A NOTE ON THE FAIR DOMINATION NUMBER
IN OUTERPLANAR GRAPHS

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Abstract

For \( k \geq 1 \), a \( k \)-fair dominating set (or just \( k \)FD-set), in a graph \( G \) is a dominating set \( S \) such that \( |N(v) \cap S| = k \) for every vertex \( v \in V - S \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \)FD-set. A fair dominating set, abbreviated FD-set, is a \( k \)FD-set for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of \( G \) that is not the empty graph, is the minimum cardinality of an FD-set in \( G \). In this paper, we present a new sharp upper bound for the fair domination number of an outerplanar graph.

Keywords: fair domination, outerplanar graph, unicyclic graph.

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1. Introduction

For notation and graph theory terminology not given here, we follow [13]. Specifically, let \( G \) be a simple graph with vertex set \( V(G) = V \) of order \( |V| = n \) and let \( v \) be a vertex in \( V \). The open neighborhood of \( v \) is \( N_G(v) = \{u \in V \mid uv \in E(G)\} \) and the closed neighborhood of \( v \) is \( N_G[v] = \{v\} \cup N_G(v) \). If the graph \( G \) is
clear from the context, then we simply write \( N(v) \) rather than \( N_G(v) \). The degree of a vertex \( v \), is \( \deg(v) = |N(v)| \). A vertex of degree one is called a leaf and its neighbor a support vertex. A strong support vertex is a support vertex adjacent to at least two leaves, and a weak support vertex is a support vertex adjacent to precisely one leaf. For a set \( S \subseteq V \), its open neighborhood is the set \( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighborhood is the set \( N[S] = N(S) \cup S \). The distance \( d(u, v) \) between two vertices \( u \) and \( v \) in a graph \( G \) is the minimum number of edges of a path from \( u \) to \( v \). A graph \( G \) of order at least three is 2-connected if the deletion of any vertex does not disconnect the graph. A cut-vertex in a connected graph is a vertex whose removal disconnect the graph. A maximal connected subgraph without a cut-vertex is called a block. A graph \( G \) is outerplanar if it can be embedded in the plane such that all vertices lie on the boundary of its exterior region. A graph \( G \) is Hamiltonian if there is a spanning cycle in \( G \). For a subset \( S \) of vertices of \( G \), we denote by \( G[S] \) the subgraph of \( G \) induced by \( S \).

A subset \( S \subseteq V \) is a dominating set of \( G \) if every vertex not in \( S \) is adjacent to a vertex in \( S \). The domination number of \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). A vertex \( v \) is said to be dominated by a set \( S \) if \( N[v] \cap S = \emptyset \).

Caro et al. [1] studied the concept of fair domination in graphs. For \( k \geq 1 \), a \( k \)-fair dominating set, abbreviated \( k \)FD-set, in \( G \) is a dominating set \( S \) such that \( |N(v) \cap D| = k \) for every vertex \( v \in V - D \). The \( k \)-fair domination number of \( G \), denoted by \( fd_k(G) \), is the minimum cardinality of a \( k \)FD-set. A \( k \)FD-set of \( G \) of cardinality \( fd_k(G) \) is called a \( fd_k(G) \)-set. A fair dominating set, abbreviated FD-set, in \( G \) is a \( 1 \)FD-set for some integer \( k \geq 1 \). The fair domination number, denoted by \( fd(G) \), of a graph \( G \) that is not the empty graph is the minimum cardinality of an FD-set in \( G \). An FD-set of \( G \) of cardinality \( fd(G) \) is called a \( fd(G) \)-set. The concept of fair domination in graphs was further studied in [9, 10, 11]. There is a close relation between the fair domination number and variant, namely perfect domination number of a graph. A perfect dominating set in a graph \( G \) is a dominating set \( S \) such that every vertex in \( V(G) - S \) is adjacent to exactly one vertex in \( S \). Hence a 1FD-set is precisely a perfect dominating set. The concept of perfect domination was introduced by Cockayne et al. in [4], and Fellows et al. [8] with a different terminology which they called semiperfect domination. This concept was further studied, see for example, [2, 3, 5, 6, 12].

