ON THE METRIC DIMENSION OF DIRECTED AND UNDIRECTED CIRCULANT GRAPHS

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Abstract

The undirected circulant graph $C_n(±1, ±2, \ldots, ±t)$ consists of vertices $v_0, v_1, \ldots, v_{n-1}$ and undirected edges $v_iv_{i+j}$, where $0 \leq i \leq n-1, 1 \leq j \leq t$ ($2 \leq t \leq \frac{n}{2}$), and the directed circulant graph $C_n(1, t)$ consists of vertices $v_0, v_1, \ldots, v_{n-1}$ and directed edges $v_iv_{i+1}, v_iv_{i+t}$, where $0 \leq i \leq n-1$ ($2 \leq t \leq n-1$), the indices are taken modulo $n$. Results on the metric dimension of undirected circulant graphs $C_n(±1, ±t)$ are available only for special values of $t$. We give a complete solution of this problem for directed graphs $C_n(1, t)$ for every $t \geq 2$ if $n \geq 2t^2$. Grigorious et al. [On the metric dimension of circulant and Harary graphs, Appl. Math. Comput. 248 (2014) 47–54] presented a conjecture saying that $\dim(C_n(±1, ±2, \ldots, ±t)) = t + p - 1$ for $n = 2tk + t + p$, where $3 \leq p \leq t + 1$. We disprove it by showing that $\dim(C_n(±1, ±2, \ldots, ±t)) \leq t + \frac{t+1}{2}$ for $n = 2tk + t + p$, where $t \geq 4$ is even, $p$ is odd, $1 \leq p \leq t + 1$ and $k \geq 1$.

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1. Introduction

Let $V(G)$ be vertex set of a connected (undirected or directed) graph $G$. The distance $d(u, v)$ between two vertices $u, v$ in an undirected graph is the number of edges in a shortest path between $u$ and $v$. In a directed graph $G$ the distance $d(u, v)$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest directed path from $u$ to $v$. 
A vertex \( w \) resolves two vertices \( u \) and \( v \) if \( d(u, w) \neq d(v, w) \). For an ordered set of vertices \( W = \{w_1, w_2, \ldots, w_z\} \), the representation of distances of \( v \) with respect to \( W \) is the ordered \( z \)-tuple

\[
r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_z)).
\]

A set \( W \subset V(G) \) is a resolving set of \( G \) if every two distinct vertices of \( G \) have different representations of distances with respect to \( W \) (if every two vertices of \( G \) are resolved by some vertex in \( W \)). The metric dimension of \( G \) is the number of vertices in a smallest resolving set and it is denoted by \( \text{dim}(G) \). The \( i \)-th coordinate in \( r(v|W) \) is 0 if and only if \( v = w_i \). Thus in order to prove that \( W \) is a resolving set of \( G \), it suffices to show that \( r(u|W) \neq r(v|W) \) for every two different vertices \( u, v \in V(G) \setminus W \).

The metric dimension is an invariant, which has applications in robot navigation [9], pharmaceutical chemistry [2], pattern recognition and image processing [10]. It has been extensively studied. For example, Imran [5] studied barycentric subdivisions of Cayley graphs and Saputro et al. [12] gave bounds on the metric dimension of the lexicographic product of graphs.

Let \( n, m \) and \( a_1, a_2, \ldots, a_m \) be positive integers such that \( 1 \leq a_1 < a_2 < \cdots < a_m \leq \frac{n}{2} \). The undirected circulant graph \( C_n(\pm a_1, \pm a_2, \ldots, \pm a_m) \) consists of the vertices \( v_0, v_1, \ldots, v_{n-1} \) and undirected edges \( v_iv_{i+a_j} \), where \( 0 \leq i \leq n-1 \), \( 1 \leq j \leq m \); the indices are taken modulo \( n \).

For generators \( a_1, a_2, \ldots, a_m \) such that \( 1 \leq a_1 < a_2 < \cdots < a_m \leq n-1 \), the directed circulant graph \( C_n(a_1, a_2, \ldots, a_m) \) consists of the vertices \( v_0, v_1, \ldots, v_{n-1} \) and directed edges \( v_iv_{i+a_j} \), where \( 0 \leq i \leq n-1 \), \( 1 \leq j \leq m \); the indices are taken modulo \( n \). The directed circulant graph \( C_n(-a_1, -a_2, \ldots, -a_m) \) contains the directed edges \( v_iv_{i-a_j} \).

