ON 2-ABSORBING FILTERS OF LATTICES

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Abstract

Let $L$ be a lattice with 1. In this paper we study the concept of 2-absorbing filter which is a generalization of prime filter. A proper filter $F$ of $L$ is called a 2-absorbing filter (resp. a weakly 2-absorbing) if whenever $x_1 \lor x_2 \lor x_3 \in F$ (resp. $1 \neq x_1 \lor x_2 \lor x_3 \in F$), for $x_1, x_2, x_3 \in L$, then there are 2 of the $x_i$’s whose join is in $F$. A basic number of results concerning 2-absorbing filters and weakly of 2-absorbing filters are given in the case when $L$ is distributive.

Keywords: lattice, filter, 2-absorbing filter, weakly 2-absorbing filter.

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1. Introduction

Recently, the study of the 2-absorbing property in the rings, modules, and semigroups has become quite popular. In many ways this program began with the paper in 2007, by Ayman Badawi, [2]. He introduced, for a commutative ring $R$, the notion of 2-absorbing ideals of $R$. A proper ideal $I$ of $R$ is called a 2-absorbing ideal if whenever $x_1x_2x_3 \in I$ for $x_1, x_2, x_3 \in R$, then there are 2 of the $x_i$’s whose product is in $I$. There have been several generalizations and extensions of this concept in the literature (see e.g. [1, 3, 5], and [10]).

In this paper, we are interested in investigating 2-absorbing filters to use other notions of 2-absorbing and associate which exist in the literature as laid
forth in [2]. Now we summarize the content of the paper. Among many results in this paper, in Section 2, it is shown (Theorem 2.2) that the only weakly 2-absorbing filters of \( L \) that are not 2-absorbing can only be \( \{1\} \) (so if \( L \) is an \( L \)-domain, then a filter is 2-absorbing if and only if it is weakly 2-absorbing), and \( F \) is a 2-absorbing filter of \( L \) if and only if whenever \( F_1 \lor F_2 \lor F_3 \subseteq F \) for some filters \( F_1, F_2, F_3 \) of \( L \), then \( F_1 \lor F_2 \subseteq F \) or \( F_1 \lor F_3 \subseteq F \) or \( F_2 \lor F_3 \subseteq F \) (Theorem 2.5). It is shown (Theorem 2.8) that If \( F \) is a 2-absorbing filter of \( L \), then either \( F \) is a prime filter or \( F = p \cap q = p \lor q \), where \( p, q \) are the only distinct filters of \( L \) that are minimal over \( F \). Let \( G \) be a 2-absorbing subfilter of a filter \( F \) of \( L \). It is shown (Theorem 2.14 and Theorem 2.15) that either \( \text{Ass}_L(G :_L F) \) is a totally ordered set or \( \text{Ass}_L(G :_L F) \) is the union of two totally ordered sets. Payrovi and Babaei [10], using the technique of efficient covering of submodules (see [8]) proved the avoidance theorem for 2-absorbing submodules. They proved that if a submodule \( N \) of a module is contained in the union of a finite number of 2-absorbing submodules with some conditions, then \( N \) must be contained in one of them. Section 3 is devoted to prove that the 2-absorbing avoidance theorem. More precisely, let \( F, F_1, F_2, \ldots, F_n (n \geq 2) \) be filters of \( L \) such that at most two of \( F_1, F_2, \ldots, F_n \) are not 2-absorbing. If \( F \subseteq \bigcup_{i=1}^{n} F_i \) and \( F_i \not\subseteq (F_j :_L x) \) for all \( x \in L \setminus F_j \) whenever \( i \neq j \), then \( F \subseteq F_i \) for some \( i \) with \( 1 \leq i \leq n \) (Theorem 3.4).