Among other results, Caro et al. [1] proved that \( fd(G) < 17n/19 \) for any maximal outerplanar graph \( G \) of order \( n \), and among open problems posed by Caro et al. [1], one asks to find \( fd(G) \) for other families of graphs.

In this paper, we study fair domination in outerplanar graphs. We present a new sharp upper bound for the fair domination number of outerplanar graphs.

We call a block \( K \) in an outerplanar graph \( G \) a strong-block if \( K \) contains
at least three vertices. We call a vertex \( w \) in a strong-block \( K \) of an outerplanar graph \( G \) a *special cut-vertex* if \( w \) belongs to a shortest path from \( K \) to a strong-block \( K' \neq K \). We call a strong-block \( K \) in an outerplanar graph \( G \) a *leaf-block* if \( K \) contains exactly one special cut-vertex. We denote by \( r(G) \) the number of strong-blocks of a graph \( G \). The following is straightforward.

**Observation 1.** Every outerplanar graph with at least two strong-blocks contains at least two leaf-blocks.

We make use of the following.

**Observation 2** (Caro et al. [1]). Every 1FD-set in a graph contains all its strong support vertices.

**Theorem 3** (Leydolda et al. [14]). An outerplanar graph \( G \) is Hamiltonian if and only if it is 2-connected.

**Theorem 4** (Hajian et al. [9]). If \( G \) is a unicyclic graph of order \( n \), then \( fd_1(G) \leq (n+1)/2 \).

## 2. Main Result

**Theorem 5.** If \( G \) is an outerplanar graph of order \( n \) and size \( m \) with \( r \geq 1 \) strong-blocks, then \( fd(G) \leq (4m - 3n + 3)/2 - r \). This bound is sharp.

**Proof.** Let \( G \) be an outerplanar graph of order \( n \) and size \( m \) with \( r \geq 1 \) strong-blocks. We prove that \( fd_1(G) \leq (4m - 3n + 3)/2 - r \). The result follows from Theorem 4 if \( G \) is a unicyclic graph. Thus assume that \( G \) is not a unicyclic graph. Suppose to the contrary that \( fd_1(G) > (4m - 3n + 3)/2 - r \). Assume that \( G \) has the minimum order, and among all such graphs, we may assume that the size of \( G \) is as minimum as possible. Let \( K_1, K_2, \ldots, K_r \) be the \( r \) strong-blocks of \( G \). By Theorem 3, \( K_j \) is Hamiltonian, for \( 1 \leq j \leq r \). Let \( C' = c_0c_1 \cdots c_{l_i}c_0 \) be a Hamiltonian cycle for \( K_i \), for \( 1 \leq i \leq r \). We proceed with the following Claims 1 and 2.

**Claim 1.** For any \( 1 \leq i \leq r \), if \( c_j \) is a vertex of \( C' \), for some \( j \in \{0,1,\ldots,l_i\} \), such that \( \deg_G(c_j) = 2 \), then \( \deg_G(c_{j+1}) \geq 3 \) and \( \deg_G(c_{j-1}) \geq 3 \), where the calculations in \( j+1 \) and \( j-1 \) are taken modulo \( l_i \).

**Proof.** Assume that \( \deg_G(c_j) = 2 \) for some \( j \in \{0,1,\ldots,l_i\} \). Suppose that \( \deg_G(c_{j+1}) = 2 \). Let \( G' = G - c_jc_{j+1} \). Clearly \( r - 1 \leq r(G') \leq r \). By the choice of \( G \), \( fd_1(G') \leq (4m(G') - 3n(G') + 3)/(2 - r(G')) \leq (4(m-1) - 3n+3)/2 - (r-1) = (4m - 3n + 3)/2 - r - 1 \). Let \( S' \) be a \( fd_1(G') \)-set. If \( |S' \cap \{c_j, c_{j+1}\}| \in \{0,2\} \),
then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $|S' \cap \{c_j, c_{j+1}\}| = 1$. Assume that $c_j \in S'$. Then $c_{j+1} \notin S'$, and $c_{j+2} \in S'$, since $S'$ is a dominating set. Thus $\{c_{j+1}\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Next assume that $c_{j+1} \notin S'$. Then $c_{j} \notin S'$ and $c_{j-1} \in S'$. Thus $\{c_{j}\} \cup S'$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$. So $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $\deg_G(c_{j+1}) \geq 3$. Similarly, $\deg_G(c_{j-1}) \geq 3$. □

