \Delta(G)$. A graph $G'$ is smaller than $G$ if one of the following holds:

1. $|E(G')| < |E(G)|$,
2. $|E(G')| = |E(G)|$ and $(n_t(G'), n_{t-1}(G'), \ldots, n_1(G'))$ precedes $(n_t(G), n_{t-1}(G), \ldots, n_1(G))$ with respect to the standard lexicographic order, where $t = \max \{\Delta(G), \Delta(G')\}$.

A graph is minimum for a property if no smaller graph satisfies it.

Given a graph $G$ and a positive integer $k$, a proper total $k$-coloring of $G$ is a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for each pair of adjacent or incident elements $x, y \in V(G) \cup E(G)$. Let $f(v) = \sum_{uv \in E(G)} \phi(uv) + \phi(v)$, $v \in V(G)$. If $f(u) \neq f(v)$ for each edge $uv \in E(G)$, then $\phi$ is a neighbor sum distinguishing total $k$-coloring, or $k$-tnsd-coloring for simplicity. The smallest number $k$ is the neighbor sum distinguishing total chromatic number of $G$, denoted by $\chi^\Sigma_k(G)$. For $k$-tnsd-coloring, Pilśniak and Woźniak gave the following conjecture.

Conjecture 1 [11]. For any graph $G$, $\chi^\Sigma_k(G) \leq \Delta(G) + 3$.

Pilśniak and Woźniak confirmed Conjecture 1 for bipartite graphs, complete graphs, cycles and subcubic graphs. Dong et al. [3] showed that Conjecture 1 holds for some sparse graphs. Yao et al. [21, 22] considered tnsd-coloring of degenerate graphs. Li et al. [9] proved that Conjecture 1 holds for $K_4$-minor free graphs. Song et al. [15] determined $\chi^\Sigma_k(G)$ for $K_4$-minor free graph $G$ with $\Delta(G) \geq 5$. For planar graph, it was proved that this conjecture holds with $\Delta(G) \geq 13$ by Li et al. [7] and $\Delta(G) \geq 11$ by Qu et al. [12]. For planar graph, it was proved that $\chi^\Sigma_k(G) \leq \Delta(G) + 2$ holds with $\Delta(G) \geq 14$ by Cheng et al. [2], $\Delta(G) \geq 12$ by Song et al. [14] and $\Delta(G) \geq 11$ by Yang et al. [20]. The bound $\Delta(G) + 2$ is sharp. Some results about planar graphs with cycle restrictions can be seen in [5, 8, 10] and [16–19]. More references on tnsd-coloring can be seen in [4] and [13].
Recently, Ge et al. [6] got the following result.

**Theorem 2** [6]. Let $G$ be a planar graph without 5-cycles. Then \( \chi''_{\Sigma}(G) \leq \max\{ \Delta(G) + 3, 10 \} \).

In this paper, we prove the following results.

**Theorem 3.** Let $G$ be a planar graph without 5-cycles. Then \( \chi''_{\Sigma}(G) \leq \max\{ \Delta(G) + 2, 10 \} \).

**Theorem 4.** Let $G$ be a planar graph without 5-cycles and without adjacent \( \Delta(G) \)-vertices. Then \( \chi''_{\Sigma}(G) \leq \max\{ \Delta(G) + 1, 10 \} \).

Clearly, \( \chi''_{\Sigma}(G) \geq \Delta(G) + 1 \) for any graph $G$. If $G$ has adjacent \( \Delta(G) \)-vertices, then \( \chi''_{\Sigma}(G) \geq \Delta(G) + 2 \). Thus we get the following corollary.

**Corollary 5.** Let $G$ be a planar graph without 5-cycles and \( \Delta(G) \geq 9 \). If $G$ has no adjacent \( \Delta(G) \)-vertices, then \( \chi''_{\Sigma}(G) = \Delta(G) + 1 \), otherwise \( \chi''_{\Sigma}(G) = \Delta(G) + 2 \).