Circulant graphs form an important family of Cayley graphs. The metric dimension of undirected circulant graphs \( C_n(\pm 1, \pm t) \) was studied for special values of \( t \). Javaid, Rahim and Ali [8] proved that if \( n \equiv 0, 2, 3 \) (mod 4), then \( \text{dim}(C_n(\pm 1, \pm 2)) = 3 \). Borchert and Gosselin [1] showed that if \( n \equiv 1 \) (mod 4), then \( \text{dim}(C_n(\pm 1, \pm 2)) = 4 \). The undirected circulant graphs \( C_n(\pm 1, \pm 3) \) were considered in [7] and the graphs \( C_n(\pm 1, \pm \frac{n}{2}) \) for even \( n \) were investigated in [11]. We study the metric dimension for directed circulant graphs with 2 generators. We give a complete solution of this problem for directed graphs \( C_n(1, t) \) for every \( t \geq 2 \) if \( n \geq 2t^2 \).

Exact values of the metric dimension of undirected graphs \( C_n(\pm 1, \pm 2, \pm 3) \) were given in [1] and [6]. Grigorious et al. [4] showed that \( t+1 \) vertices \( v_0, v_1, \ldots, v_t \) resolve the graph \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) if \( n \equiv r \) (mod \( 2t \)), where \( 2 \leq r \leq t+2 \) and they gave the bound \( \text{dim}(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq r - 1 \) if \( n \equiv r \) (mod \( 2t \)), where \( t+3 \leq r \leq 2t+1 \). They presented a conjecture saying that \( \text{dim}(C_n(\pm 1, \pm 2, \ldots, \pm t)) = t + p - 1 \) for \( n = 2tk + t + p \), where
3 \leq p \leq t + 1. We disprove it for even \( t \geq 4 \) and odd \( p \geq 5 \) by showing that \( \dim(C_n(\pm1, \pm2, \ldots, \pm t)) \leq t + \frac{p-1}{2} \) for \( n = 2tk + t + p \) where \( t \geq 4 \) is even, \( p \) is odd, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \). Note that Chau and Gosselin [3] recently proved that \( \dim(C_n(\pm1, \pm2, \ldots, \pm t)) = t + 1 \) if \( n \equiv 2 \) (mod \( 2t \)) and \( n \equiv t + 1 \) (mod \( 2t \)). They also showed that \( \dim(C_n(\pm1, \pm2, \ldots, \pm t)) = \dim(C_n)(\pm1, \pm2, \ldots, \pm t)) = \dim(C_n+2t)(\pm1, \pm2, \ldots, \pm t) \) for large \( n \), which implies that the metric dimension of the graphs \( C_n(\pm1, \pm2, \ldots, \pm t) \) is completely determined by the congruence class of \( n \) modulo \( 2t \).

2. Directed Circulant Graphs

We study the metric dimension of directed circulant graphs \( C_n(1, t) \). It is easy to see that the graph \( C_n(1, t) \) is isomorphic to the graph \( C_n(-1, -t) \) for \( 2 \leq t \leq n-1 \). We present Theorems 1 and 2 for the graph \( C_n(-1, -t) \), because it is easier to express distances from vertices in a graph to vertices in chosen resolving sets if we consider \( C_n(-1, -t) \) (especially in the proof of Theorem 2).

The distance from the vertex \( v_j \) to the vertex \( v_i \) in \( C_n(-1, -t) \), where \( i, j \in \{0, 1, \ldots, n-1\} \), is

\[
\begin{align*}
(1) \quad d(v_j, v_i) &= \left\lfloor \frac{j-i}{t} \right\rfloor + p, \quad p \equiv (j-i) \text{ (mod } t), \quad \text{if } j \geq i, \\
(2) \quad d(v_j, v_i) &= \left\lfloor \frac{n+j-i}{t} \right\rfloor + p, \quad p \equiv (n+j-i) \text{ (mod } t), \quad \text{if } j < i,
\end{align*}
\]

where \( 0 \leq p \leq t - 1 \).