Let us briefly review some definitions and tools that will be used later. A lattice is a poset \((L, \leq)\) in which every couple elements \( x, y \) has a g.l.b. (called the meet of \( x \) and \( y \), and written \( x \land y \)) and a l.u.b. (called the join of \( x \) and \( y \), and written \( x \lor y \)). A lattice \( L \) is complete when each of its subsets \( X \) has a l.u.b. and a g.l.b. in \( L \). Setting \( X = L \), we see that any nonvoid complete lattice contains a least element 0 and greatest element 1 (in this case, we say that \( L \) is a lattice with 0 and 1). A lattice \( L \) is called a distributive lattice if \((a \lor b) \land c = (a \land c) \lor (b \land c)\) for all \( a, b, c \) in \( L \) (equivalently, \( L \) is distributive if \((a \land b) \lor c = (a \lor c) \land (b \lor c)\) for all \( a, b, c \) in \( L \)). A non-empty subset \( F \) of a lattice \( L \) is called a filter, if for \( a \in F, b \in L, a \leq b \) implies \( b \in F \), and \( x \land y \in F \) for all \( x, y \in F \) (so if \( L \) is a lattice with 1, then 1 \( \in F \) and \( \{1\} \) is a filter of \( L \)). A lattice \( L \) with 1 is called \( L \)-domain if \( a \lor b = 1 \) (\( a, b \in L \)), then \( a = 1 \) or \( b = 1 \). A proper filter \( F \) of \( L \) is called prime if \( x \lor y \in F \), then \( x \in F \) or \( y \in F \). Let \( L \) be a lattice with 0 and 1. If \( a \in L \), then a complement of \( a \) in \( L \) is an element \( b \in L \) such that \( a \land b = 0 \) and \( a \lor b = 1 \). The lattice \( L \) is complemented if every element of \( L \) has a complement in \( L \) [4]. First we need the following well-known lemma.

**Lemma 1.1.** Let \( L \) be a lattice.

(i) A non-empty subset \( F \) of \( L \) is a filter of \( L \) if and only if \( x \lor z \in F \) and \( x \land y \in F \) for all \( x, y, z \in F \), \( z \in L \) (so \( 0 \in F \) if and only if \( F = L \)). Moreover, since \( x = x \lor (x \land y) \) and \( y = y \lor (x \land y) \), \( F \) is a filter and \( x \land y \in F \) gives \( x, y \in F \) for all \( x, y \in L \).
If $F_1, \ldots, F_n$ are filters of $L$ and $a \in L$, then $\bigvee_{i=1}^n F_i = \{ \bigvee_{i=1}^n a_i : a_i \in F_i \}$ and $a \vee F_i = \{ a \vee a_i : a_i \in F_i \}$ are filters of $L$.

If $D$ is an arbitrary non-empty subset of $L$, then the set $T(D)$ consisting of all elements of $L$ of the form $(a_1 \land a_2 \land \cdots \land a_n) \lor x$ (with $a_i \in D$ for all $1 \leq i \leq n$ and $x \in L$) is a filter of $L$ containing $D$ (so if $D = \{a\}$, then $T(\{a\}) = T(a) = \{ a \lor t : t \in L \}$).

If $L$ is distributive, $F, G$ are filters of $L$, and $x \in L$, then $(G :_L F) = \{ x \in L : x \lor F \subseteq G \}$ and $(F :_L \{x\}) = (F :_L x) = \{ a \in L : a \lor x \in F \}$ are filters of $L$.

If $\{F_i\}_{i \in \Delta}$ is a chain of filters of $L$, then $\bigcup_{i \in \Delta} F_i$ is a filter of $L$.

2. Some basic properties of 2-absorbing filters

In this section, we collect some properties concerning 2-absorbing filters of a lattice $L$. Throughout this paper, we shall assume unless otherwise stated, that $L$ is a distributive lattice with 1 and 0.

**Definition 2.1.** A proper filter $F$ of $L$ is called a 2-absorbing (resp. a weakly 2-absorbing) filter if whenever $a, b, c \in L$ and $a \lor b \lor c \in F$ (resp. $1 \neq a \lor b \lor c \in F$), then $a \lor b \in F$ or $a \lor c \in F$ or $b \lor c \in F$.

Clearly, every 2-absorbing filter of $L$ is a weakly 2-absorbing. However, since $\{1\}$ is always weakly 2-absorbing "by definition," a weakly 2-absorbing filter need not be 2-absorbing.

**Theorem 2.2.** If $F$ is a weakly 2-absorbing of $L$ that is not 2-absorbing, then $F = \{1\}$. In particular, the only weakly 2-absorbing filters of $L$ that are not 2-absorbing can only be $\{1\}$.