**2. The Proof of Theorem 3**

We will prove it by contradiction. Let $G$ be a minimum counterexample to Theorem 3 which is embedded in the plane. Set $k = \max\{ \Delta(G) + 2, 10 \}$. By the choice of $G$, any planar graph $G'$ without 5-cycles which is smaller than $G$ has a $k$-tnsd-coloring $\phi'$. In the following, we will choose some $G'$ and extend the coloring $\phi'$ of $G'$ to a desired coloring $\phi$ of $G$ to get a contradiction. Unless otherwise stated, for any $x \in (V(G) \cup E(G)) \cap (V(G') \cup E(G'))$, set $\phi(x) = \phi'(x)$.

In the following proof, we will omit the coloring of all 3\(^{-}\)-vertices. Since they have at most 9 forbidden colors and $k \geq 10$, they can be colored easily.

In Figure 1, we draw a vertex $x$ in black if it has no other neighbors than the ones already depicted, and a vertex $x$ in white if it might have more neighbors than the ones shown in the figure.

**Claim 1.** These configurations of $F_1$, $F_2$, $F_3$ and $F_4$ in Figure 1 are reducible.

**Proof.** (1) Suppose to the contrary that $G$ contains configuration $F_1$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F'_1$ in Figure 1). Thus $G'$ is a planar graph without 5-cycles which is smaller than $G$. Hence $G'$ admits a $k$-tnsd-coloring $\phi'$. We can stick $u_i$, $v_i$ together properly for $i = 1, 2$ (if necessary, exchange the colors of $uu_1$ and $uu_2$), and then recolor $u_i$, $v_i$, thus we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(2) Suppose to the contrary that $G$ contains configuration $F_2$. We obtain a smaller graph $G'$ by splitting $v_i$ into $u_i$, $v_i$ for $i = 1, 2$ (see $F'_2$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$.
(i) If $\phi'(wu_1) \neq \phi'(uu_2)$ or $\phi'(wu_1) = \phi'(uu_2) \notin \{\phi'(vv_1), \phi'(vv_2)\}$, then we can stick $u_i, v_i$ together for $i = 1, 2$ (if necessary, exchange the colors of $vv_1$ and $vv_2$).

(ii) If $\phi'(wu_1) = \phi'(uu_2) \in \{\phi'(vv_1), \phi'(vv_2)\}$, without loss of generality, suppose that $\phi'(uu_2) = \phi'(vv_1)$. Exchange the colors of $vv_1 (uu_2)$ and $uv$. Therefore, we can stick $u_i, v_i$ together for $i = 1, 2$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(3) Suppose to the contrary that $G$ contains configuration $F_3$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}, v_{i2}$ for $i = 1, 3$ (see $F_3'$ in Figure 1) without producing 5-cycles. Thus $G'$ has a $k$-tnsd-coloring $\phi'$.

(i) If $\phi'(ww_{12}) \neq \phi'(ww_{32})$ or $\phi'(ww_{12}) = \phi'(ww_{32}) \notin \{\phi'(vv_{11}), \phi'(vv_{31})\}$, then we can stick $v_{i1}, v_{i2}$ together for $i = 1, 3$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{31}$).

(ii) If $\phi'(ww_{12}) = \phi'(ww_{32}) \in \{\phi'(vv_{11}), \phi'(vv_{31})\}$, without loss of generality, suppose that $\phi'(ww_{12}) = \phi'(vv_{11})$. Then we exchange the colors of $w_{12}$ and $w_{32}$. Therefore, we can stick $v_{i1}, v_{i2}$ together for $i = 1, 3$. Thus, by recoloring, we can obtain a $k$-tnsd-coloring $\phi$ of $G$, a contradiction.

(4) Suppose to the contrary that $G$ contains configuration $F_4$. We obtain a smaller graph $G'$ by splitting $v_i$ into $v_{i1}, v_{i2}$ for $i = 1, 4$ (see $F_4'$ in Figure 1) without producing 5-cycles. Thus $G'$ admits a $k$-tnsd-coloring $\phi'$.