**Theorem 1.** Let \( t \geq 2 \) and \( n \geq 2t^2 \). Then \( \dim(C_n(-1, -t)) \geq t \).

**Proof.** We prove the result by contradiction. Assume that \( \dim(C_n(-1, -t)) \leq t - 1 \). Let \( W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{t-1}}\} \) be a resolving set of \( C_n(-1, -t) \), where \( 0 \leq i_1 \leq i_2 \leq \cdots \leq i_{t-1} \). Since we have at most \( t - 1 \) different vertices in \( W \) and the graph \( C_n(-1, -t) \) has at least \( 2t^2 \) vertices, \( C_n(-1, -t) \) contains a set of \( 2t \) consecutive vertices \( V' = \{v_j, v_{j+1}, \ldots, v_{j+2t-1}\} \), where \( 0 \leq j \leq n - 1 \), such that no vertex of \( W \) is in \( V' \). Without loss of generality we can assume that \( j = n - 2t \), which means that \( V' = \{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-1}\} \) and \( i_{t-1} < n - 2t \).

Since \( |W| \leq t - 1 \), there is a \( k \in \{0, 1, \ldots, t-1\} \), such that no vertex \( v_i \in W \) satisfies \( i \equiv k \) (mod \( t \)). So we can write any vertex of \( W \) in the form \( v_{tr+s} \), where \( 0 \leq s \leq t - 1 \), \( s \neq k \) and \( r \geq 0 \).

Let \( v_l \) be any vertex in the set of \( t \) vertices \( \{v_{n-2t}, v_{n-2t+1}, \ldots, v_{n-t-1}\} \), such that \( l \equiv k \) (mod \( t \)). Then we can write \( l = tx + k \), where \( 0 \leq k \leq t - 1 \). We show that the vertices \( v_{lx+k}, v_{lx+k+t-1} \in V' \) are not resolved by \( W \). Note that
Let us present an upper bound on the metric dimension of directed circulant graphs with 2 generators.

**Theorem 2.** Let $2 \leq t < n$. Then $\dim(C_n(-1, -t)) \leq t$.

**Proof.** We prove that $W = \{v_0, v_1, \ldots, v_{n-1}\}$ is a resolving set of $C_n(-1, -t)$. First we find all vertices $v_j$ $(1 \leq j \leq n-1)$ of $C_n(-1, -t)$ such that $d(v_j, v_0) = x$ for any $x \geq 1$. We can write $j = tr + p$ where $r \geq 0$ and $0 \leq p \leq t - 1$. Since by (1), $d(v_{tr+p}, v_0) = r + p$, we have $r + p = x$. Thus $r = x - p \geq 0$ and then $v_{l(x-p)+p}$ for $0 \leq p \leq t - 1$ and $1 \leq l(x-p)+p \leq n-1$ are the vertices of $C_n(1, t)$ such that $d(v_{l(x-p)+p}, v_0) = x$.

It remains to show that these vertices are resolved by $v_i$, $i = 1, 2, \ldots, t - 1$. It suffices to consider only those vertices $v_{l(x-p)+p}$ which are not in $W$, so we can assume that $t(x-p)+p > i$. For $i = 1, 2, \ldots, t - 1$, by (1),

\[
d(v_{l(x-p)+p}, v_i) = \begin{cases} x - p + \left\lfloor \frac{p - i}{t} \right\rfloor + p - i = x - i & \text{if } p \geq i, \\ x - p + \left\lfloor \frac{p - i}{t} \right\rfloor + p - i + t = x + t - 1 - i & \text{if } p < i. \end{cases}
\]