**Claim 2.** If $c_j$ is a vertex of $C^i$, for some $j \in \{0, 1, \ldots, l_i\}$, such that $\deg_G(c_j) = 2$, then non of $c_{j+1}$ and $c_{j-1}$ is a support vertex of $G$.

**Proof.** Assume that $\deg_G(c_j) = 2$ for some $j \in \{0, 1, \ldots, l_i\}$. Suppose that $c_{j+1}$ is a support vertex of $G$. Let $G' = G - c_j c_{j+1}$. Clearly $r - 1 \leq r(G') \leq r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 3n + 3)/2 - r - 1 = (4m - 3n + 3)/2 - r - 1$. Let $S'$ be a $fd_1(G')$-set. By Observation 2, $c_{j+1} \in S'$, since $c_{j+1}$ is a strong support vertex of $G'$. If $c_{j-1} \notin S'$, then $S'$ is a 1FD-set for $G$ of cardinality at most $(4m - 3n + 3)/2 - r - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - r - 1$, a contradiction. Thus $c_{j-1} \in S'$ and so $\{c_j\} \cup S'$ is an 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - r$, and so $fd_1(G) \leq (4m - 3n + 3)/2 - r$, a contradiction. Hence $c_{j+1}$ is not a support vertex of $G$. Similarly, $c_{j-1}$ is not a support vertex of $G$. □

We consider the following cases.

**Case 1.** $r = 1$. First assume that $V(G) = \{c_{0}, c_{1}, c_{2}, \ldots, c_{l_i}\}$ and so $n = l_i + 1$. By Claim 1, at least $\lfloor n/2\rfloor$ vertices of $C^i$ are of degree at least 3. Now, we can easily see that $m = \lfloor 1/2 \rfloor \sum_{v \in V(G)} \deg(v) \geq n + \lfloor n/2\rfloor / 2$. (Since $\delta(G) \geq 2$ and at least $\lfloor n/2\rfloor$ vertices of $G$ are of degree at least 3, we have $\sum_{v \in V(G)} \deg(v) \geq 2n + \lfloor n/2\rfloor$.)

Thus $m \geq n + \lfloor n/2\rfloor / 2$. If $n$ is even, then $n \leq (4m - 3n)/2$ and if $n$ is odd, then $n \leq (4m - 3n - 1)/2$. We thus obtain that $n \leq (4m - 3n + 3)/2 - 1$. Now $V(G)$ is a 1FD-set in $G$ of cardinality $n$, and thus $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. We deduce that $V(G) \neq \{c_{0}, c_{1}, c_{2}, \ldots, c_{l_i}\}$. Since $r = 1$, there is a vertex of degree one in $G$. Let $v_d$ be a leaf of $G$ such that $d(v_d, C^i)$ is maximum. Let $v_0 v_1 \cdots v_d$ be the shortest path from $v_d$ to a vertex $v_0 \in C^i$. Clearly, $\{v_0, v_1, \ldots, v_d\} \cap V(C^i) = \{v_0\}$.