**Proof.** We suppose that $F \neq \{1\}$, and look for a contradiction. Let $x \lor y \lor z \in F$. If $x \lor y \lor z \neq 1$, then $F$ weakly 2-absorbing gives $x \lor y \in F$ or $y \lor z \in F$ or $x \lor z \in F$; so $F$ is 2-absorbing which is a contradiction. So assume that $x \lor y \lor z = 1$. Since $F \neq \{1\}$, there exists $b \in F$ with $b \neq 1$. Then $1 \neq b = b \land 1 = b \land (x \lor y \lor z) = ((b \land (x \lor y)) \lor ((b \land (x \lor z)) \lor ((b \land (y \lor z)))) \in F$, so $b \land (x \lor y) \in F$ or $b \land (x \lor z) \in F$ or $b \land (y \lor z) \in F$. Thus $x \lor y \in F$ or $x \lor z \in F$ or $y \lor z \in F$ by Lemma 1.1 (i), and so $F$ is 2-absorbing, a contradiction. Thus $F = \{1\}$. The "in particular" statement is clear.

**Remark 2.3.** (i) If $F, F_1, F_2$ are filters of $L$ with $F \subseteq F_1 \cup F_2$, then we show that either $F \subseteq F_1$ or $F \subseteq F_2$. Suppose that $F \subseteq F_1 \cup F_2$ such that $F \not\subseteq F_1$; we show that $F \subseteq F_2$. Let $a \in F$ be such that $a \not\in F_1$. Let $x \in F \cap F_1$. Then $F$ is a filter gives $a \land x \in F \subseteq F_1 \cup F_2$; so $a, x \in F_2$. Therefore $F \cap F_1 \subseteq F_2$. Thus $F = F \cap (F_1 \cup F_2) = (F \cap F_1) \cup (F \cap F_2) \subseteq F_2$.
(ii) Assume that \( m \) is a maximal filter of a lattice \( L \) with 0 and let \( a \lor b \notin m \) with \( a, b \notin m \) for some \( a, b \in L \). Then \( T(m \cup \{a\}) = T(m \cup \{b\}) = L \) since \( m \) is maximal. An inspection will show that 0 \( \notin L \) implies that \( L = F \) which is a contradiction. Thus every maximal filter of \( L \) is prime [6].

(iii) If \( F \) is a filter of a \( L \)-domain \( L \), then \( F \) is 2-absorbing if and only if it is weakly 2-absorbing.

**Proposition 2.4.** Let \( F_1, F_2, F \) be filters of \( L \) such that \( F \) is 2-absorbing.

(i) If \( a, b \in L \) and \( (a \lor b) \lor F_1 \subseteq F \), then \( a \lor b \in F \) or \( a \lor F_1 \subseteq F \) or \( b \lor F_1 \subseteq F \).

(ii) If \( a \in L \) and \( a \lor (F_1 \lor F_2) \subseteq F \), then \( a \lor F_1 \subseteq F \) or \( a \lor F_2 \subseteq F \) or \( F_1 \lor F_2 \subseteq F \).

**Proof.** (i) Let \( a \lor b \notin F \) and \( a \lor F_1 \notin F \). Then there is an element \( c \in F_1 \) such that \( a \lor c \notin F \). Now \( a \lor b \lor c \in F \) gives \( b \lor c \in F \) since \( F \) is 2-absorbing. We have to show that \( b \lor F_1 \subseteq F \). Let \( d \) be an arbitrary element of \( F_1 \). Then \( (d \land c) \lor (a \lor b) = (a \lor b \lor c) \lor (a \lor b \lor d) \in F \) since \( F \) is a filter; so either \( (d \land c) \lor a = (a \lor c) \lor (a \lor d) \in F \) or \( (d \land c) \lor b = (b \lor c) \lor (b \lor d) \in F \). If \( (d \land c) \lor a \in F \), then \( a \lor c \subseteq F \) by Lemma 1.1 (i) that is a contradiction. If \( (d \land c) \lor b \in F \), then \( b \lor d \in F \). Thus \( b \lor F_1 \subseteq F \).