We know that the first entry of $r(v_{l(x-p)+p}|W)$ is $x$. From (3) it follows that the next $p$ entries (where $0 \leq p \leq t - 1$) are $x - i$ and the last $t - 1 - p$ entries of $r(v_{l(x-p)+p}|W)$ are $x + t - 1 - i$. So if $p = 0$ (and if $v_{ix}$ exists), the first entry of $r(v_{lx}|W)$ is $x$ and the other entries are $x + t - 1 - i$ which means that $r(v_{lx}|W) = (x, x + t - 2, x + t - 3, \ldots, x + t - 1 - (t - 1))$. If $p = 1$, the first entry of $r(v_{l(x-1)+1}|W)$ is $x$, the second entry is $x - 1$ and the other entries are $x + t - 1 - i$, so $r(v_{l(x-1)+1}|W) = (x, x + t - 2, x + t - 3, \ldots, x + t - 1 - (t - 1)), x - 1, x + t - 1 - i, \ldots, x + t - 1 - 1).$
\[(x, x - 1, x + t - 3, x + t - 4, \ldots, x + t - 1 - (t - 1)). \text{ Similarly } r(v_{t(x-2)+2}|W) = (x, x - 1, x - 2, x + t - 4, \ldots, x + t - 1 - (t - 1)), \ldots, r(v_{t(x-(t-1))+1}+1|W) = (x, x - 1, x - 2, \ldots, x - (t - 1)).\]

Since all vertices \(v_j, 1 \leq j \leq n - 1\), such that \(d(v_j, v_0) = x\) are resolved by \(W\), we have \(\dim(C_n(-1, -t)) \leq |W| = t\). \(\blacksquare\)

From Theorems 1 and 2 we obtain Corollary 3.

**Corollary 3.** Let \(t \geq 2\) and \(n \geq 2t^2\). Then \(\dim(C_n(-1, -t)) = t\).

Since the graphs \(C_n(-1, -t)\) and \(C_n(1, t)\) are isomorphic, we get the following corollary.

**Corollary 4.** Let \(t \geq 2\) and \(n \geq 2t^2\). Then \(\dim(C_n(1, t)) = t\).

### 3. Undirected Circulant Graphs

We give an upper bound on the metric dimension of undirected circulant graphs \(C_n(\pm 1, \pm 2, \ldots, \pm t)\) for \(n \equiv r \mod 2t\), where \(r = 1\) and \(r = t+1, t+3, \ldots, 2t-1\).

The distance between two vertices \(v_i\) and \(v_j\) in \(C_n(\pm 1, \pm 2, \ldots, \pm t)\), where \(0 \leq i < j < n\), is

\[(4) \quad d(v_i, v_j) = \min \left\{ \left\lfloor \frac{j-i}{t} \right\rfloor, \left\lceil \frac{n-(j-i)}{t} \right\rceil \right\}.
\]

This equation can be simplified as

\[(5) \quad d(v_i, v_j) = \begin{cases} \left\lfloor \frac{j-i}{t} \right\rfloor & \text{if } 0 \leq j-i \leq \frac{n}{2}, \\ \left\lceil \frac{n-(j-i)}{t} \right\rceil & \text{if } \frac{n}{2} < j-i < n. \end{cases}
\]

**Theorem 5.** Let \(n = 2tt + t + p\) where \(t \geq 4\) is even, \(p\) is odd, \(1 \leq p \leq t + 1\) and \(k \geq 1\). Then
\[
\dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p+1}{2}.
\]

**Proof.** Let \(n = 2tk + t + p\) where \(k \geq 1, t \geq 4\) is even and \(p = 1, 3, \ldots, t+1\). Let

\[
W_1 = \{v_0, v_2, \ldots, v_{t-2}\}, \quad W_2 = \{v_{t-1}, v_{t+1}, \ldots, v_{2t-3}\}, \\
W_3 = \{v_{tk+t-1}, v_{tk+t+1}, \ldots, v_{tk+t+p-2}\}.
\]

We have \(|W_1| = |W_2| = \frac{t}{2}\) and \(|W_3| = \frac{p+1}{2}\). Let us prove that \(W = W_1 \cup W_2 \cup W_3\) is a resolving set of the graph \(C_n(1, 2, \ldots, t)\).
We divide the vertex set of \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) into four disjoint sets:

\[
V_1 = \{v_0, v_1, \ldots, v_t\}, \quad V_2 = \{v_{t+1}, v_{t+2}, \ldots, v_{tk+t}\},
\]

\[
V_3 = \{v_{tk+t+1}, v_{tk+t+2}, \ldots, v_{tk+t+p-1}\}, \quad V_4 = \{v_{tk+t+p}, v_{tk+t+p+1}, \ldots, v_{n-1}\}.
\]