(i) If $\phi'(ww_{12}) \neq \phi'(ww_{42})$ or $\phi'(ww_{12}) = \phi'(ww_{42}) \notin \{\phi'(vv_{11}), \phi'(vv_{41})\}$, then we can stick $v_{i1}, v_{i2}$ together for $i = 1, 4$ (if necessary, exchange the colors of $vv_{11}$ and $vv_{41}$).

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Figure 1. Illustration of Claim 1.

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(ii) If \( \phi'(uv_{12}) = \phi'(zv_{42}) \in \{ \phi'(uv_{11}), \phi'(uv_{41}) \} \), without loss of generality, suppose that \( \phi'(uv_{12}) = \phi'(zv_{42}) = \phi'(uv_{11}) \). Since \( \phi'(uv_{2}) \neq \phi'(uv_{3}) \), suppose that \( \phi'(uv_{2}) \neq \phi'(uv_{12}) \). We exchange the colors of \( uv_{12} \) and \( uv_{2} \). Therefore, we can stick \( v_{i1}, v_{i2} \) together for \( i = 1, 4 \). Thus, by recoloring, we can obtain a \( k \)-tnd-coloring \( \phi \) of \( G \), a contradiction.

It is easy to see that the following claim given in [16] also holds with the graph \( G \) in our proof.

**Claim 2** [16]. In the graph \( G \), the following results holds.

1. Each \( t^- \)-vertex is not adjacent to any \( (7-t^-) \)-vertex, where \( t = 4, 5 \).
2. For each vertex \( v \in V(G) \), if \( d_G^1(v) \geq 1 \), then \( d_G^2(v) = 0 \); if \( d_G^2(v) \geq 2 \), then \( d_G^3(v) = 0 \).
3. If \( d_G^1(v) = 5 \), then \( d_G^2(v) \leq 1 \).
4. If \( d_G^2(v) = 6 \), then \( d_G^3(v) \leq 2 \). Furthermore, if \( d_G^3(v) \geq 1 \), then \( d_G^4(v) \leq 1 \).
5. If \( d_G^3(v) = 7 \), then \( d_G^4(v) \leq 2 \). Furthermore, if \( d_G^4(v) \geq 1 \), then \( d_G^5(v) \leq 2 \).
6. If \( d_G^4(v) = l \) (\( l \geq 8 \)), then \( d_G^5(v) < \lceil \frac{l}{2} \rceil \).
7. If \( d_G^4(v) = l \) (\( l \geq 8 \)) and \( d_G^5(v) \geq 1 \), then \( d_G^6(v) + d_G^5(v) \leq l - 1 \).
8. Each 3-face in \( G \) is a \( (2, 6^+, 6^+) \)-face, a \( (3, 5^+, 5^+) \)-face or a \( (4^+, 4^+, 4^+) \)-face.

**Claim 3.** Each 4-face in \( G \) is a \( (2, 6^+, 3^+, 6^+) \)-face, a \( (3, 6^+, 3, 6^+) \)-face, a \( (3, 5^+, 4^+, 5^+) \)-face or a \( (4^+, 4^+, 4^+, 4^+) \)-face.

