ON TOTAL $H$-IRREGULARITY STRENGTH OF THE DISJOINT UNION OF GRAPHS

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Abstract

A simple graph $G$ admits an $H$-covering if every edge in $E(G)$ belongs to at least one subgraph of $G$ isomorphic to a given graph $H$. For the subgraph $H \subseteq G$ under a total $k$-labeling we define the associated $H$-weight as the sum of labels of all vertices and edges belonging to $H$. The total $k$-labeling is called the $H$-irregular total $k$-labeling of a graph $G$ admitting

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an $H$-covering if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The \textit{total $H$-irregularity strength} of a graph $G$ is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling.

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

\textbf{Keywords:} $H$-covering, $H$-irregular labeling, total $H$-irregularity strength, copies of graphs, union of graphs.

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1. \textbf{Introduction}

Consider a simple and finite graph $G$ with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called labels. If the domain is $V(G) \cup E(G)$ then we call the labeling a \textit{total labeling}. For a total $k$-labeling $\psi : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ the associated total vertex-weight of a vertex $x$ is

\[ wt_\psi(x) = \psi(x) + \sum_{xy \in E(G)} \psi(xy) \]

and the associated total edge-weight of an edge $xy$ is

\[ wt_\psi(xy) = \psi(x) + \psi(xy) + \psi(y). \]

A total $k$-labeling $\psi$ is defined to be an \textit{edge irregular total $k$-labeling} of the graph $G$ if for every two different edges $xy$ and $x'y'$ of $G$ there is $wt_\psi(xy) \neq wt_\psi(x'y')$ and to be a \textit{vertex irregular total $k$-labeling} of $G$ if for every two distinct vertices $x$ and $y$ of $G$ there is $wt_\psi(x) \neq wt_\psi(y)$. This concept was given by Bača, Jendrol’, Miller and Ryan in [8].

The minimum $k$ for which the graph $G$ has an edge irregular total $k$-labeling is called the \textit{total edge irregularity strength} of the graph $G$, $tes(G)$. Analogously, we define the \textit{total vertex irregularity strength} of $G$, $tvs(G)$, as the minimum $k$ for which there exists a vertex irregular total $k$-labeling of $G$.

The following lower bound on the total edge irregularity strength of a graph $G$ is given in [8].

\begin{equation}
tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)| + 2}{3} \right\rceil, \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil \right\},
\end{equation}

where $\Delta(G)$ is the maximum degree of $G$. This lower bound is tight for paths, cycles and complete bipartite graphs of the form $K_{1,n}$. 
Ivančo and Jendrol’ [12] posed a conjecture that for an arbitrary graph $G$ different from $K_5$ with maximum degree $\Delta(G)$, $\text{tes}(G) = \max \{\lfloor |E(G)|/2 \rfloor, \lfloor (\Delta(G) + 1)/2 \rfloor \}$. This conjecture has been verified for complete graphs and complete multipartite graphs in [13, 14], for the categorical product of two cycles and two paths in [2, 4], for generalized Petersen graphs in [11], for generalized prisms in [9], for the corona product of a path with certain graphs in [16] and for large dense graphs with $(|E(G)| + 2)/3 \leq (\Delta(G) + 1)/2$ in [10].

The bounds for the total vertex irregularity strength are given in [8] as follows.

\begin{equation}
\left[ \frac{|V(G)| + \delta(G)}{\Delta(G) + 1} \right] \leq \text{tvs}(G) \leq |V(G)| + \Delta(G) - 2\delta(G) + 1,
\end{equation}

where $\delta(G)$ is the minimum degree of $G$.