(ii) Let \( a \lor F_1 \notin F \) and \( a \lor F_2 \notin F \). We have to show that \( F_1 \lor F_2 \subseteq F \). Suppose that \( x \in F_1 \) and \( y \in F_2 \). By hypothesis, there exist \( z \in F_1 \setminus F \) and \( w \in F_2 \setminus F \) such that \( a \lor z \notin F \) and \( a \lor w \notin F \). As \( a \lor z \lor w \in a \lor (F_1 \lor F_2) \subseteq F \), we get \( z \lor w \in F \). Now \( z \land x \in F_1 \) and \( y \land w \in F_2 \) gives \( a \lor (z \land x) \lor (y \land w) \in F \); so \( (z \land x) \lor (y \land w) \in F \) since \( F \) is 2-absorbing (see Lemma 1.1 (ii)). It follows that \( (z \land x) \lor y \in F \); hence \( x \lor y \in F \) by Lemma 1.1 (i). Therefore, \( F_1 \lor F_2 \subseteq F \).

**Theorem 2.5.** Let \( F \) be a proper filter of \( L \). The following statements are equivalent:

(i) \( F \) is a 2-absorbing filter of \( L \).

(ii) If \( F_1 \lor F_2 \lor F_3 \subseteq F \) for some filters \( F_1, F_2, F_3 \) of \( L \), then \( F_1 \lor F_2 \subseteq F \) or \( F_1 \lor F_3 \subseteq F \) or \( F_2 \lor F_3 \subseteq F \).

**Proof.** (i)\( \Rightarrow \) (ii) Suppose that \( F_1 \lor F_2 \lor F_3 \subseteq F \) for some filters \( F_1, F_2, F_3 \) of \( L \) and \( F_1 \lor F_2 \notin F \). Then by Proposition 2.4 for all \( a \in F_3 \) either \( a \lor F_1 \subseteq F \) or \( a \lor F_2 \subseteq F \). If \( a \lor F_1 \subseteq F \), for all \( a \in F_3 \) we are done. Similarly, if \( a \lor F_2 \subseteq F \), for all \( a \in F_3 \) we are done. Assume that \( a, b \in L \) are such that \( a \lor F_1 \notin F \) and \( b \lor F_2 \notin F \). It follows that \( b \lor F_1 \subseteq F \) and \( a \lor F_2 \subseteq F \). Since \( (a \land b) \lor (F_1 \lor F_2) \subseteq F \), we get either \( (a \land b) \lor F_1 \subseteq F \) or \( (a \land b) \lor F_2 \subseteq F \) by Proposition 2.4. If \( (a \land b) \lor F_1 \subseteq F \), then \( z \lor (a \land b) = (z \lor a) \lor (z \lor b) \in F \) for all \( z \in F_1 \) which implies that \( a \lor z \in F \) by Lemma 1.1 (i); so \( a \lor F_1 \subseteq F \) which is a contradiction. Similarly, if \( (a \land b) \lor F_2 \subseteq F \), we get a contradiction. Thus either \( F_1 \lor F_3 \subseteq F \) or \( F_2 \lor F_3 \subseteq F \).

(ii)\( \Rightarrow \) (i) Let \( a, b, c \in L \) with \( a \lor b \lor c \in F \). Then by (ii), \( T(a) \lor T(b) \lor T(c) \subseteq F \) gives \( a \lor b \in T(a) \lor T(b) \subseteq F \) or \( a \lor c \in T(a) \lor T(c) \subseteq F \) or \( b \lor c \in T(b) \lor T(c) \subseteq F \). Thus \( F \) is 2-absorbing.
We say that a subset $D \subseteq L$ is join closed if $0 \in D$ and $a \vee b \in D$ for all $a, b \in D$. Clearly, if $p$ is a prime filter of $L$, then $L \setminus p$ is a join closed subset of $L$. The set of all prime filters of $L$ is denoted by $\text{Spec}(L)$. If $F$ is a filter of $L$, then we set $\text{var}(F) = \{p \in \text{Spec}(L) : F \subseteq p\}$, and the set of all prime filters of $L$ that are minimal over $F$ is denoted by $\text{min}(F)$.

**Lemma 2.6.** (i) Assume that $F$ is a filter of $L$ and let $S$ be a join closed set of $L$ such that $S \cap F = \emptyset$. Then the set $\sum = \{K : F \subseteq K, K \cap S = \emptyset\}$ of filters under the relation of inclusion has at least one maximal element, and any such maximal element of $\sum$ is a prime filter.

