DECOMPOSITIONS OF CUBIC TRACEABLE GRAPHS

WENZHONG LIU

AND

PANPAN LI

Department of Mathematics
Nanjing University of Aeronautics and Astronautics
Nanjing 210016, China

e-mail: wzhiu7502@muaa.edu.cn
252366887@qq.com

Abstract

A traceable graph is a graph with a Hamilton path. The 3-Decomposition Conjecture states that every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching. We prove the conjecture for cubic traceable graphs.

Keywords: decomposition, cubic traceable graph, spanning tree, matching, 2-regular graph.

2010 Mathematics Subject Classification: 05C70, 05C75.

1. Introduction

In the paper all graphs are finite and simple. The reader can refer to [3, 18] for concepts not defined here. A graph $G$ is cubic if every vertex in $G$ is of degree 3. A spanning tree of $G$ is an acyclic connected subgraph containing all vertices of $G$. A graph that consists of pairwise disjoint edges is called a matching. A $k$-regular spanning subgraph of $G$ is called a $k$-factor. A 1-factor of $G$ is also called a perfect matching. An edge $e$ of $G$ is called a chord of a cycle $C$ in $G$ if the two endpoints of $e$ are on $C$ but $e$ is not itself an edge of $C$. A cycle $C$ is separating in a cubic graph $G$ if either $C$ has a chord, or $G - V(C)$ is disconnected; otherwise, non-separating. A Hamilton cycle is a cycle in $G$ containing all vertices of $G$. A graph with a Hamilton cycle is called a Hamiltonian graph. A Hamilton path
is a path in $G$ containing all vertices of $G$. A graph with a Hamilton path is called a traceable graph. Assume that $H$ is a Hamilton path in $G$. Each edge $e \in E(G) \setminus E(H)$ is called a chord of $H$. For every chord $e = vu$ of $H$, there exists a unique cycle $C_e$ consisting of $e$ and the subpath $vHu$. We call $C_e$ the associated cycle of $e$. A chord $e = st$ of $H$ is minimal if there is no other chord of $H$ whose two endpoints are on the subpath $sHt$.

A decomposition of a graph $G$ consists of pairwise edge-disjoint subgraphs whose union is $G$. It is a canonical problem in structural graph theory to decompose cubic graphs into subgraphs with certain properties. Such a problem can be traced back to the Petersen Theorem [16] that every bridgeless cubic graph has a 1-factor, which implies that each bridgeless cubic graph can be decomposed into a 1-factor and a 2-factor. The Vizing Theorem [17] on proper edge-coloring shows that every cubic graph admits a decomposition consisting of four matchings.

Decompositions of cubic graphs into paths are related to the Fan-Raspaud conjecture [9] that every 2-edge-connected cubic graph contains three perfect matchings with empty intersection. It is interesting to decompose a cubic graph into a spanning tree and other subgraphs. Malkevitch [14] asked which cubic graphs admit a decomposition into a spanning tree and a 2-regular subgraph, that is, a decomposition with a HIST (a homeomorphically irreducible spanning tree is a spanning tree without a 2-degree vertex). Many researchers characterized graphs with a HIST (see [1, 2, 5, 6, 7]). Douglas [8] proved that it is NP-complete to decide whether a graph with maximum degree 3 contains a HIST, which positively solves the problem presented by Albertson, Berman, Hutchinson and Thomassen [2]. It is clear that the complete graph $K_4$ can be decomposed into a HIST (a star) and a 2-regular subgraph (a triangle) while the cube $Q_3$ has no HIST. However, we can decompose $Q_3$ into a spanning tree (with two 2-degree vertices), a 2-regular subgraph (a 4-cycle) and a matching (an edge). See Figure 1. Relaxing the restriction that the spanning tree does not contain a vertex of degree 2, Hoffmann-Ostenhof presented the following conjecture.

Figure 1. A decomposition of $K_4$ with a star (thin line) and a triangle (dot line) in (a) while a decomposition of $Q_3$ with a spanning tree (thin line), a 4-cycle (dot line) and a matching (thick line) in (b).
Conjecture 1 (3-Decomposition Conjecture). Every connected cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching.