Przybyło in [17] proved that $\text{tvs}(G) < 32|V(G)|/\delta(G) + 8$ in general and $\text{tvs}(G) < 8|V(G)|/r + 3$ for $r$-regular graphs. This was then improved by Anholcer, Kalkowski and Przybyło [5] in the following way

\begin{equation}
\text{tvs}(G) \leq 3 \left[ \frac{|V(G)|}{\delta(G)} \right] + 1 \leq \frac{3|V(G)|}{\delta(G)} + 4.
\end{equation}

Recently, Majerski and Przybyło [15] based on a random ordering of the vertices proved that if $\delta(G) \geq (|V(G)|)^{0.5}\ln |V(G)|$, then

\begin{equation}
\text{tvs}(G) \leq \frac{(2+o(1))|V(G)|}{\delta(G)} + 4.
\end{equation}

The exact values for the total vertex irregularity strength for circulant graphs and unicyclic graphs are determined in [1, 6] and [3], respectively.

An edge-covering of $G$ is a family of subgraphs $H_1, H_2, \ldots, H_t$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, \ldots, t$. Then it is said that $G$ admits an $(H_1, H_2, \ldots, H_t)$-edge covering. If every subgraph $H_i$ is isomorphic to a given graph $H$, then the graph $G$ admits an $H$-covering.

Let $G$ be a graph admitting an $H$-covering. For the subgraph $H \subseteq G$ under the total $k$-labeling $\psi$, we define the associated $H$-weight as

$$wt_\psi(H) = \sum_{v \in V(H)} \psi(v) + \sum_{e \in E(H)} \psi(e).$$

A total $k$-labeling $\psi$ is called to be an $H$-irregular total $k$-labeling of the graph $G$ if all subgraphs of $G$ isomorphic to $H$ have distinct weights. The total $H$-irregularity strength of a graph $G$, denoted $\text{ths}(G, H)$, is the smallest integer $k$ such that $G$ has an $H$-irregular total $k$-labeling. This definition was introduced by Ashraf, Bača, Lásicsáková and Semaničová-Feňovčíková [7]. If $H$ is isomorphic to $K_2$, then the $K_2$-irregular total $k$-labeling is isomorphic to the edge irregular total $k$-labeling and thus the total $K_2$-irregularity strength of a graph $G$ is equivalent to the total edge irregularity strength; that is $\text{ths}(G, K_2) = \text{tes}(G)$.

The next theorem gives a lower bound for the total $H$-irregularity strength.
Theorem 1 [7]. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Then
\[ \text{ths}(G, H) \geq \left[ 1 + \frac{t-1}{|V(H)|+|E(H)|} \right]. \]

If $H$ is isomorphic to $K_2$ then from Theorem 1 the lower bound on the total edge irregularity strength given in (1) follows immediately.

The next theorem proves that the lower bound in Theorem 1 is tight.

Theorem 2 [7]. Let $r, s$, $2 \leq s \leq r$, be positive integers. Then
\[ \text{ths}(P_r, P_s) = \left\lceil \frac{s+r-1}{2s-1} \right\rceil. \]

In this paper, we estimate lower and upper bounds on the total $H$-irregularity strength for the disjoint union of multiple copies of a graph and the disjoint union of two non-isomorphic graphs. We also prove the sharpness of the upper bounds.

2. Copies of Graphs

By the symbol $mG$ we denote the disjoint union of $m$ copies of a graph $G$. Immediately from Theorem 1 we obtain a lower bound for the $H$-irregularity strength of $m$ copies of a graph $G$.

Corollary 3. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$ and let $m$ be a positive integer. Then
\[ \text{ths}(mG, H) \geq \left[ 1 + \frac{mt-1}{|V(H)|+|E(H)|} \right]. \]

In the next theorem we give an upper bound for $\text{ths}(mG, H)$.

Theorem 4. Let $G$ be a graph having an $H$-irregular total $\text{ths}(G, H)$-labeling $f$. Let $m$ be a positive integer. Then
\[ \text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left[ \frac{wt_{\text{max}}(H) - wt_{\text{min}}(H) + 1}{|V(H)|+|E(H)|} \right], \]

where $wt_{\text{max}}(H)$ and $wt_{\text{min}}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G, H)$-labeling $f$ of $G$.