(ii) If $F$ is a filter of $L$, then $F = \cap_{p \in \text{var}(F)} p$.

(iii) Let $F, p$ be filters of $L$ with $p$ prime and $F \subseteq p$. Then there exists a minimal prime filter $q$ of $F$ with $q \subseteq p$.

(iv) If $F$ is a filter of $L$, then $F = \cap_{p \in \text{min}(F)} p$.

**Proof.** (i) Since $F \in \sum$, $\sum \neq \emptyset$. Of course, the relation of inclusion, $\subseteq$, is a partial order on $\sum$. Now $\sum$ is easily seen to be inductive under inclusion, so by Zorn’s Lemma $\sum$ has a maximal element $q$ with $q \cap S = \emptyset$ and $F \subseteq q$. It suffices to show that $q$ is prime. Now let $x, x' \in L \setminus q$; we must show that $x \vee x' \notin q$. Since $x \notin q$, we have $F \subseteq q \not\subseteq T(q \cup \{x\})$. By the maximality of $q$, we have $T(q \cup \{x\}) \cap S \neq \emptyset$, and so there exist $s \in S, c \in L$ and $q \in q$ such that $s = (q \wedge x) \vee c$. Similarly, $s' = (q' \wedge x') \vee c'$ for some $s' \in S, q' \in q$ and $c' \in L$. Set $z = c \vee c'$. Then $s \vee s' = (q \wedge x) \vee (q' \wedge x') \vee z = [(q \wedge x) \vee x'] \wedge [(q \wedge x) \vee q'] \vee z = [(x \vee x') \wedge (q \vee q')] \vee z$. As $(q \wedge x) \vee q', q \wedge x' \in q, S \cap q = \emptyset$ and $q$ is a filter, we have $x \vee x' \notin q$. Thus $q$ is a prime filter.

(ii) It is enough to show that $\cap_{p \in \text{var}(F)} p \subseteq F$. Let $a \in \cap_{p \in \text{var}(F)} p$. We suppose that $a \notin F$, and look for a contradiction. Set $S = \{0, a\}$. Then $S$ is a join closed set of $L$ with $S \cap F = \emptyset$. Hence, by (i), there exists a prime filter $q$ of $L$ such that $F \subseteq q$ and $q \cap S = \emptyset$. It follows that $q \in \text{var}(F)$, so that $a \in S \cap q$, a contradiction.

(iii) Set $\Delta = \{q \in \text{Spec}(L) : F \subseteq q \subseteq p\}$. Then $p \in \Delta$, and so $\Delta \neq \emptyset$. By an argument like that in (i) (take $S = L \setminus p$), the set $\Delta$ of prime filters of $L$ has a minimal member with respect to inclusion (by partially ordering $\Delta$ by reverse inclusion and using Zorn’s Lemma) which is prime. (iv) follows from (iii) (since every prime filter in $\text{var}(F)$ contains a minimal prime filter of $F$). \[\Box\]

Compare the next Proposition with Theorem 2.1, p. 2 in [7].

**Proposition 2.7.** Let $F \subseteq p$ be filters of $L$, where $p$ is a prime filter. Then the following conditions are equivalent:

(i) $p$ is a minimal prime filter of $F$.

(ii) $L \setminus p$ is a join closed set that is maximal with $(L \setminus p) \cap F = \emptyset$. 
(iii) For each $x \in p$, there is a $y \notin p$ such that $y \lor x \in F$.

**Proof.** (i)$\Rightarrow$(ii) Since $(L \setminus p) \cap F = \emptyset$, the set $\Delta$ of all join closed sets, say $H$, with $H \cap F = \emptyset$ is not empty. Of course, the relation of inclusion, $\subseteq$, is a partial order on $\Delta$. Now $\Delta$ is easily seen to be inductive under inclusion, so by Zorn’s Lemma $\Delta$ has a maximal element $S$. Again by Zorn’s Lemma, there is a filter $q$ of $L$ containing $F$ that is maximal with respect to being disjoint from $S$ which is prime by Lemma 2.6 (i). Note that $q$ is disjoint from $L \setminus p$ which implies that $p = q$. Thus $S = L \setminus p$.