Conjecture 1 was first posed in [10] (see also [4, Problem BCC 22.12] and [13]). Ozeki and Ye [15] showed that Conjecture 1 holds for 3-connected cubic graphs on the plane and the projective plane. Hoffmann-Ostenhof, Kaiser and Ozeki [12] proved that Conjecture 1 holds for all connected planar cubic graphs. In [1, 11] it was proved that a cubic Hamiltonian graph admits such a desired decomposition. It was informed that Ye [19] showed Conjecture 1 for 3-connected cubic graphs on the Klein bottle and the torus. In the paper, we prove Conjecture 1 for traceable cubic graphs.

Theorem 2. Every traceable cubic graph can be decomposed into a spanning tree, a 2-regular graph and a matching.

The proof of Theorem 2 consists of four cases (see Section 2). The first case discusses cubic Hamiltonian graphs. The second and third cases are more extensive analyses than the first case. A new technique is used to deal with the fourth case.

2. Proof of Theorem 2

Assume that $G$ is a cubic graph with a Hamilton path $H$. Let the vertices $v_1$ and $v_6$ be the two endpoints of $H$. Then $v_1$ and $v_6$ are incident with two chords of $H$, every other vertex on $H$ is incident with only one chord. If $v_1$ is adjacent to $v_6$ by a chord of $H$, then let the vertex $v_2$ be a neighbor of $v_1$ and $v_5$ be a neighbor of $v_6$ such that the two pairs of vertices are jointed by chords of $H$, respectively. Otherwise, let the vertices $v_2, v_3$ be two neighbors of $v_1$ jointed by chords of $H$ such that these vertices are ordered as $v_1, v_2, v_3$ on $H$, and let the vertices $v_4, v_5$ be two neighbors of $v_6$ jointed by chords of $H$ with the order as $v_4, v_5, v_6$ on $H$.

Lemma 3. Assume that $C$ is a 2-regular non-separating subgraph of $G$ that is the union of associated cycles of chords of $H$, and assume that each of $v_1$ and $v_6$ is joined by a chord of $H$ to at least one vertex of $V(C) \cup \{v_1, v_6\}$. Then there is a decomposition of $G - E(C)$ into a spanning tree of $G$ and a matching.

Proof. Since $C$ is a 2-regular non-separating subgraph of $G$, $G - E(C)$ is connected and has a spanning tree. Let $T$ be a spanning tree of $G - E(C)$ that contains the forest $H - E(C)$, and let $M$ be the subgraph of $G$ induced by $E(G - E(C \cup T))$. Then $M$ is a matching of $G - E(C)$. Thus $G - E(C)$ admits a decomposition consisting of the spanning tree $T$ and the matching $M$. ■
**Proof of Theorem 2.** Let $G$ and $H$ be defined as above. Considering the symmetry of the position of the vertex $v_i (i = 1, 2, \ldots, 6)$ on $H$, we have the following four cases.

*Case 1.* $v_1$ is adjacent to $v_6$ by a chord of $H$.

*Case 2.* $v_4$ is on the subpath $v_1Hv_2$.

*Case 3.* $v_4$ is on the subpath $v_2Hv_3$.

*Case 4.* $v_4$ is on the subpath $v_3Hv_5$.

It is sufficient to show that each case admits a desired decomposition of $G$. See Figure 2.

*Case 1.* $v_1$ is adjacent to $v_6$ by a chord of $H$. In this case, $G$ is a Hamiltonian cubic graph. For completeness we give a proof similar to [1, 11].

Since $G$ is a simple cubic graph, there are other chords of $H$ besides the chord $v_1v_6$. Then there exists a minimal chord $e$ of $H$. Let $C_e$ be the associated cycle of $e$. Then $C_e$ is a non-separating cycle. From Lemma 3, $G - E(C_e)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. So there is a decomposition of $G$ with the 2-regular subgraph $C_e$, the spanning tree $T$ and the matching $M$.

*Case 2.* $v_4$ is on the subpath $v_1Hv_2$. Let $C_1^2 = v_1Hv_4v_6Hv_3v_1$ and $C_2^2 = v_1v_2Hv_3v_1$ be the cycles (see (2) of Figure 2).
Decompositions of Cubic Traceable Graphs

Suppose that $C_1^2$ is a non-separating cycle of $G$. From Lemma 3, $G - E(C_1^2)$ admits a decomposition consisting of a spanning tree $T$ and a matching $M$. Thus we can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_1^2$ and the matching $M$. Otherwise, $C_1^2$ is a separating cycle of $G$. Then there is at least one chord of $C_1^2$ (and of $H$ also) locating on the subpath $v_1Hv_4$, locating on the subpath $v_3Hv_6$, or linking the subpaths $v_1Hv_4$ and $v_3Hv_6$.