Proof. Let $G$ be a graph that admits an $H$-covering given by $t$ subgraphs isomorphic to $H$. We denote these subgraphs as $H^1, H^2, \ldots, H^t$. Assume that $f$ is an $H$-irregular total $k$-labeling of a graph $G$ with $\text{ths}(G, H) = k$. The smallest
weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^{\text{min}}(H)$. Evidently
\begin{equation}
wt_f^{\text{min}}(H) \geq |V(H)| + |E(H)|. \tag{5}
\end{equation}
Analogously, the largest weight of a subgraph $H$ under the total $k$-labeling $f$ is denoted by the symbol $wt_f^{\text{max}}(H)$. It holds that
\begin{equation}
wt_f^{\text{max}}(H) \geq wt_f^{\text{min}}(H) + t - 1 \tag{6}
\end{equation}
and
\begin{equation}
wt_f^{\text{max}}(H) \leq (|V(H)| + |E(H)|)k. \tag{7}
\end{equation}
Thus $f : V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ and
\begin{equation}
\{wt_f(H^j) : j = 1, 2, \ldots, t\} \subset \{wt_f^{\text{min}}(H), wt_f^{\text{min}}(H) + 1, \ldots, wt_f^{\text{max}}(H)\}. \tag{8}
\end{equation}
By the symbol $x_i$, $i = 1, 2, \ldots, m$, we denote an element (a vertex or an edge) in the $i$th copy of $G$, denoted by $G_i$, corresponding to the element $x$ in $G$, i.e., $x \in V(G) \cup E(G)$. Analogously, let $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, be the subgraph in the $i$th copy of $G$ corresponding to the subgraph $H^j$ in $G$.
Let us define the total labeling $g$ of $mG$ in the following way. For $i = 1, 2, \ldots, m$ let
\begin{equation*}
g(x_i) = f(x) + (i - 1)\left\lfloor \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\end{equation*}
Evidently, all the labels are at most
\begin{equation*}
k + (m - 1)\left\lfloor \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor.
\end{equation*}
For the weight of every subgraph $H^j_i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, t$, isomorphic to the graph $H$ under the labeling $g$ we have
\begin{align*}
wt_g(H^j_i) &= \sum_{v \in V(H^j_i)} g(v) + \sum_{e \in E(H^j_i)} g(e) \\
&= \sum_{v \in V(H^j_i)} \left( f(v) + (i - 1)\left\lfloor \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right) \\
&\quad + \sum_{e \in E(H^j_i)} \left( f(e) + (i - 1)\left\lfloor \frac{wt_f^{\text{max}}(H) - wt_f^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rfloor \right)
\end{align*}
\[
\sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e) + |V(H)| (i - 1) \left( \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right) \\
+ |E(H)| (i - 1) \left( \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right) \\
= wt_f(H) + (|V(H)| + |E(H)|) (i - 1) \left( \frac{wt_f^\text{max}(H) - wt_f^\text{min}(H) + 1}{|V(H)| + |E(H)|} \right). 
\]

This means that in the given copy of \( G \) the \( H \)-weights are distinct.

According to (8) we get that the largest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \( i \)th copy of \( G \), \( i = 1, 2, \ldots, m \), denoted by \( wt^\text{max}_g(H : H \subset G_i) \), is at most

\[
wt^\text{max}_g(H : H \subset G_i) \leq wt^\text{max}_f(H) + (|V(H)| + |E(H)|) (i - 1) \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right)
\]

and the smallest weight of a subgraph isomorphic to \( H \) under the total labeling \( g \) in the \((i + 1)\)th copy of \( G \), \( i = 1, 2, \ldots, m - 1 \), denoted by \( wt^\text{min}_g(H : H \subset G_{i+1}) \), is at least

\[
wt^\text{min}_g(H : H \subset G_{i+1}) \geq wt^\text{min}_f(H) + (|V(H)| + |E(H)|) i \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right). 
\]