Assume that $d \geq 2$. Suppose that $\deg_G(v_{d-1}) = 2$. Let $G' = G - \{v_d, v_{d-1}\}$. Clearly $r(G') = r$. By the choice of $G$, $fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 3n + 3)/2 - 1 = (4m - 3n + 3)/2 - 1$. Let $S'$ be a $fd_1(G')$-set. If $v_{d-2} \notin S'$, then $S' \cup \{v_d\}$ is a 1FD-set in $G$ of cardinality at most $(4m - 3n + 3)/2 - 1$ and so $fd_1(G) \leq (4m - 3n + 3)/2 - 1$, a contradiction. Thus $v_{d-2} \in S'$. Then $S' \cup \{v_{d-1}\}$ is a 1FD-set in $G$ of cardinality at most
(4m - 3n + 3)/2 - 1 and so \(fd_1(G)\) ≤ (4m - 3n + 3)/2 - 1, a contradiction. Thus assume that \(\deg_G(v_{d-1}) \geq 3\). Clearly any vertex of \(N_G(v_{d-1}) - \{v_{d-2}\}\) is a leaf. Let \(G'\) be obtained from \(G\) by removing all leaves adjacent to \(v_{d-1}\). Clearly \(r(G') = r\). By the choice of \(G\), \(fd_1(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4(m - 2) - 3(n - 2) + 3)/2 - 1 = (4m - 3n + 3)/2 - 2\). Let \(S'\) be a \(fd_1(G')\)-set. If \(v_{d-1} \in S'\), then \(S'\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - 2\) and so \(fd_1(G) \leq (4m - 3n + 3)/2 - 2\), a contradiction. Thus assume that \(v_{d-1} \notin S'\). Then \(v_{d-2} \in S'\). Now \(S' \cup \{v_{d-1}\}\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - 1\) and so \(fd_1(G) \leq (4m - 3n + 3)/2 - 1\), a contradiction.

We next assume that \(d = 1\). Let \(D_1 = \{c_1^j \mid \deg_G(c_1^j) = 2\}\) and \(D_2 = \{c_2^j \mid \deg_G(c_2^j) \geq 3\}\). Clearly \(D_1 + |D_2| = n + |D_3|/2\). Observe that \(m = \frac{1}{2} \sum_{v \in V(G)} \deg(v) \geq n + |D_3|/2\). Clearly \(n \geq l_i + 1 + |D_2|\). Thus

\[
\frac{(4m - 3n + 3)}{2} - 1 \geq \frac{(4(n + |D_3|)/2 - 3n + 3} / 2 - 1
\geq \frac{(l_i + 1 + |D_2| + |D_3| + 3)}{2} - 1
\geq \frac{(l_i + 1 + |D_1| + |D_2| + |D_3| + 3)}{2} - 1
= l_i + 3/2 > l_i + 1.
\]

Evidently, \(\{c_0^1, \ldots, c_0^i\}\) is a \(fd_1(G)\)-set of cardinality \(l_i + 1\). Thus \(fd_1(G) < (4m - 3n + 3)/2 - r\), a contradiction.

**Case 2.** \(r \geq 2\). By Observation 1, \(G\) has at least two leaf-blocks. Let \(K_i\) be a leaf-block of \(G\), where \(i \in \{1, 2, \ldots, r\}\). By relabeling of the vertices of \(C^i\) we may assume that \(c_0^i\) is a special cut-vertex of \(G\). Let \(G'\) be the graph obtained by removal of all edges \(c_0^i c_j^i\), with \(c_j^i \in \{c_1^i, \ldots, c_l^i\}\). Clearly \(G'\) has two components. Let \(G_1'\) be the component of \(G'\) containing \(c_1^i\) and \(G_2'\) be the component of \(G'\) containing \(c_0^i\). Clearly, \(\{c_1^i, c_2^i, \ldots, c_l^i\} \subseteq V(G_1')\). We consider the following subcases.

**Subcase 2.1.** \(V(G_1') = \{c_1^i, c_2^i, \ldots, c_l^i\}\). Let \(G_1^* = G[V(G_1')] \cup \{c_0^i\}\). Clearly \(n(G_1^*) = l_i + 1\). By Claim 1, at least \([l_i/2]\) vertices of \(C^i - c_0^i\) are of degree at least 3.