Further suppose that $C_2^2$ is a non-separating cycle of $G$. Let $M$ be a set of all chords of $H$ whose two endpoints are not both on $C_2^2$ except the chord $v_1v_6$, and let $T = G - E(C_2^2) - M$. Then $M$ and $T$ are a matching and a spanning tree of $G$ respectively. $T \cup M \cup C_2^2$ forms a desired decomposition of $G$. Otherwise, $C_2^2$ is a separating cycle of $G$. Then there is at least one chord of $C_2^2$ on the subpath $v_2Hv_3$. Now, we discuss three subcases as follows.

Subcase 2.1. There is at least one chord of $C_3^2$ on the subpath $v_1Hv_4$. Since there is at least one chord of $C_3^2$ on the subpath $v_1Hv_4$, we can pick a minimal chord $e_1 = u_1u_2$ of $H$ such that the right endpoint $u_2$ of $e_1$ is the closest to the vertex $v_4$ among all minimal chords of $H$ on the subpath $v_1Hv_4$. Let $C_{e_1}$ be the associated cycle of $e_1$. Similarly, since there is at least one chord of $C_3^2$ on the subpath $v_2Hv_3$, we have a minimal chord $e_2 = u_3u_4$ of $H$ such that the left endpoint $u_3$ of $e_2$ is closest to the vertex $v_2$ among all minimal chords of $H$ on the subpath $v_2Hv_3$. Let $C_{e_2}$ be the associated cycle of $e_2$. Suppose that there is no chord of $H$ which links $C_{e_1}$ and $C_{e_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_1}$ and on $C_{e_2}$ except the chords $v_1v_3$ and $v_4v_6$. Thus $M$ becomes a matching of $G$. Let $T = G - E(C_{e_1} \cup C_{e_2}) - E(M)$. Then $T$ is a spanning tree of $G$. We can give a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_1} \cup C_{e_2}$, and the matching $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{e_1}$ and $C_{e_2}$. Let $u_5u_6$ be such a chord of $H$, and let $C_{e_3}$ be the associated cycle of $e_3$. Suppose that $C_{e_3}$ is a non-separating cycle of $G$. From Lemma 3, $G - E(C_{e_3})$ has a decomposition consisting of a spanning tree $T$ and a matching $M$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{e_3}$ and the matching $M$. Otherwise, $C_{e_3}$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u_5Hu_6$ other than $e_3$. So there must be a minimal chord of $H$ on the subpath $u_5Hu_6$.

Let $e_4$ be a minimal chord of $H$ on the subpath $u_5Hu_6$, and let $C_{e_4}$ be the associated cycle of $e_4$. If the vertex $v_4$ is on $C_{e_4}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_4}$ except the chord $v_1v_3$. Let $T = G - E(C_{e_4}) - E(M)$. So we obtain a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{e_4}$, and the matching $M$. If the vertex $v_2$ is on $C_{e_4}$ and the vertex $v_4$ not on $C_{e_4}$, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{e_4}$ except the chord $v_4v_6$. Let $T = G - E(C_{e_4}) - E(M)$. Thus we can decompose $G$ into the spanning tree $T$,
the 2-regular subgraph \( C_{e_4} \), and the matching \( M \). See Figure 3.

![Figure 3](image-url)

Figure 3. \( v_2 \) is on \( C_{e_4} \) and \( v_4 \) not on \( C_{e_4} \).