After some manipulation we get

\[
wt^\text{min}_g(H : H \subset G_{i+1}) \\
\geq wt^\text{min}_f(H) + (|V(H)| + |E(H)|) i \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right) \\
= wt^\text{min}_f(H) + (|V(H)| + |E(H)|) (i - 1) \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right) \\
+ (|V(H)| + |E(H)|) \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right). 
\]

As

\[
\left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right) \geq \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|},
\]

we obtain

\[
wt^\text{min}_g(H : H \subset G_{i+1}) \geq wt^\text{min}_f(H) \\
+ (|V(H)| + |E(H)|) (i - 1) \left( \frac{wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1}{|V(H)| + |E(H)|} \right) \\
+ (wt^\text{max}_f(H) - wt^\text{min}_f(H) + 1)
\]
Thus in all components the \( H \)-weights are distinct. This concludes the proof. ■

We obtain the following corollary.

**Corollary 5.** Let \( G \) be a graph admitting an \( H \)-irregular total \( \text{ths}(G,H) \)-labeling \( f \). Let \( m \) be a positive integer. Then

\[
\text{ths}(mG, H) \leq m \text{ths}(G, H).
\]

**Proof.** Let \( f \) be a \( \text{ths}(G,H) \)-labeling of a graph \( G \) and let \( \text{ths}(G, H) = k \). As \( \text{wt}_f^{\min}(H) \geq |V(H)| + |E(H)| \) and \( \text{wt}_f^{\max}(H) \leq \frac{(|V(H)| + |E(H)|)k}{|V(H)| + |E(H)|} \) we get

\[
\left\lceil \frac{\text{wt}_f^{\max}(H) - \text{wt}_f^{\min}(H)+1}{|V(H)| + |E(H)|} \right\rceil \leq \left\lceil \frac{(|V(H)| + |E(H)|)k - (|V(H)| + |E(H)|) + 1}{|V(H)| + |E(H)|} \right\rceil = k.
\]

Hence, by Theorem 4,

\[
\text{ths}(mG, H) \leq \text{ths}(G, H) + (m - 1) \left\lceil \frac{\text{wt}_f^{\max}(H) - \text{wt}_f^{\min}(H)+1}{|V(H)| + |E(H)|} \right\rceil \leq k + (m - 1)k = mk.
\]

Let \( \{H^1, H^2, \ldots, H^t\} \) be the set of all subgraphs of \( G \) isomorphic to \( H \). Let \( f \) be an \( H \)-irregular total \( k \)-labeling of a graph \( G \) with \( \text{ths}(G, H) = k \) such that

\[
\{ \text{wt}_f(H^j) : j = 1, 2, \ldots, t \} = \{ \text{wt}_f^{\min}(H), \text{wt}_f^{\min}(H) + 1, \ldots, \text{wt}_f^{\min}(H) + t - 1 \}.
\]

(9)

Evidently, if the fraction

\[
\frac{\text{wt}_f^{\max}(H) - \text{wt}_f^{\min}(H)+1}{|V(H)| + |E(H)|} = \frac{t}{|V(H)| + |E(H)|}
\]

is an integer then the weights of all \( H \)-weights in \( mG \) under the total labeling \( g \) of \( mG \) defined in the proof of Theorem 4 constitute the set

\[
\{ \text{wt}_f^{\min}(H), \text{wt}_f^{\min}(H) + 1, \ldots, \text{wt}_f^{\min}(H) + mt - 1 \}.
\]