Assume that \(l_i\) is even. Thus at least \(l_i/2\) vertices of \(C^i - c_0^i\) are of degree at least 3. Now, we can easily see that \(m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1')} \deg(v) \geq l_i + 1 + l_i/4\). Let \(G_2^* = G[V(G_2') \cup \{c_1^i, c_i^j\}] - \{c_0^i, c_0^i\}\). Clearly \(n = n(G_2^*) + l_i - 2\), \(m = m(G_2^*) + m(G_1^*) - 2\) and \(r(G_2^*) = r - 1\). By the choice of \(G\), \(fd_1(G_2^*) \leq (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*)\). Let \(S''\) be a \(fd_1(G_2^*)\)-set. By Observation 2, \(c_0^i \in S''\), since \(c_0^i\) is a strong support vertex of \(G_2^*\). Then \(S'' \cup \{c_1^i, c_2^i, \ldots, c_l^i\}\) is
Thus $fd_d(G)$ is even. Thus are of degree at least 3. Now, we can easily see that $m$.

Let $A = \{c_1', c_2', \ldots , c_{l_i}' \} \cup \{c_1, c_2, \ldots , c_{l_i} \}$. Clearly $n = |A| = l_i + 1 + (l_i - 1)/4$. Since $l_i - 1/2$ vertices of $C^i - c_0'$ are of degree at least 3, we thus obtain that precisely $(l_i - 1)/2$ vertices of $C^i - c_0'$ are of degree 3, and so $(l_i + 1)/2$ vertices of $C^i - c_0'$ are of degree two. Now Claim 1 implies that $\deg_G(c_1') = \deg_G(c_0') = 2$.

Thus we obtain that $\deg_{G^i}(c_0') = 2$. Let $A_1 = \{c_j \mid \deg_G(c_j') = 2$ for $1 \leq j \leq l_i \}$ and $A_2 = \{c_1', c_2', \ldots , c_{l_i}' \} \cup A_1$. Clearly $|A_1| = (l_i + 1)/2$ and $|A_2| = (l_i - 1)/2$. Note that $|A_2|$ is even, since the number of odd vertices in every graph (here $G^i$) is even. Thus $|A_1|$ is odd, since $l_i$ is odd and $|A_1| + |A_2| = l_i$. Then $|A_1| \geq 3$, since $c_1', c_i' \in A_1$. Now Claim 1 implies that $A_1 = \{c_1', c_3', \ldots , c_{l_i + 1}'/2, \ldots , c_{l_i}' \}$ and $A_2 = \{c_2', c_4', \ldots , c_{l_i - 1}' \}$.

**Fact 1.** There are two adjacent vertices $c_s', c_t' \in A_2$ such that $|s - t| = 2$.

**Proof.** Note that $l_i \equiv 1 \pmod{4}$, since $\frac{l_i - 1}{2}$ is even. If $l_i = 5$, then $c_2, c_4' \in A_2$ are the desired vertices, since they are the only vertices of $G^*_i$ of degree three. Thus assume that $l_i \geq 9$. If $\{c_{l_i + 1}' + 1, c_{l_i + 1}' - 3 \} \cap N(c_{l_i + 1}' - 1) \neq \emptyset$, then the desired pairs

$$
\begin{align*}
(4m - 3n + 3)/2 - r & \geq (4(m(G^*_2) + m(G^*_1) - 2) - 3(n(G^*_2) + n(G^*_1) - 3) + 3)/2 - r \\
& = (4m(G^*_2) - 3n(G^*_2) + 3)/2 - r(G^*_2) + (4m(G^*_1) - 3(l_i + 1) + 1)/2 - 1 \\
& \geq |S''| + (4(l_i + 1 + l_i/4) - 3l_i - 2)/2 - 1 = |S''| + l_i.
\end{align*}
$$