So we suppose that neither \( v_2 \) nor \( v_4 \) is on \( C_{e_4} \). According to the choices of \( e_1 \) and \( e_2 \), we deduce that \( e_4 \) must locate on the subpath \( v_4Hv_2 \). Thus there is at least one minimal chord on the subpath \( v_4Hv_2 \) (for example, the minimal chord \( e_4 \)). We pick up a minimal chord, denoted by \( e_4^* \), on the subpath \( v_4Hv_2 \) such that the right endpoint \( u^* \) of \( e_4^* \) is the closed to the vertex \( v_2 \) among all minimal chords of \( H \) on the subpath \( v_4Hv_2 \). Let \( C_{e_4} \) be the associated cycle of \( e_4^* \). Further suppose that there is no chord of \( H \) which links \( C_{e_4} \) and \( C_{e_2} \). Let \( M \) be the set of all chords of \( H \) none of whose two endpoints are on \( C_{e_4}^* \) and on \( C_{e_2} \) except the chords \( v_1v_2 \) and \( v_4v_6 \). Let \( T = G - E(C_{e_4} \cup C_{e_2}) - E(M) \). So we obtain a desired decomposition of \( G \) with \( T \), \( C_{e_4} \cup C_{e_2} \), and \( M \). Otherwise, there is at least one chord of \( H \) which links \( C_{e_4} \) and \( C_{e_2} \). Since neither the subpath \( u^*Hv_2 \) nor the subpath \( v_2Hv_3 \) has any chord, there must exist a minimal chord \( e_5 \) of \( H \) such that its associated cycle \( C_{e_5} \) contains the vertex \( v_2 \). We can employ the same means to get a desired decomposition of \( G \) as the case that \( v_2 \) is on \( C_{e_4} \) and \( v_4 \) not on \( C_{e_4} \). See Figure 4.

Subcase 2.2. There is at least one chord of \( C_2^2 \) on the subpath \( v_3Hv_6 \). Since there is at least one chord of \( C_2^2 \) on the subpath \( v_3Hv_6 \), we choose a minimal chord \( e_1' = u_1'u_2' \) of \( H \) such that the left endpoint \( u_1' \) of \( e_1' \) is the closest to the vertex \( v_3 \) among all minimal chords of \( H \) on the subpath \( v_3Hv_6 \). Let \( C_{e_1} \) be the associated cycle of \( e_1' \). Similarly, since there is at least one chord of \( C_2^2 \) on the subpath \( v_2Hv_3 \), there exists a minimal chord \( e_2' = u_3'u_4' \) of \( H \) such that the right endpoint \( u_3' \) of \( e_2' \) is the closest to the vertex \( v_3 \) among all minimal chords of \( H \) on the subpath \( v_2Hv_3 \). Let \( C_{e_2} \) be the associated cycle of \( e_2' \). Suppose that there is no chord of \( H \) which links \( C_{e_1} \) and \( C_{e_2} \). Let \( M \) be the set of all chords of \( H \) none of whose two endpoints are on \( C_{e_1} \) and on \( C_{e_2} \) except the chords \( v_1v_3 \) and \( v_4v_6 \). Thus \( M \) becomes a matching of \( G \). Let \( T = G - E(C_{e_1} \cup C_{e_2}) - E(M) \). Then \( T \) is a spanning tree of \( G \). We obtain a desired decomposition of \( G \) with
Decompositions of Cubic Traceable Graphs

Let $T$, $C_{v_1}^e$, and $M$. Otherwise, there is at least one chord of $H$ which links the cycles $C_{v_1}^e$ and $C_{v_2}^e$. Let $e_3' = u_5'v_6'$ be such a chord of $H$, and let $C_{v_3}^e$ be the associated cycle of $e_3'$. Suppose that $C_{v_3}^e$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{v_3}^e$ except the chord $v_4v_6$, and let $T = G - E(C_{v_3}^e) - E(M)$. We can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{v_3}^e$ and the matching $M$. Otherwise, $C_{v_3}^e$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $u_5'Hu_6'$ other than $e_3'$. So there must be a minimal chord of $H$ on the subpath $u_5'Hu_6'$. Let $e_4'$ be a minimal chord of $H$ on the subpath $u_5'Hu_6'$, and let $C_{v_3}^e$ be the associated cycle of $e_4'$. According to the definitions of $e_3'$ and $e_4'$, we deduce that $e_4'$ is incident with the vertex $v_3$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{v_3}^e$ except the chord $v_4v_6$, and let $T = G - E(C_{v_3}^e) - E(M)$. So $G$ has the decomposition with the spanning tree $T$, 2-regular subgraph $C_{v_3}^e$ and the matching $M$.

Subcase 2.3. There exists at least one chord of $C_1^2$ which links the subpaths $v_1Hv_4$ and $v_3Hv_6$. From Subcase 2.1 and Subcase 2.2, we only need to consider that neither the subpath $v_1Hv_4$ nor the subpath $v_3Hv_6$ has any chord of $C_1^2$ in the subcase. Since there exists at least one chord of $C_1^2$ which links the subpaths $v_1Hv_4$ and $v_3Hv_6$, we can pick a chord $e_6 = u_7u_8$ whose left endpoint $u_7$ is the closest to the vertex $v_1$ among all chords of $C_1^2$ which link the subpaths $v_1Hv_4$ and $v_3Hv_6$. Since neither the subpath $v_1Hv_4$ nor the subpath $v_3Hv_6$ has any chord of $C_1^2$, so do the subpaths $v_1Hv_7$ and $v_3Hv_8$. Then, we can deduce the cycle $C_3^2 = v_1Hv_4v_3Hv_1$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3^2$ except the chord $v_4v_6$. Let $T = G - E(C_3^2) - E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. So we get a desired decomposition of $G$ with $T$, $C_3^2$, and $M$, see Figure 5.