**Proof.** Let \( T = v_{12}v_{23}v_{34}v_{1} \) be a 4-face of \( G \), and assume that \( d_G(v_1) \leq d_G(v_i) \), where \( i = 2, 3, 4 \). If \( d_G(v_1) = 2 \), by Claim 2(1), \( d_G(v_2) \geq 6 \). By Claim 1, \( F_1 \) is reducible, thus \( T \) is a \( (2, 6^+, 3^+, 6^+) \)-face. If \( d_G(v_1) = d_G(v_3) = 3 \), by Claim 2(1) and Claim 2(3), \( d_G(v_2) \geq 6 \) and \( d_G(v_4) \geq 6 \), thus \( T \) is a \( (3, 6^+, 3, 6^+) \)-face. If \( d_G(v_1) = 3 \) and \( d_G(v_3) \geq 4 \), by Claim 2(1), \( d_G(v_2) \geq 5 \) and \( d_G(v_4) \geq 5 \), thus \( T \) is a \( (3, 5^+, 4^+, 5^+) \)-face. If \( d_G(v_1) \geq 4 \) and \( d_G(v_3) \geq 4 \), by Claim 2(1), \( d_G(v_2) \geq 4 \) and \( d_G(v_4) \geq 4 \), thus \( T \) is a \( (4^+, 4^+, 4^+, 4^+) \)-face.

Let \( H \) be the graph obtained from \( G \) by removing all 1-vertices. By Claims 1–3, we have the following facts.

**Fact 1.** For the graph \( H \), we have \( \delta(H) \geq 2 \); \( d_H(v) = d_G(v) \), for \( 2 \leq d_G(v) \leq 5 \). If \( d_G(v) \geq 6 \), then \( d_H(v) \geq 5 \).

**Fact 2.**

1. In the graph \( H \), each \( 3^- \)-vertex is not adjacent to any \( 4^- \)-vertex.
2. If \( d_H(v) = 5 \), then \( d_H^2(v) = 0 \) and \( d_H^3(v) \leq 1 \).
3. If \( d_H(v) = 6 \), then \( d_H^3(v) \leq 1 \); furthermore, if \( d_H^2(v) = 1 \), then \( d_H^3(v) = 0 \); if \( d_H^2(v) = 0 \), then \( d_H^3(v) \leq 2 \).
(4) If \( d_H(v) = 7 \), then \( d_H^2(v) \leq 2 \); furthermore, if \( d_H^2(v) = 2 \), then \( d_H^3(v) = 0 \); if \( d_H^2(v) = 1 \), then \( d_H^3(v) \leq 1 \).

(5) If \( d_H(v) = l \) (\( l \geq 8 \)), then \( d_H^2(v) \leq l - 1 \).

**Fact 3.**

(1) Each 3-face in \( H \) is a \((2, 6^+, 6^+)^{-}\)-face, a \((3, 5^+, 5^+)^{-}\)-face or a \((4^+, 4^+, 5^+)^{-}\)-face.

(2) Each 4-face in \( H \) is a \((2, 6^+, 3^+, 6^+)^{-}\)-face, a \((3, 6^+, 3^+, 6^+)^{-}\)-face, a \((3, 5^+, 4^+, 5^+)^{-}\)-face or a \((4^+, 4^+, 4^+, 4^+)^{-}\)-face.

A \((2, 6^+, 6^+)^{-}\)-face or a \((3, 5^+, 5^+)^{-}\)-face is called a *bad* 3-face. A \((4^+, 5^+, 5^+)^{-}\)-face is called a *normal* 3-face. A \((2, 6^+, 3^+, 6^+)^{-}\)-face or a \((3, 6^+, 3^+, 6^+)^{-}\)-face is called a *bad* 4-face, and other 4-face is a *normal* 4-face. We use \( n'_i(v) \), \( n''_i(v) \) to denote the number of bad \( i \)-faces and the number of normal \( i \)-faces incident with \( v \) in \( H \), respectively, \( i = 3, 4 \).

Since \( G \) has no 5-cycles, we have the following fact.

**Fact 4.** These configurations are reducible to \( H \):

(1) a 5-face,

(2) a 3-face adjacent to two 3-faces,

(3) a 3-face adjacent to a 4-face, and they are sharing only one edge.

By Fact 4, we have the following fact.