In particular, this implies that the upper bound for \( \text{ths}(mG, H) \) given in Theorem 4 is tight if \( G \) is a graph that satisfies the conditions mentioned above.
Theorem 6. Let $G$ be a graph admitting an $H$-covering given by $t$ subgraphs isomorphic to $H$. Let $f$ be an $H$-irregular total $\text{ths}(G, H)$-labeling of $G$ such that 
\[
\{wt_f(H^j): j = 1, 2, \ldots, t\} = \{wt_f^{\min}(H), wt_f^{\min}(H) + 1, \ldots, wt_f^{\min}(H) + t - 1\}.
\]
If the fraction $\frac{t}{|V(H)| + |E(H)|}$ is an integer then
\[
\text{ths}(mG, H) \leq \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|}.
\]
Moreover, if $\text{ths}(G, H) = \left[1 + \frac{t}{|V(H)| + |E(H)|}\right] = 1 + \frac{t}{|V(H)| + |E(H)|}$ then
\[
\text{ths}(mG, H) = \text{ths}(G, H) + \frac{(m-1)t}{|V(H)| + |E(H)|} = 1 + \frac{mt}{|V(H)| + |E(H)|}.
\]

Theorem 2 gives the exact value for the total $P_r$-irregularity strength for a path $P_r$. Moreover, the $P_r$-irregular total $(\{(s + r - 1)/(2s - 1)\})$-labeling of $P_r$ described in the proof of Theorem 2 in [7] has the property that the set of $P_r$-weights consists of $t$ consecutive integers, where $t = r - s + 1$ is the number of all subgraphs in $P_r$ isomorphic to $P_s$. As $|V(P_s)| = s$ and $|E(P_s)| = s - 1$ and if the number $(r - s + 1)/(2s - 1)$ is an integer then according to Theorem 6 we get that
\[
\text{ths}(mP_r, P_s) = \text{ths}(P_r, P_s) + (m - 1)\frac{r-s+1}{2s-1} = \left[\frac{r-s+1+2s-1-1}{2s-1}\right] + (m - 1)\frac{r-s+1}{2s-1}
\]
\[
\quad = \left[\frac{r-s+1}{2s-1} + 1 - \frac{1}{2s-1}\right] + (m - 1)\frac{r-s+1}{2s-1}
\]
\[
\quad = \frac{r-s+1}{2s-1} + 1 + (m - 1)\frac{r-s+1}{2s-1} = m\frac{r-s+1}{2s-1} + 1.
\]

Thus we obtain the following result.

Corollary 7. Let $m, r, s, m \geq 1, 2 \leq s \leq r$, be positive integers. If $2s - 1$ divides $r - s + 1$, then
\[
\text{ths}(mP_r, P_s) = \frac{m(r-s+1)}{2s-1} + 1.
\]

If $H$ is isomorphic to $K_2$ then $\text{ths}(G, K_2) = \text{tes}(G)$. Immediately from Theorem 4 the next corollary follows.

Corollary 8. Let $m$ be a positive integer. Then
\[
\left\lfloor \frac{m|E(G)|+2}{3} \right\rfloor \leq \text{ths}(mG, K_2) = \text{tes}(mG) \leq \text{tes}(G) + (m - 1)\left\lfloor \frac{wt^{\max}_f - wt^{\min}_f + 1}{3} \right\rfloor,
\]
where $wt^{\max}_f$ and $wt^{\min}_f$ are the largest and smallest edge weights under a total $\text{tes}(G)$-labeling $f$ of $G$. 
3. Disjoint Union of Two Non-Isomorphic Graphs

In this section we will deal with the total $H$-irregularity strength of two graphs $G_1$ and $G_2$ admitting an $H$-covering. From Theorem 1 we immediately obtain

**Corollary 9.** Let $G_i$, $i = 1, 2$, be a graph admitting an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. Then

$$\text{ths}(G_1 \cup G_2, H) \geq \left[ 1 + \frac{t_1 + t_2 - 1}{|V(H)| + |E(H)|} \right].$$

The next theorem gives an upper bound for $\text{ths}(G_1 \cup G_2, H)$.