Figure 4. The minimal chord $e_5$ of $H$ links $C_{v_4}^e$ and $C_{v_2}^e$, and its associated cycle $C_{v_5}$ contains $v_2$. 
Case 3. $v_4$ is on the subpath $v_2Hv_3$. Suppose that there exists a minimal chord $f$ of $H$ on the subpath $v_1Hv_3$ such that its associated cycle $C_f$ contains the vertex $v_4$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_f$ except the chord $v_1v_3$, and let $T = G - E(C_f) - E(M)$. Then $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we have a desired decomposition of $G$ with $T$, $C_f$ and $M$. Otherwise it is sufficient to consider that (3.0) the associated cycle of any minimal chord of $H$ on the subpath $v_1Hv_3$ does not contain $v_4$.

Since there is a chord of $H$ on the subpath $v_1Hv_2$ (for example, the chord $v_1v_2$), we can pick a minimal chord $f_1 = t_1t_2$ of $H$ such that the right endpoint $t_2$ is the closest to the vertex $v_2$ among all minimal chords of $H$ on the subpath $v_1Hv_2$. Note if $f_1$ is the chord $v_1v_2$, then let $t_i = v_i$ ($i = 1, 2$). Let $C_{f_1}$ be the associated cycle of $f_1$. Similar to the subpath $v_5Hv_6$, we can pick a minimal chord $f_2 = t_3t_4$ of $H$ such that the left endpoint $t_3$ is the closest to the vertex $v_5$ among all minimal chords of $H$ on the subpath $v_5Hv_6$. If $f_2$ is the chord $v_5v_6$, then let $t_3 = v_5$ and $t_4 = v_6$. Let $C_{f_2}$ be the associated cycle of $f_2$. Suppose that there is no chord of $H$ which links the cycles $C_{f_1}$ and $C_{f_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_1}$ and on $C_{f_2}$ except the chords $v_1v_3$ and $v_4v_6$. Then $M$ is a matching of $G$. Let $T = G - E(C_{f_1} \cup C_{f_2}) - E(M)$. $T$ is a spanning tree of $G$. So we can decompose $G$ into the spanning tree $T$, the 2-regular subgraph $C_{f_1} \cup C_{f_2}$, and the matching $M$. Otherwise, there exists at least one chord of $H$ which links $C_{f_1}$ and $C_{f_2}$. We can assume that a chord $f_3 = t_5t_6$ of $H$ links $C_{f_1}$ and $C_{f_2}$ and $t_5$ is the left endpoint of $f_3$. Let $C_3^3 = v_1v_2Hv_3v_1$. If $C_3^3$ is a non-separating cycle of $G$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3^3$ except the chord $f_3$, and $T = G - E(C_3^3) - E(M)$. It is clear that $M$ and $T$ are a matching and a spanning tree of $G$, respectively. Thus we obtain a desired decomposition of $G$ with $T$, $C_3^3$ and $M$. Otherwise, $C_3^3$ is a separating cycle of $G$. Then, there is at least one chord of $H$ on the subpath $v_2Hv_3$. Let $f_4$ be any minimal chord of $H$ on the subpath $v_2Hv_3$, and let $C_{f_4}$ be
the associated cycle of $f_4$. From (3.0), $C_{f_4}$ does not contain the vertex $v_4$.