**Fact 5.** If \( d_H(v) = l \) and \( n'_3(v) > 0 \), then \( n'_3(v) + n'_4(v) \leq l - 2 \).

By Euler’s formula, we have

\[
\sum_{v \in V(H)} (2d_H(v) - 6) + \sum_{f \in F(H)} (d_H(f) - 6) = -12.
\]

We will use the discharging method to obtain a contradiction. First, we give an initial charge function: \( w(v) = 2d_H(v) - 6 \) for each \( v \in V(H) \); \( w(f) = d_H(f) - 6 \) for each \( f \in F(H) \). Next, we will design some discharging rules. Let \( w' \) be the new charge after the discharging process. It suffices to show that \( w'(x) \geq 0 \) for each \( x \in V(H) \cup F(H) \), which leads to a contradiction.

In the following, a \( k \)-face means a \( k \)-face in \( H \), the discharging rules are defined as follows.

**R1** Every 2-vertex \( v \) in \( H \) takes 1 from each neighbor.

**R2** Every 4-vertex \( v \) in \( H \) gives 1 to each incident 3-face, gives \( \frac{1}{2} \) to each incident 4-face.

**R3** Every 5\(^+\)-vertex \( v \) in \( H \) gives \( \frac{3}{2} \) to each incident bad 3-face, gives 1 to each incident normal 3-face.
R4 Every $5^+$-vertex $v$ in $H$ gives 1 to each incident bad 4-face, gives $\frac{3}{4}$ to each incident normal 4-face.

We will verify the new charge of each $x \in V(H) \cup F(H)$. In the following, we use $d(v), d_i(v), n_i(v)$ and $d(f)$ to denote $d_H(v), d'_H(v), n_H^i(v)$ and $d_H(f)$, respectively. We first consider the new charge of each $f \in F(H)$.

- $d(f) = 3$. If $f$ is a bad 3-face, by R3, $w'(f) = 3 - 6 + \frac{3}{2} \cdot 2 = 0$; otherwise, by R2 and R3, $w'(f) = 3 - 6 + 1 \cdot 3 = 0$.
- $d(f) = 4$. If $f$ is a bad 4-face, by R4, $w'(f) = 4 - 6 + 1 \cdot 2 = 0$. If $f$ is a $(2, 6^+, 4^+, 6^+)$-face or a $(3, 5^+, 4^+, 5^+)$-face, by R2 and R4, $w'(f) = 4 - 6 + \frac{3}{2} \cdot 2 + \frac{1}{2} = 0$. If $f$ is a $(4^+, 4^+, 4^+, 4^+)$-face, by R2 and R4, $w'(f) \geq 4 - 6 + \frac{1}{2} \cdot 4 = 0$.
- $d(f) = t (t \geq 6)$. $w'(f) = w(f) = t - 6 \geq 0$.

Next we will consider the new charge of each $v \in V(H)$.