Thus $fd_d(G) \leq (4m - 3n + 3)/2 - r$, a contradiction.
are \(c_{i+1}^{d} + 1\) and the vertex of \(\{c_{i+1}^{d}, c_{i+1}^{d+1}\} \cap N(c_{i+1}^{d+1})\). Thus assume that \(\{c_{i+1}^{d}, c_{i+1}^{d+1}\} \cap N(c_{i+1}^{d+1}) = \emptyset\). Clearly there is a vertex \(c_i^d \in A_2\) such that \(c_i^d\) is adjacent to \(c_{i+1}^{d+1}\). Without loss of generality, assume that \(t < \frac{h+1}{2} - 3\). Since \(G\) is an outerplanar graph, \(|A_2 \cap \{c_h^d : t + 2 \leq h \leq \frac{h+1}{2} - 3\}|\) is even. Furthermore, since \(G\) is an outerplanar graph, any vertex of \(A_2 \cap \{c_h^d : t + 2 \leq h \leq \frac{h+1}{2} - 3\}\) is adjacent to a vertex of \(A_2 \cap \{c_h^d : t + 2 \leq h \leq \frac{h+1}{2} - 3\}\). Consequently, there are two pairs \(c_{h1}^d, c_{h2}^d \in A_2 \cap \{c_h^d : t + 2 \leq h \leq \frac{h+1}{2} - 3\}\) such that \(c_{h1}^d \in N(c_{h2}^d)\) and \(|h_1 - h_2| = 2\). 

Let \(c_i^d\) and \(c_{i+2}^d\) be two adjacent vertices of \(A_2\) according to Fact 1. Clearly, \(\deg(c_{i+1}^{d}) = 2\). Let \(G^* = G - c_i c_{i-1} c_{i+1} c_{i+2}\). Clearly \(n(G^*) = n, m(G^*) = m - 2\) and \(r - 1 \leq r(G^*) \leq r\). By the choice of \(G\), \(f_{d1}(G^*) \leq (4m(G) - 3n(G) + 3)/2 - r(G^*) \leq (4m - 3n + 3)/2 - r - 3\). Let \(S^*\) be a \(f_{d1}(G^*)\)-set. Since \(c_{i+2}^d\) is a strong support vertex of \(G^*\), by Observation 2, we have \(c_{i+2}^d \in S^*\). If \(c_{i-1}^d \notin S^*\), then \(S^*\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - r - 3\) and \(f_{d1}(G) \leq (4m - 3n + 3)/2 - r - 3\), a contradiction. Thus \(c_{i-1}^d \in S^*\). Then \(S^* \cup \{c_{i}^d, c_{i+1}^d\}\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - r - 1\) and \(f_{d1}(G) \geq (4m - 3n + 3)/2 - r - 1\), a contradiction.

Subcase 2.2. \(V(G') \neq \{c_{i1}, c_{i2}, \ldots, c_{i_j}\}\). Since \(K_i\) is a leaf-block of \(G\), \(G' - C_i\) has some vertex of degree at most one. Let \(v_d\) be a leaf of \(G'\) such that \(d(v_d, C_i - c_{i_j})\) is as maximum as possible, and the shortest path from \(v_d\) to \(C_i\) does not contain \(c_{i_j}\). Let \(v_0 v_1 \ldots v_d\) be the shortest path from \(v_d\) to a vertex \(v_0 \in C_i\).

Suppose that \(d \geq 2\). Assume that \(d_{G}(v_{d-1}) = 2\). Let \(G' = G - \{v_d, v_{d-1}\}\). Clearly \(r(G') = r\). By the choice of \(G\), \(f_{d1}(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') = (4m - 3n + 3)/2 - r = (4m - 3n + 3)/2 - r - 1\). Let \(S'\) be a \(f_{d1}(G')\)-set. If \(v_{d-2} \notin S'\), then \(S' \cup \{v_d\}\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - r\) and \(f_{d1}(G) \leq (4m - 3n + 3)/2 - r\), a contradiction. Thus \(v_{d-2} \in S'\). Then \(S' \cup \{v_{d-1}\}\) is a 1FD-set in \(G\) of cardinality at most \((4m - 3n + 3)/2 - r\) and \(f_{d1}(G) \leq (4m - 3n + 3)/2 - r\), a contradiction. We deduce that \(d_{G}(v_{d-1}) \geq 3\). Clearly any vertex of \(N_{G}(v_{d-1}) - \{v_{d-2}\}\) is a leaf. Let \(G'\) be obtained from \(G\) by removing all leaves adjacent to \(v_{d-1}\). Clearly \(r(G') = r\). By the choice of \(G\), \(f_{d1}(G') \leq (4m(G') - 3n(G') + 3)/2 - r(G') \leq (4m - 3n + 3)/2 - r = (4m - 3n + 3)/2 - r - 1\). Let \(S'\) be a \(f_{d1}(G')\)-set.