Suppose that there is not any chord of $H$ which links the cycles $C_{f_4}$ and $C_{f_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_4}$ and on $C_{f_2}$ except the chords $v_1v_3$ and $v_1v_6$. Let $T = G - E(C_{f_4} \cup C_{f_2}) - E(M)$. Then $G$ has the desired decomposition $\{T, C_{f_4} \cup C_{f_2}, M\}$. Otherwise, there is a chord of $H$ which links $C_{f_4}$ and $C_{f_2}$. Of course, there is at least one chord of $H$ which links the subpath $t_5 Hv_3$ and $C_{f_2}$. Let $f_5 = t_7 t_8$ be a chord of $H$ linking the subpath $t_5 Hv_3$ and $C_{f_2}$ such that the left endpoint $t_7$ is the closest to the vertex $t_5$ among all chords of $H$ linking the subpath $t_5 Hv_3$ and $C_{f_2}$. Let $C_2^3 = t_5 H t_7 t_8 H t_6 t_5$. If $C_2^3$ is a non-separating cycle of $G$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_2^3$ except the chords $v_1v_3$ and $v_5v_6$. Let $T = G - E(C_2^3) - E(M)$. So we get a desired decomposition of $G$ with $T$, $C_2^3$ and $M$. Otherwise, $C_2^3$ is a separating cycle of $G$. Then there must be at least one chord of $H$ on the subpath $t_5 H t_7$. Let $f_6$ be a minimal chord of $H$ on the subpath $t_5 H t_7$, and let $C_{f_6}$ be the associated cycle of $f_6$. From (3.0), we have that $C_{f_6}$ does not contain the vertex $v_4$. According to the choice of $f_5$, there is no chord of $H$ which links $C_{f_6}$ and $C_{f_2}$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{f_6}$ and on $C_{f_2}$ except the chords $v_1v_3$ and $v_5v_6$. Let $T = G - E(C_{f_6} \cup C_{f_2}) - E(M)$. Thus we have a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{f_6} \cup C_{f_2}$, and the matching $M$, see Figure 6.

![Figure 6](image)

**Figure 6.** Case 3 is illustrated.

**Case 4.** $v_4$ is on the subpath $v_3 Hv_5$. Since there are chords of $H$ on the subpath $v_1 Hv_3$ (for example, the chords $v_1v_2$ and $v_1v_3$), we can choose a minimal chord $g_1 = s_1s_2$ of $H$ on the subpath $v_1 Hv_3$. If $g_1$ is the chord $v_1v_2$, then $s_i = v_i$ ($i = 1, 2$). Let $C_{g_1}$ be the associated cycle of $g_1$. Similarly, let $g_2 = s_3s_4$ be a minimal chord of $H$ on the subpath $v_3 Hv_6$. If $g_2$ is the chord $v_5v_6$, then $s_3 = v_5$ and $s_4 = v_6$. Let $C_{g_2}$ be the associated cycle of $g_2$. If there is no chord of $H$
which links the cycles $C_{g_1}$ and $C_{g_2}$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_{g_1}$ and on $C_{g_2}$ except the chords $v_1v_3$ and $v_4v_6$. Let $T = G - E(C_{g_1} \cup C_{g_2}) - E(M)$. Thus we have a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_1} \cup C_{g_2}$ and the matching $M$. Otherwise, we suppose that

(4.0) the associated cycle of any minimal chord of $H$ on the subpath $v_1Hv_3$ is linked by a chord of $H$ with the associated cycle of each minimal chord of $H$ on the subpath $v_4Hv_6$.

Since the subpath $v_1Hv_2$ has at least one chord of $H$, there is a minimal chord $g_3$ of $H$. If the subpath $v_1Hv_2$ only has the chord $v_1v_2$, then $g_3 = v_1v_2$. Let $C_{g_3}$ be the associated cycle of $g_3$. Similarly, there exists a minimal chord $g_4$ of $H$ on the subpath $v_5Hv_6$. If the subpath $v_5Hv_6$ only has the chord $v_5v_6$, then $g_4 = v_5v_6$. Let $C_{g_4}$ be the associated cycle of $g_4$. From (4.0), there is at least one chord of $H$ which links $C_{g_3}$ and $C_{g_4}$. Let $g_5 = s_5s_6$ be such a chord of $H$. We discuss the following two subcases.

**Subcase 4.1.** There are at least two chords of $H$ which link the subpaths $v_1Hv_3$ and $v_4Hv_6$. Let $g_6 = s_7s_8$ be a chord of $H$ linking the subpaths $v_1Hv_3$ and $v_4Hv_6$ different from $g_3$ such that the left endpoint $s_7$ is the closest to the vertex $s_5$ among all chords of $H$ linking such two subpaths. Suppose that the cycle $C_4^1 = s_5s_7s_8Hs_6s_5$ is a non-separating cycle of $G$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_4^1$ except the chords $v_1v_3$ and $v_4v_6$, and let $T = G - E(C_4^1) - E(M)$. Thus $G$ can be decomposed into the spanning tree $T$, the 2-regular subgraph $C_4^1$ and the matching $M$, see Figure 7.