First we prove that any two vertices of \( V_2 \) have different representations of distances with respect to \( W \). For \( x = 1, 2, \ldots, k - 1; \ j = 1, 2, \ldots, t; \ i = 0, 2, \ldots, t - 2 \), we have \( v_i \in W_1 \) and by (5),

\[
d(v_{tx+j}, v_i) = x + \left\lfloor \frac{j-i}{t} \right\rfloor = \begin{cases} 
  x+1 & \text{if } i < j, \\
  x & \text{if } i \geq j,
\end{cases}
\]

and if \( x = k; \ j = 1, 2, \ldots, t \), by (4), we get

\[
d(v_{tk+j}, v_i) = \min\left\{ \left\lfloor \frac{(tk+j) - i}{t} \right\rfloor, \left\lfloor \frac{n - [(tk+j) - i]}{t} \right\rfloor \right\},
\]

\[
= \min\left\{ k + \left\lfloor \frac{j-i}{t} \right\rfloor, k+1 + \left\lfloor \frac{p+i-j}{t} \right\rfloor \right\} = \begin{cases} 
  k+1 & \text{if } i < j, \\
  k & \text{if } i \geq j.
\end{cases}
\]

Since \( j \) (where \( 1 \leq j \leq t \)) is greater than \( \left\lfloor \frac{1}{2} \right\rfloor \) elements from the set \( \{0, 2, \ldots, t - 2\} \), the first \( \left\lfloor \frac{1}{2} \right\rfloor \) entries of \( r(v_{tx+j}|W_1) \) for \( x = 1, 2, \ldots, k \) are equal to \( x+1 \) and the other \( \frac{1}{2} - \left\lfloor \frac{1}{2} \right\rfloor \) entries are \( x \); \( r(v_{tx+j}|W_1) = (x+1, \ldots, x+1, x, \ldots, x) \). Therefore the only vertices in \( V_2 \) with the same representations of distances with respect to \( W_1 \) are the pairs \( (v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \ldots, (v_{tk+t-1}, v_{tk+t}) \). But since for \( x = 1, 2, \ldots, k \) and \( j = 1, 3, \ldots, t - 3 \), we obtain \( v_{tx+j} \in W_2 \) and by (5),

\[
d(v_{tx+j}, v_{tx+j}) = x - 1, \quad d(v_{tx+j+1}, v_{tx+j}) = x - 1 + \left\lfloor \frac{1}{t} \right\rfloor = x,
\]

and for \( v_{t-1} \in W_2 \), we have

\[
d(v_{tx+t-1}, v_{t-1}) = x, \quad d(v_{tx+t}, v_{t-1}) = x + \left\lfloor \frac{1}{t} \right\rfloor = x + 1,
\]

vertices in \( W_2 \) resolve the pairs \( (v_{t+1}, v_{t+2}), (v_{t+3}, v_{t+4}), \ldots, (v_{tk+t-1}, v_{tk+t}) \). Thus no two vertices in \( V_2 \) have the same representations of distances with respect to \( W \).

Let us study representations of distances of the vertices in \( V_4 \). For \( x = 1, 2, \ldots, k - 1; \ j = 0, 1, \ldots, t - 1; \ i = 0, 2, \ldots, t - 2 \); we have \( v_i \in W_1 \) and by (6),

\[
d(v_{n-tx+j}, v_i) = \left\lfloor \frac{n - [(n-tx+j) - i]}{t} \right\rfloor = x + \left\lfloor \frac{i-j}{t} \right\rfloor = \begin{cases} 
  x & \text{if } i \leq j, \\
  x+1 & \text{if } i > j.
\end{cases}
\]
and if \( x = k \), we get

\[
d(v_{n-tk+j}, v_i) = \min \left\{ \left\lceil \frac{(n-tk+j) - i}{t} \right\rceil, \left\lceil \frac{n - (n-tk+j) - i}{t} \right\rceil \right\}
\]

\[
= \min \left\{ k + 1 + \left\lceil \frac{p - j}{t} \right\rceil, k + \left\lceil \frac{i - j}{t} \right\rceil \right\} = \begin{cases} k & \text{if } i \leq j, \\ k + 1 & \text{if } i > j. \end{cases}
\]