- $d(v) = 2$. By R1, $w'(v) = 2 \cdot 2 - 6 + 1 \cdot 2 = 0$.
- $d(v) = 3$. No rule applies to $v$, $w'(v) = 2 \cdot 3 - 6 = 0$.
- $d(v) = 4$. By Fact 2(1), $d_2(v) = d_3(v) = 0$. If $n_3(v) = 0$, by R2, $w'(v) = 2 \cdot 4 - 6 - \frac{3}{2} \cdot n_4(v) \geq 2 - \frac{3}{2} \cdot 4 = 0$. If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 2$.
  By R2, $w'(v) = 2 \cdot 4 - 6 - 1 \cdot n_3(v) - \frac{3}{2} \cdot n_4(v) \geq 2 - 1 \cdot 2 = 0$.
- $d(v) = 5$. By Fact 2(2), $d_2(v) = 0, d_3(v) \leq 1$, so we have $n_3^i(v) \leq 2$ and $n_4^i(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n_4^0(v) \geq 4 - \frac{3}{2} \cdot 5 = \frac{1}{4} > 0$.
  If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4^0(v) \geq 3$. By R3 and R4, $w'(v) = 2 \cdot 5 - 6 - \frac{3}{2} \cdot n_3^0(v) - \frac{3}{4} \cdot n_4^0(v) \geq 4 - \frac{3}{2} \cdot 2 - 1 = 0$.
- $d(v) = 6$. By Fact 2(3), $d_2(v) \leq 1$.
  (a) $d_2(v) = 1$. By Fact 2(3), $d_3(v) = 0$, so we have $n_3^i(v) \leq 1$ and $n_4^i(v) = 0$.
  If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_4^0(v) \geq 6 - 1 - \frac{3}{2} \cdot 6 = \frac{1}{2} > 0$.
  If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4^0(v) \leq 4$.
  By R1, R3 and R4, $w'(v) = 2 \cdot 6 - 6 - d_2(v) - \frac{3}{2} \cdot n_3^0(v) - \frac{3}{2} \cdot n_4^0(v) \geq 6 - 1 - \frac{3}{2} \cdot 1 - 1 = \frac{1}{2} > 0$.
  (b) $d_2(v) = 0$. If $n_3(v) = 0$, by R4, $w'(v) \geq 2 \cdot 6 - 6 - 1 \cdot n_4(v) \geq 6 - 1 = 0$.
  If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 4$.
  By R3 and R4, $w'(v) \geq 2 \cdot 6 - 6 - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 6 - \frac{3}{2} \cdot 4 = 0$.
- $d(v) = 7$. By Fact 2(4), $d_2(v) \leq 2$.
  (a) $d_2(v) = 2$. By Fact 2(4), $d_3(v) = 0$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3^i(v) = 0$. If $n_3(v) = 0$, by R1 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - \frac{3}{4} \cdot n_4^0(v) \geq 8 - 2 - \frac{3}{4} \cdot 7 = \frac{1}{4} > 0$.
  If $n_3(v) > 0$, by Fact 5, $n_3(v) + n_4(v) \leq 5$.
  Noting that $n_3^0(v) = n_4^0(v) = 0$.
  By R1, R3 and R4, $w'(v) = 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_3^0(v) - \frac{3}{4} \cdot n_4^0(v) \geq 8 - 2 - 1 \cdot 5 = 0$.
  (b) $d_2(v) \leq 1$. If $n_3(v) = 0$, by R1 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - 1 \cdot n_4(v) \geq 8 - 1 - 7 = 0$.
  If $n_3(v) > 0$, by Fact 4 and Fact 5, $n_3(v) \leq 4$ and $n_3(v) + n_4(v) \leq 5$.
  By R1, R3 and R4, $w'(v) \geq 2 \cdot 7 - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 8 - \frac{3}{2} \cdot 4 - 1 = 0$.
- $d(v) = l (l \geq 8)$, by Fact 2(5), $d_2(v) \leq l - 1$. 
(a) $d_2(v) = l - 1$. By Claim 1, $F_1$ and $F_2$ are reducible, so we have $n_3(v) = 0$ and $n_4(v) \leq 2$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 1) - 1 \cdot 2 = l - 7 > 0$.

(b) $d_2(v) = l - 2$.

(b1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 4$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 2) - 4 = l - 8 \geq 0$.

(b2) $n_3(v) > 0$. By Claim 1, $F_1$ and $F_2$ are reducible, and by Fact 4, we have $n_3(v) = 1$ and $n_4(v) = 0$. By R1 and R3, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) \geq 2l - 6 - (l - 2) - \frac{3}{2} = l - \frac{11}{2} > 0$.

(c) $d_2(v) = l - 3$.

(c1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 6$.

If $n_4(v) = 6$, by Claim 1, $F_3$ is reducible, so we have $n_3(v) = 0$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - \frac{3}{4} \cdot n_3'(v) = 2l - 6 - (l - 3) - \frac{3}{4} \cdot 6 = l - \frac{15}{2} > 0$.

If $n_4(v) \leq 5$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - 1 \cdot 5 = l - 8 \geq 0$.