![Figure 7. The cycle $C_4^1 = s_5s_7s_8Hs_6s_5$ is a non-separating cycle of $G$ (dot line).](image-url)

Otherwise, $C_4^1$ is a separating cycle of $G$. From (4.0) and the choice of $g_6$, we can deduce that there exists at least one chord of $H$ on the subpath $s_8Hs_6$. 


Then we pick a minimal chord $g_7$ of $H$ on the subpath $s_8Hs_6$ such that the right endpoint of $g_7$ is the closest to the vertex $s_6$ among all chords of $H$ on the subpath $s_8Hs_6$. Let $C_{g_7}$ be the associated cycle of $g_7$. According to (4.0), there is at least one chord of $H$ which links the cycles $C_{g_3}$ and $C_{g_7}$. Let $g_8 = s_9s_{10}$ be a chord of $H$ linking $C_{g_3}$ and $C_{g_7}$ such that the right endpoint $s_{10}$ is the closest to the vertex $s_6$ among such all chords of $H$. Then, the cycle $C^4_{g_3} = s_9Hs_8s_6Hs_{10}s_9$ is a non-separating cycle. Since both $s_5$ and $s_9$ are on the associated cycle $C_{g_3}$ of the minimal chord $g_3$, there is no chord of $H$ on the subpath $s_9Hs_5$. Let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C^4_{g_3}$ except the chords $v_1v_3$ and $v_4v_6$, and let $T = G - E(C^4_{g_3}) - E(M)$. So we have a desired decomposition of $G$ with $T$, $C^4_{g_3}$ and $M$, see Figure 8.

**Figure 8.** The cycle $C^4_{g_3} = s_9Hs_8s_6Hs_{10}s_9$ is a non-separating cycle of $G$ (dot line).

**Subcase 4.2.** The chord $g_5$ is the only one chord of $H$ which links the subpaths $v_1Hv_3$ and $v_4Hv_6$. According to (4.0), it can be deduced that the vertex $s_5$ locates between two endpoints of any minimal chord of $H$ on the subpath $v_1Hv_3$. If not, there is a minimal chord of $H$ such that its associate cycle is not incident with $s_5$. Then there is a chord of $H$ different from $g_5$ which links the associated cycle of this minimal chord and $C_{g_4}$, contradiction. Further we can obtain that $s_5$ locates between two endpoints of each chord of $H$ on the subpath $v_1Hv_3$.

Only for convenience, we give a drawing of the graph $G$ here. Except that the chord $g_5$ is arranged on one side of $H$, all chords of $H$ are arranged on the other side of $H$. We discuss two cases as follows.

**Subcase 4.2.1.** There exists a chord $g$ of $H$ such that $g$ intersects at least two chords of $H$ on the subpath $v_1Hv_3$. Let $g_9 = s_{11}s_{12}$ and $g_{10} = s_{13}s_{14}$ be two chords of $H$ intersecting $g$ such that the left endpoint $s_{11}$ of $g_9$ is the closest to the left endpoint $s_{13}$ of $g_{10}$ among such all chords of $H$ on the subpath $v_1Hv_3$. Let the cycle $C^4_3 = s_{11}s_{12}Hs_{14}s_{13}Hs_{11}$. Then $C^4_3$ is a non-separating cycle of $G$. If
none of $g_9$ and $g_{10}$ is the chord $v_1v_3$, then let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3^4$ and on $C_{g_4}$ except the chords $v_1v_3$, $v_4v_6$, and $g$; otherwise, let $M$ be the set of all chords of $H$ none of whose two endpoints are on $C_3^4$ and on $C_{g_4}$ except the chords $v_4v_6$ and $g$. Let $T = G - E(C_3^4 \cup C_{g_4}) - E(M)$. Thus the graph $G$ can be decomposed into the spanning tree $T$, the 2-regular subgraph $C_3^4 \cup C_{g_4}$ and the matching $M$, see Figure 9.

Figure 9. (1) $g$ intersects the chords $g_9$ and $g_{10}$ on the subpath $v_1Hv_3$, and none of $g_9$ and $g_{10}$ is the chord $v_1v_3$; (2) $g$ intersects the chords $g_9$ and $g_{10}$ on the subpath $v_1Hv_3$, and one of $g_9$ and $g_{10}$ is the chord $v_1v_3$.