Since \( j \) (where \( 0 \leq j \leq t - 1 \)) is greater than or equal to \( \left\lceil \frac{j}{2} \right\rceil + 1 \) elements from the set \( \{0, 2, \ldots, t-2\} \), the first \( \left\lceil \frac{j}{2} \right\rceil + 1 \) entries of \( r(v_{n-tx+j}|W_1) \) (for \( x = 1, 2, \ldots, k \)) are equal to \( x \) and the other entries are \( x + 1 \). The only vertices in \( V_4 \) with the same representations of distances with respect to \( W_1 \) are the pairs \((v_{n-tk}, v_{n-tk+1}), (v_{n-tk+2}, v_{n-tk+3}), \ldots, (v_{n-2}, v_{n-1})\). We show that most of these pairs are resolved by vertices in \( W_2 \). For \( x = 1, 2, \ldots, k-1 \) and \( j = 1, 3, \ldots, t-3 \), we have \( v_{t+j} \in W_2 \) and by (6),

\[
d(v_{n-tx+j}, v_{t+j}) = x + 1, \quad d(v_{n-tx+j-1}, v_{t+j}) = x + 1 + \left\lceil \frac{1}{2} \right\rceil = x + 2,
\]

and for \( v_{t-1} \in W_2, x = 1, 2, \ldots, k, \) by (6),

\[
d(v_{n-tx+t-1}, v_{t-1}) = x, \quad d(v_{n-tx+t-2}, v_{t-1}) = x + \left\lceil \frac{1}{2} \right\rceil = x + 1,
\]

so vertices of \( W_2 \) resolve all pairs of vertices \((v_{n-tk+t-2}, v_{n-tk+t-1}), (v_{n-tk+t}, v_{n-tk+t+1}), \ldots, (v_{n-2}, v_{n-1})\), which are the pairs \((v_{tk+2t+p-2}, v_{tk+2t+p-1}), (v_{tk+2t+p}, v_{tk+2t+p+1}), \ldots, (v_{n-2}, v_{n-1})\). It remains to resolve the pairs \((v_{tk+t+p}, v_{tk+t+p+1}), (v_{tk+t+p+2}, v_{tk+t+p+3}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})\).

For \( j = 0, 2, \ldots, t-p-3 \), we have \( v_{t+p+j} \in W_2 \) and by (5),

\[
d(v_{tk+t+p+j}, v_{t+p+j}) = k, \quad d(v_{tk+t+p+j+1}, v_{t+p+j}) = k + \left\lceil \frac{1}{2} \right\rceil = k + 1,
\]

so the pairs \((v_{tk+t+p}, v_{tk+t+p+1}), \ldots, (v_{tk+2t-3}, v_{tk+2t-2})\) are resolved by \( W_2 \).

For \( j = t-p-1, t-p+1, \ldots, t-4 \), we have \( v_{tk+p+j} \in W_3 \) and by (5),

\[
d(v_{tk+t+p+j}, v_{tk+p+j}) = 1, \quad d(v_{tk+t+p+j+1}, v_{tk+p+j}) = 1 + \left\lceil \frac{1}{2} \right\rceil = 2,
\]

so the pairs \((v_{tk+2t-1}, v_{tk+2t}), \ldots, (v_{tk+2t+p-4}, v_{tk+2t+p-3})\) are resolved by \( W_3 \). Thus all pairs of vertices in \( V_4 \) are resolved by \( W \).

A vertex \( v \in V_2 \) and a vertex in \( V_3 \) can have the same representations of distances with respect to \( W_1 \) only if all entries of \( r(v|W_1) \) are the same numbers. For \( x = 1, 2, \ldots, k \), we have \( v_{tx+t-1}, v_{tx+t} \in V_2 \) and \( r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = (x+1, \ldots, x+1) \). For \( v_{n-tx+t-2}, v_{n-tx+t-1} \in V_4 \) we have \( r(v_{n-tx+t-2}|W_1) = r(v_{n-tx+t-1}|W_1) = (x, \ldots, x) \), which implies that for \( x = 1, 2, \ldots, k-1 \), we obtain \( r(v_{tx+t-1}|W_1) = r(v_{tx+t}|W_1) = r(v_{n-tx-2}|W_1) = r(v_{n-tx-1}|W_1) \). Since for \( v_{2t-3} \in V_2 \), by (5),