**Theorem 10.** Let $G_i$, $i = 1, 2$, be a graph having an $H$-irregular total $\text{ths}(G_i, H)$-labeling $f_i$. Then

$$\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \frac{|w_{f_2}^\text{max}(H) - w_{f_1}^\text{min}(H)| + 1}{|V(H)| + |E(H)|} \right\}, \right.$$

$$\left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \frac{|w_{f_1}^\text{max}(H) - w_{f_2}^\text{min}(H)| + 1}{|V(H)| + |E(H)|} \right\} \right\},$$

where $w_{f_i}^\text{max}(H)$ and $w_{f_i}^\text{min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G_i, H)$-labeling $f_i$ of $G_i$.

**Proof.** Let $G_i$, $i = 1, 2$, be a graph that admits an $H$-covering given by $t_i$ subgraphs isomorphic to $H$. We denote these subgraphs as $H_i^1, H_i^2, \ldots, H_i^{t_i}$. Assume that $f_i$ is an $H$-irregular total $k_i$-labeling of a graph $G_i$ with $\text{ths}(G_i, H) = k_i$. The smallest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $w_{f_i}^\text{min}(H)$. Evidently

(10) $$w_{f_i}^\text{min}(H) \geq |V(H)| + |E(H)|.$$

Analogously, the largest weight of a subgraph $H$ under the total $k_i$-labeling $f_i$ is denoted by the symbol $w_{f_i}^\text{max}(H)$. It holds that

(11) $$w_{f_i}^\text{max}(H) \geq w_{f_i}^\text{min}(H) + t_i - 1$$

and

(12) $$w_{f_i}^\text{max}(H) \leq (|V(H)| + |E(H)|)k_i.$$

Thus $f_i : V(G_i) \cup E(G_i) \to \{1, 2, \ldots, k_i\}$ and

(13) $$\{ w_{f_i}(H_i^j) : j = 1, 2, \ldots, t_i \} \subset \{ w_{f_i}^\text{min}(H), w_{f_i}^\text{min}(H) + 1, \ldots, w_{f_i}^\text{max}(H) \}.$$
Let us define the total labeling $g$ of $G_1 \cup G_2$ in the following way.

\[
g(x) = \begin{cases} f_1(x) & \text{if } x \in V(G_1) \cup E(G_1), \\ f_2(x) + \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil & \text{if } x \in V(G_2) \cup E(G_2). \end{cases}
\]

Evidently, all the labels are not greater than

\[
\max \left\{ k_1, k_2 + \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}.
\]

For the weight of the subgraph $H_1^j$, $j = 1, 2, \ldots, t_1$, isomorphic to the graph $H$ under the labeling $g$ we get

\[
w_t^g(H_1^j) = \sum_{v \in V(H_1^j)} g(v) + \sum_{e \in E(H_1^j)} g(e) = \sum_{v \in V(H_1^j)} f_1(v) + \sum_{e \in E(H_1^j)} f_1(e) = w_{f_1}(H_1^j).
\]

For the weight of the subgraph $H_2^j$, $j = 1, 2, \ldots, t_2$, isomorphic to the graph $H$ under the labeling $g$ we get

\[
w_t^g(H_2^j) = \sum_{v \in V(H_2^j)} g(v) + \sum_{e \in E(H_2^j)} g(e) = \sum_{v \in V(H_2^j)} f_2(v) + \sum_{e \in E(H_2^j)} f_2(e) + |V(H)| \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil + |E(H)| \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil
\]

\[
= w_{f_2}(H_2^j) + \left\lfloor |V(H)| + |E(H)| \right\rfloor \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.
\]