(c2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so we have $n_4(v) \leq 2$. By Claim 1, $F_1$ is reducible, and by Fact 4, we have $n_4(v) \leq 2$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{4} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 3) - \frac{3}{4} \cdot 2 - 2 = l - 8 \geq 0$.

(d) $d_2(v) = l - 4$.

(d1) $n_3(v) = 0$. By Claim 1, $F_1$ is reducible, so we have $n_4(v) \leq 8$.

$n_4(v) = i \ (i = 7, 8)$. By Claim 1, $F_3$ is reducible, so we have $n_3'(v) \leq 8 - i$.

By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_3''(v) \geq 2l - 6 - (l - 4) - 1 \cdot (8 - i) - \frac{3}{4} \cdot (i - (8 - i)) = l - 4 - \frac{i}{2} \geq 0$.

$n_4(v) \leq 6$. By R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - 1 \cdot 6 = l - 8 \geq 0$.

(d2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, so each 2-neighbor of $v$ is not incident with a 3-face. And note that each 3-face is not adjacent to two 3-faces, so we have $n_3(v) \leq 2$.

$n_3(v) = i \ (i = 1, 2)$. By Claim 1, $F_1$ and $F_2$ are reducible, and note that each 3-face is not adjacent to a 4-face, we have $n_4(v) \leq 6 - 2i$. By R1, R3 and R4, $w'(v) \geq 2l - 6 - d_2(v) - \frac{3}{2} \cdot n_3(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 4) - \frac{3}{2} \cdot i - 1 \cdot (6 - 2i) = l - 8 + \frac{i}{2} > 0$.

(e) $d_2(v) = l - 5$.

(e1) $n_3(v) = 0$. If $n_4(v) \leq l - 1$, by R1 and R4, $w'(v) \geq 2l - 6 - d_2(v) - 1 \cdot n_4(v) \geq 2l - 6 - (l - 5) - 1 \cdot (l - 1) = 0$.

Now suppose that $n_4(v) = l$. By Claim 1, $F_1$ is reducible, so we have $d_2(v) \leq \left\lfloor \frac{l}{2} \right\rfloor$. Noting that $d_2(v) = l - 5$, we have $8 \leq l \leq 10$.

By Claim 1, $F_1$, $F_3$ and $F_4$ are reducible, so we have $n_4' \leq 4$. By R1 and R4, $w'(v) = 2l - 6 - d_2(v) - 1 \cdot n_4'(v) - \frac{3}{4} \cdot n_4''(v) \geq 2l - 6 - (l - 5) - 1 \cdot 4 - \frac{3}{4} \cdot (l - 4) = \frac{l}{4} - 2 \geq 0$.

(e2) $n_3(v) > 0$. By Claim 1, $F_2$ is reducible, and by Fact 4, we have $n_3(v) \leq 3$. 

extend the coloring \( \phi \) of a \( \Delta \)-planar graph as the proof of Theorem 3.

Theorem 4 is almost the same as the proof of Theorem 3 except for some details. Let \( G \) be a minimum counterexample to Theorem 4 which is embedded in the plane. Set \( k = \max\{\Delta(G) + 1, 10\} \). By the choice of \( G \), any planar graph \( G' \) without 5-cycles and without adjacent \( \Delta(G) \)-vertices which is smaller than \( G \) has a \( k \)-tsul-coloring \( \phi' \). Similarly, we will choose some \( G' \) and extend the coloring \( \phi' \) of \( G' \) to a desired coloring \( \phi \) of \( G \) to get a contradiction. It is easy to see that all the claims in the proof of Theorem 3 except for Claim 2(6) and Claim 2(7) also hold here. The proof of Claim 2(6) and Claim 2(7) can be seen in [5]. The rest of the proof including the discharging method is the same as the proof of Theorem 3.

### References


doi:10.1016/j.dam.2015.03.013


doi:10.1007/s10114-014-4254-7


Received 5 December 2017
Revised 8 February 2018
Accepted 7 March 2018