\[
d(v_{tx+t-1}, v_{2t-3}) = x - 1 + \left\lceil \frac{2}{2} \right\rceil = x, \quad d(v_{tx+t}, v_{2t-3}) = x - 1 + \left\lceil \frac{3}{2} \right\rceil = x,
\]
and by (6),
\[ d(v_{n-tx-2}, v_{2t-3}) = x + 2 + \left\lceil \frac{-1}{t} \right\rceil = x + 2, \]
\[ d(v_{n-tx-1}, v_{2t-3}) = x + 2 + \left\lceil \frac{-2}{t} \right\rceil = x + 2, \]
any vertex in \( V_2 \) and any vertex in \( V_4 \) have different representations of distances with respect to \( W \).

We study representations of the vertices in \( V_3 \). For \( j = 1, 2, \ldots, p - 1 \) and \( i = 0, 2, \ldots, t - 2 \), we have \( v_i \in W_1 \) and by (4),
\[ d(v_{tk+t+j}, v_i) = \min \left\{ k + 1 + \left\lceil \frac{j-i}{t} \right\rceil, k + \left\lceil \frac{p+i-j}{t} \right\rceil \right\} = k + 1, \]
thus \( r(v_{tk+t+j}|W_i) = (k + 1, \ldots, k + 1) \). The only vertices in \( V_2 \cup V_4 \) with the same representations of distances with respect to \( W_1 \) are \( v_{tk+t-1} \) and \( v_{tk+t} \).

Let us prove that any two vertices in \( V_3 \cup \{ v_{tk+t-1}, v_{tk+t} \} \) have different representations with respect to \( W \). It suffices to consider the vertices in \( V' = (V_3 \cup \{ v_{tk+t-1}, v_{tk+t} \}) \setminus W_3 = \{ v_{tk+t}, v_{tk+t+2}, \ldots, v_{tk+t+p-1} \} \). For \( j = 0, 2, \ldots, p - 1 \) and \( i = 1, 3, \ldots, t - 3 \), we have \( v_{t+i} \in W_2 \) and by (5)
\[ d(v_{tk+t+j}, v_{t+i}) = k + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} k & \text{if } i \geq j, \\ k + 1 & \text{if } i < j. \end{cases} \]

Since \( j \) (for \( j \leq t - 2 \)) is greater than \( \frac{t}{2} \) elements from the set \( \{1, 3, \ldots, t - 3\} \), the first \( \frac{t}{2} \) elements of \( r(v_{tk+t+j}|W_2) \) where \( W_2 = W_2 \setminus \{v_{t-1}\} \) are equal to \( k + 1 \) and the other \( \frac{t}{2} - \frac{t}{2} - 1 \) entries are \( k \). If \( p = t + 1 \) and \( j = t \), we obtain \( r(v_{tk+t+j}|W_2) = r(v_{tk+2t}|W_2) = (k + 1, \ldots, k + 1) \). It follows that the only vertices of \( V' \) having the same representations of distances with respect to \( W_2 \) are \( v_{tk+2t} \) and \( v_{tk+2t-2} \) if \( p = t + 1 \). These vertices are resolved by \( v_{tk+t-1} \in W_3 \), since by (5),
\[ d(v_{tk+2t}, v_{tk+t-1}) = 1 + \left\lceil \frac{1}{t} \right\rceil = 2 \text{ and } d(v_{tk+2t-2}, v_{tk+t-1}) = 1 + \left\lceil \frac{-1}{t} \right\rceil = 1. \]
Thus all vertices of \( V_3 \) are resolved by \( W_2 \).