We thus assume that \(d = 1\). Let \(D_1 = \{c_j^d \mid \deg_G(c_j^d) = 2\}\), \(D_2 = \{c_j^d \mid c_j^d \notin D_1\}\).
is a support vertex of \( G \) \} and \( D_3 = \{ c_i^j \mid \deg_G(c_i^j) \geq 3 \) and \( c_i^j \) is not a support vertex of \( G \). Clearly \(| D_1 | + | D_2 | + | D_3 | = l_i \). Observe that \(| D_2 | \geq 1 \), since \( d = 1 \). Thus by Claims 1 and 2, \(| D_1 | \leq | D_3 | \). Let \( G_1^* = G[D_1 \cup \{ c_0 \}] \). Observe that \( m(G_1^*) = \frac{1}{2} \sum_{v \in V(G_1^*)} \deg(v) \geq n(G_1^*) + | D_3 | / 2 \). Then \( m(G_1^*) \geq l_i + 1 + | D_2 | \). Let \( G_2^* = \{ G_2^* \cup \{ c_1^j, c_2^j \} \} - \{ c_i^j \} \). Clearly \( n = n(G_2^*) + n(G_1^*) - 3 \), \( m = m(G_2^*) + m(G_1^*) - 2 \) and \( r(G_2^*) = r - 1 \). By the choice of \( G \), \( fd_1(G_2^*) \leq (4m(G_1^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) \). Let \( S'' \) be a \( fd_1(G_2^*) \)-set. By Observation 2, \( c_0^i \in S'' \), since \( c_0^i \) is a strong support vertex of \( G_2^* \). Then \( S'' \cup \{ c_1^j, c_2^j, \ldots, c_i^j \} \) is a 1FD-set for \( G \) of cardinality \( | S'' | + l_i \). On the other hand
\[
(4m - 3n + 3)/2 - r \\
\geq (4m(G_2^*) + m(G_1^*) - 2) - 3(n(G_2^*) + n(G_1^*) - 3) + 3)/2 - r \\
= (4m(G_2^*) - 3n(G_2^*) + 3)/2 - r(G_2^*) + (4m(G_1^*) - 3n(G_1^*) + 1)/2 - 1 \\
\geq | S'' | + (4(n(G_1^*) + | D_3 | / 2) - 3n(G_1^*) + 1)/2 - 1 \\
= | S'' | + (n(G_1^*) + 2| D_3 | + 1)/2 - 1 \\
\geq | S'' | + (l_i + 1 + | D_2 | + 2| D_3 | + 1)/2 - 1 \\
\geq (l_i + | D_2 | + | D_3 | + | D_1 |)/2 \geq | S'' | + l_i.
\]
Thus \( fd_1(G) \leq | S'' | + l_i \leq (4m - 3n + 3)/2 - r \), a contradiction.

To the sharpness, consider a cycle \( C_5 \).

3. Concluding Remarks

As it is noted, Caro et al. [1] proved that \( fd(G) < 17n/19 \) for any maximal outerplanar graph \( G \) of order \( n \). They also proved that \( fd(G) \leq n - 2 \) for any connected graph \( G \) of order \( n \geq 3 \). It is worth-noting that the bound of Theorem 5 improves the bound \( n - 2 \) when \( 4m < 5n + 2r - 7 \). It is also known that every maximal outerplanar graph \( G \) of order at least 3 is 2-connected [7], and thus \( r(G) = 1 \). Therefore, the bound of Theorem 5 improves the bound \( 17n/19 \) when \( 4m < \frac{9n}{19} - 1 \). We have the following conjecture.

Conjecture 6. If \( G \) is a graph of order \( n \) and size \( m \) with \( r \geq 1 \) strong-blocks, then \( fd(G) \leq (4m - 3n + 3)/2 - r \).

References

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