(ii)$\Rightarrow$(iii) Assume that $1 \neq x \in p$ and let $S = \{y \lor (\bigwedge_{i=1}^{0} x) : y \in L \setminus p, i = 0, 1, \ldots \}$ (Note that $\bigwedge_{i=1}^{0} x$ is interpreted as 0, and clearly, $\bigwedge_{i=1}^{0} x = x$). Then $S$ is a join closed set that properly contains $L \setminus p$; so $F \cap S \neq \emptyset$ by maximality of $L \setminus p$. Thus there exists $y \in L \setminus p$ such that $x \lor y \in F$.

(iii)$\Rightarrow$(i) Let $q$ be a prime filter such that $F \not\subseteq q \subseteq p$. If $p \neq q$, then there is an element $x \in p$ with $x \notin q$; so $x \lor y \in F \not\subseteq q$ for some $y \notin p$ which is a contradiction. Therefore $p = q$.  

The following theorem is a lattice counterpart of Theorem 2.4 in [2] describing the structure of 2-absorbing ideals.

**Theorem 2.8.** (i) If $F$ is a 2-absorbing filter of $L$, then there exist at most two prime filters of $L$ that are minimal over $F$.

(ii) If $F$ is a 2-absorbing filter of $L$, then either $F$ is a prime filter of $L$ or $F = p \cap q = p \lor q$, where $p$, $q$ are the only distinct filters of $L$ that are minimal over $F$.

(iii) If either $F$ is a prime filter of $L$ or $F$ is an intersection of two prime filter of $L$, then $F$ is 2-absorbing.

**Proof.** (i) Assume that that $\Delta$ is the set of prime filters of $L$ which are minimal over $F$ and let $\Delta$ has at least three elements. Let $p, q \in \Delta$ with $p \neq q$. Then there exist $x_1, x_2 \in L$ such that $x_1 \in p \setminus q$ and $x_2 \in q \setminus p$. First we show that $x_1 \lor x_2 \in F$. By Proposition 2.7, there exist $a \notin p$ and $b \notin q$ such that $a \lor x_1, b \lor x_2 \in F$. Since $x_1, x_2 \notin p \cap q$ and $a \lor x_1, b \lor x_2 \in F \subseteq p \cap q$, we conclude that $a \in q \setminus p$ and $b \in p \setminus q$; so $a, b \notin p \cap q$. Since $a \lor x_1, b \lor x_2 \in F$, we have $(a \land b) \lor (x_1 \lor x_2) = [(a \lor x_1) \lor x_2] \land [(b \lor x_2) \lor x_1] \in F$ since $F$ is a filter. By Lemma 1.1 (i), $a \land b \notin p$ and $a \lor b \notin q$. Since $(a \land b) \lor x_1 \notin q$ and $(a \land b) \lor x_2 \notin p$, $F$ is a 2-absorbing filter gives $x_1 \lor x_2 \in F$. Now suppose there is a $r \in \Delta$ such that $r$ is neither $p$ nor $q$. Then we can choose $z_1 \in p \setminus (q \cup r)$, $z_2 \in q \setminus (p \cup r)$, and $z_3 \in r \setminus (p \cup q)$. By an argument like that as above, we have $z_1 \lor z_2 \in F$. Since $F \subseteq p \cap q \cap r$ and $z_1 \lor z_2 \in F$, we get either $z_1 \in r$ or $z_2 \in r$ that is a contradiction, as required.
(ii) By (i) and Lemma 2.6 (iv), we conclude that either \( F \) is a prime filter or \( F = p \cap q \), where \( p, q \) are the only distinct filters of \( L \) that are minimal over \( F \). An inspection will show that \( p \cap q = p \vee q \).

(iii) The first assertion is clear. Let \( p \) and \( q \) be two prime filters of \( L \); we have to show that \( F = p \cap q \) is a 2-absorbing filter of \( L \). Let \( a, b, c \in L \) such that 
\[
    a \lor b \lor c \in p \land q.
\]
Therefore \( a \lor b \lor c \in p \) and \( a \lor b \lor c \in q \). If \( a \in p \cap q \), then 
\[
    a \lor b \in p \land q.
\]
If \( a \in p \) and \( b \in p \), then \( a \lor b \in p \land q \) since \( p \) and \( q \) are filters of \( L \). The other cases we do the same.