We consider the vertices in \( V_1 \). For \( j = 1, 3, \ldots, t - 1 \) and \( t; i = 0, 2, \ldots, t - 2 \), we have \( v_i \in W_1 \) and \( d(v_j, v_i) = \left\lceil \frac{j-i}{t} \right\rceil = 1 \), thus \( r(v_j|W_1) = (1, \ldots, 1) \) for \( v_j \in V_1 \setminus W_1 \). From the previous part of this proof we know that the only vertices in \( V_2 \cup V_3 \cup V_4 \) having the representation with respect to \( W_1 \) equal to \( (1, \ldots, 1) \) are \( v_{n-2} \) and \( v_{n-1} \). Since \( v_{t-1} \in W_2 \), it remains to resolve all pairs of vertices in the set \( V'' = \{ v_1, v_3, \ldots, v_{t-3}; v_t, v_{n-2}, v_{n-1} \} \).

We study their representations with respect to \( W_2 \). For \( j = 1, 3, \ldots, t - 3 \) and \( i = -1, 1, \ldots, t - 3 \), we have \( v_{t+i} \in W_2 \) and by (5),
\[ d(v_j, v_{t+i}) = 1 + \left\lceil \frac{i-j}{t} \right\rceil = \begin{cases} 1 & \text{if } i \leq j, \\ 2 & \text{if } i > j. \end{cases} \]
Since \( j \) is greater than or equal to \( \frac{t+3}{2} \) elements from the set \( \{-1, 1, \ldots, t - 3\} \), the first \( \frac{t+3}{2} \) entries of \( r(v_j|W_2) \) are equal to 1 and the other \( \frac{t}{2} - \frac{t+3}{2} \) entries
are 2. The first two entries of \( r(v_j|W_3) \) are always 1. For \( v_t \) and any \( v_{t+i} \in W_2 \),
\[
d(v_t, v_{t+i}) = \left\lceil \frac{|i|}{t} \right\rceil = 1,
\]
therefore \( r(v_t|W_2) = (1, \ldots, 1) \).

For \( i = -1, 1, \ldots, t-3 \), by (6),
\[
d(v_{n-1}, v_{t+i}) = 1 + \left\lceil \frac{i+1}{t} \right\rceil = \begin{cases} 
1 & \text{if } i = -1, \\
2 & \text{if } i \geq 1,
\end{cases}
\]
so \( r(v_{n-1}|W_2) = (1, 2, \ldots, 2) \). We have \( d(v_{n-2}, v_{t+i}) = 1 + \left\lceil \frac{i+2}{t} \right\rceil = 2 \), thus \( r(v_{n-2}|W_2) = (2, \ldots, 2) \).

The only pair of vertices in \( V'' \) having the same representations with respect to \( W_2 \) is \( (v_{t-3}, v_t) \), which is resolved by \( v_{tk+t-1} \in W_3 \), since by (5) we have
\[
d(v_{t-3}, v_{tk+t-1}) = k + \left\lceil \frac{2}{t} \right\rceil = k + 1 \text{ and } d(v_t, v_{tk+t-1}) = k + \left\lceil \frac{1}{t} \right\rceil = k.
\]

Every two distinct vertices of the graph \( C_n(\pm 1, \pm 2, \ldots, \pm t) \) have different representations of distances with respect to \( W \), thus \( W \) is a resolving set of \( C_n(\pm 1, \pm 2, \ldots, \pm t) \). Hence \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq |W| = t + \frac{p+1}{2} \).

\[\square\]

4. Conclusion

We studied the metric dimension of undirected and directed circulant graphs. Results on the metric dimension of undirected circulant graphs \( C_n(\pm 1, \pm t) \) are available only for special values of \( t \). In Section 2 we found exact values of the metric dimension for directed circulant graphs \( C_n(1, t) \) by showing that if \( t \geq 2 \) and \( n \geq 2t^2 \), then \( \dim(C_n(1, t)) = t \).

In Section 3 we presented a bound on the metric dimension of undirected circulant graphs. We proved that for \( n = 2tk + t + p \), where \( t \geq 4 \) is even, \( p \) is odd, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \), \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p+1}{2} \). Note that by [13], \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p}{2} \) if \( t \) and \( p \) are even, \( 2 \leq p \leq t \), thus we have \( \dim(C_n(\pm 1, \pm 2, \ldots, \pm t)) \leq t + \frac{p}{2} \) for \( n = 2tk + t + p \), where \( t \geq 4 \) is even, \( 1 \leq p \leq t + 1 \) and \( k \geq 1 \),

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References


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