**Subcase 4.2.2.** There is not any chord of $H$ that intersects two chords on the subpath $v_1Hv_3$. Suppose that there is a chord $g^1 = s^1s^2$ of $H$ such that the endpoint $s^1$ is on the subpath $v_1Hs_5$ and the endpoint $s^2$ is on the subpath $v_2Hv_3$ (the former case for short). On the subpath $v_1Hs^2$, we start from the second edge and choose every other edge along the direction from $v_1$ to $s^2$. Otherwise, there is not any chord of $H$ one of which endpoints is on the subpath $v_1Hs_5$ and the other on the subpath $v_2Hv_3$ (the latter case for short). On the subpath $v_1Hv_2$, we start from the second edge and choose every other edge along the direction from $v_1$ to $v_2$. Let $M_0$ be the set of the chosen edges in both cases. Then $M_0$ is a matching of $G$. Let $V$ be the set of vertices on the subpath $v_1Hs^2$ for the former case or the set of vertices on the subpath $v_1Hv_2$ for the latter case. We first prove the following claim.

**Claim.** Let $M_0$, $V$, the former case, and the latter case be defined as above. Then the subgraph $G[V] - E(M_0)$ is a path, where $G[V]$ is a subgraph of $G$ induced by $V$.

**Proof.** Let $G_1 = G[V] - E(M_0)$. Since the vertex $s_5$ locates between the two endpoints of each chord of $H$ on the subpath $v_1Hv_3$, $V$ consists of $s_5$ and the union of the two endpoints of each chord on the subpath $v_1Hs^2$ for the former case or on the subpath $v_1Hv_2$ for the latter case. $|V|$ is odd. Both the subpath $v_1Hs^2$ and the subpath $v_1Hv_2$ have an even number of edges. According to the choice of $M_0$, all vertices of $G_1$ are of degree 2 except two 1-degree vertices $s_5$ and $s^2$ for
the former case or two 1-degree vertices $s_5$ and $v_2$ for the latter case. It suffices to prove that $G_1$ is connected.

Suppose that $G_1$ is disconnected. The components of $G_1$ consist of one path and some cycles according to the degree condition of $G_1$. Let $C$ be a component of $G_1$ which is a cycle. In $G_1$, $s_5$ is not incident with $C$ since $s_5$ is of degree 1. Let $t_1$ and $t_2$ be two vertices of $C$ such that $t_1$ is the closest to $s_5$ among all vertices of $C$ which locate on the left side of $s_5$ and $t_2$ is the closest to $s_5$ among all vertices of $C$ which locate on the right side of $s_5$. Let the path $P = t_1Hs_5Ht_2$. Then the edges incident with $t_1$ and $t_2$ on $P$ are edges of $M_0$. So $P$ has an odd number of edges and an even number of vertices according to the choice of $M_0$. We can deduce that there is a chord $g^*$ of $H$ such that it only has one endpoint on $P$. The endpoint of $g^*$ not on $P$ can not be on $C$ according to the choice of $M_0$ and $P$. Then $g^*$ intersects at least two edges of $C$ which are chords of $H$ on the subpath $v_1Hv_3$, contraction with assumptions in Subcase 4.2.2. So $G_1$ is connected, and is a path.

Let the subpath $P_1 = s^2Hv_6$ for the former case or $P_1 = v_2Hv_6$ for the latter case. Let $M_1$ be the set of all chords of $H$ on $P_1$ none of whose two endpoints are on $C_{g_4}$ except the chord $v_4v_6$. Let $M = M_0 \cup M_1 \cup v_1v_3$, and let $T = G - E(C_{g_4}) - E(M)$. Thus we get a desired decomposition of $G$ with the spanning tree $T$, the 2-regular subgraph $C_{g_4}$ and the matching $M$, see Figure 10.

From Theorem 2, we have the following corollary.

**Corollary 4.** Let $G$ be a connected cubic graph with $n$ vertices and girth at least $(n - 1)$. Then $G$ can be decomposed into a spanning tree, a 2-regular graph and a matching.
Acknowledgements

The authors would like to thank the referees for their careful reading and useful suggestions. The first author was supported by Fundamental Research Funds for the Central Universities NO. NZ2015106.

References


Received 6 February 2017
Revised 18 January 2018
Accepted 18 January 2018