According to (13) we get that the largest weight of a subgraph $H$ under the total labeling $g$ in $G_1$, denoted by $w_{g}^{\max}(H : H \subset G_1)$, is at most

\[
w_{g}^{\max}(H : H \subset G_1) = w_{f_1}^{\max}(H)
\]

and the smallest weight of a subgraph $H$ under the total labeling $g$ in $G_2$, denoted by $w_{g}^{\min}(H : H \subset G_2)$, is at least

\[
w_{g}^{\min}(H : H \subset G_2) \geq w_{f_2}^{\min}(H) + \left\lfloor |V(H)| + |E(H)| \right\rfloor \left\lceil \frac{w_{f_1}^{\max}(H) - w_{f_2}^{\min}(H) + 1}{|V(H)| + |E(H)|} \right\rceil.
\]
Note, that when writing $H_i$ we only consider subgraphs of $G_i$ isomorphic to $H$. As

$$\left\lceil \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \geq \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|},$$

we get

$$wt_{g}^{\text{min}}(H : H \subset G_2) \geq wt_{f_2}^{\text{min}}(H) + (|V(H)| + |E(H)|)\frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|}$$

$$\geq wt_{f_2}^{\text{min}}(H) + (wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1) = wt_{f_1}^{\text{max}}(H) + 1$$

$$> wt_{f_1}^{\text{max}}(H) = wt_{g}^{\text{max}}(H : H \subset G_1).$$

Thus all the $H$-weights under the labeling $g$ in $G_1 \cup G_2$ are distinct.

Analogously we can define the total labeling $h$ of $G_1 \cup G_2$ such that

$$h(x) = f_2(x) \quad \text{if} \quad x \in V(G_2) \cup E(G_2),$$

$$h(x) = f_1(x) + \left\lceil \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \quad \text{if} \quad x \in V(G_1) \cup E(G_1).$$

Using similar arguments we can also show that under the total labeling $h$ the $H$-weights in $G_1 \cup G_2$ are distinct.

Thus $g$ and $h$ are $H$-irregular total labelings of $G$. Immediately from this fact we get

$$\text{ths}(G_1 \cup G_2, H) \leq \min \left\{ \max \left\{ \text{ths}(G_2, H), \text{ths}(G_1, H) + \left\lceil \frac{wt_{f_2}^{\text{min}}(H) - wt_{f_1}^{\text{max}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\}, \right.$$

$$\left. \max \left\{ \text{ths}(G_1, H), \text{ths}(G_2, H) + \left\lceil \frac{wt_{f_1}^{\text{max}}(H) - wt_{f_2}^{\text{min}}(H) + 1}{|V(H)| + |E(H)|} \right\rceil \right\} \right\}.$$
If $H$ is isomorphic to $K_2$ then from Theorem 10 it follows that

$$\text{ths}(G_1 \cup G_2, K_2) = \text{tes}(G_1 \cup G_2)$$

$$\leq \min \left\{ \max \left\{ \text{tes}(G_2), \text{tes}(G_1) + \left\lceil \frac{3\text{tes}(G_1)-2}{3} \right\rceil \right\}, \right.$$ 

$$\max \left\{ \text{tes}(G_1), \text{tes}(G_2) + \left\lceil \frac{3\text{tes}(G_1)-2}{3} \right\rceil \right\} \right\}$$

$$= \text{tes}(G_1) + \text{tes}(G_2)$$

which is equal to the result from Theorem 11.

4. Conclusion

In this paper, we have estimated lower and upper bounds for the total $H$-irregularity strength for the disjoint union of $m$ copies of a graph. We have proved that if a graph $G$ admits an $H$-irregular total $\text{ths}(G,H)$-labeling $f$ and $m$ is a positive integer then

$$\text{ths}(mG,H) \leq \text{ths}(G,H) + (m-1) \left\lceil \frac{w_{f}^{\max}(H)-w_{f}^{\min}(H)+1}{|V(H)|+|E(H)|} \right\rceil,$$

where $w_{f}^{\max}(H)$ and $w_{f}^{\min}(H)$ are the largest and smallest weights of a subgraph $H$ under a total $\text{ths}(G,H)$-labeling $f$ of $G$. This upper bound is tight.

We have also proved an upper bound for the total $H$-irregularity strength for the disjoint union of two non-isomorphic graphs.

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