The collection of ideals of \( Z \), the ring of integers, form a lattice under set inclusion which we shall denote by \( L(Z) \) with respect to the following definitions: 
\[ mZ \lor nZ = (m,n)Z \text{ and } mZ \land nZ = [m,n]Z \] for all ideals \( mZ \) and \( nZ \) of \( Z \), where \( (m,n) \) and \( [m,n] \) are greatest common divisor and least common multiple of \( m, n \), respectively. Note that \( L(Z) \) is a distributive complete lattice with least element the zero ideal and the greatest element \( Z \).

**Theorem 2.9.** The following hold:

(i) If \( p \) is a prime number and \( k \) is a positive integer, then the set \( F_{p,k} = \{ mZ \in L(Z) : p^k \nmid m \} \) is a prime filter of \( L(Z) \).

(ii) \( L(Z) \setminus \{ 0 \} \) is the only maximal filter of \( L(Z) \).

(iii) Every prime filter of \( L(Z) \) is of the form either \( F_{p,k} \) for some prime number \( p \) and positive integer \( k \) or \( L(Z) \setminus \{ 0 \} \).

(iv) Every 2-absorbing filter of \( L(Z) \) is of the form \( L(Z) \setminus \{ 0 \} \) or \( F_{p,m} \) or \( F_{p,m} \cap F_{q,n} \) for some positive integers \( m, n \) and prime numbers \( p, q \) with \( p \neq q \).

**Proof.** (i) Let \( mZ, nZ \in F_{p,k} \) and \( sZ \in L(Z) \). Now \( p^k \nmid m \) and \( p^k \nmid n \) gives \( p^k \nmid [m,n] \); so \( [m,n]Z \in F_{p,k} \). As \( p^k \nmid m \), we get \( p^k \nmid (m,s) \) which implies that \( (m,s)Z \in F_{p,k} \). Thus \( F_{p,k} \) is a filter of \( L(Z) \). Let \( mZ \lor nZ = (m,n)Z \in F_{p,k} \) with \( mZ \notin F_{p,k} \). Then \( p^k \nmid (m,n) \) and \( p^k \nmid n \) gives \( p^k \nmid m \); so \( nZ \in F_{p,k} \). Thus \( F_{p,k} \) is prime. 

(ii) is clear.

(iii) Let \( F \) be a prime filter of \( L(Z) \). First we show that there exist at most one prime number \( p \) and positive integer \( k \) such that for every \( mZ \in F \) implies that \( p^k \nmid m \). Otherwise, there are distinct prime numbers \( p, q \) and positive integers \( k, s \) such that for every \( mZ \in F \) implies that \( p^k \nmid m \) and \( q^s \nmid m \). Then \( p^kZ \lor q^sZ = Z \in F \) gives either \( p^kZ \in F \) or \( q^sZ \in F \) which is a contradiction. If there exists \( p^k \) such that for every \( mZ \in F \) implies that \( p^k \nmid m \). Let \( t \) be least positive integer such that for every \( mZ \in F \) implies that \( p^t \nmid m \); we show that \( F = F_{p,t} \). It suffices to show that for every \( mZ \) with \( p^t \nmid m \), \( mZ \in F \). There are distinct prime numbers \( q_1, \ldots, q_n \) such that \( m = p^l q_1^{s_1} \cdots q_n^{s_n} \), where \( 0 \leq l < t \), \( p \neq q_j \) with \( 1 \leq j \leq n \), and \( s_j \) is a positive integer for \( 1 \leq j \leq n \). As \( l < t \), there exist \( m'Z \in F \) such that \( p^t \nmid m' \), so \( m'Z \subseteq p^l Z \). Thus \( p^l Z \in F \) since \( F \) is
a filter. Moreover, \( p^iZ \vee q_i^iZ = Z \in F \) gives \( q_i^iZ \in F \) with \( 1 \leq i \leq n \). Thus \( mZ = p^iZ \wedge (\wedge_{i=1}^n q_i^iZ) \in F \). Suppose that there is not such \( p^i \); we show that \( F = L(Z) \setminus \{0\} \). Let \( m \) be a non-zero integer. It is enough to show that \( mZ \in F \).

We can write \( m = p_1^{s_1} \cdots p_n^{s_n} \), where \( p_i \neq p_j \) with \( i \neq j \) and for each \( i \), \( s_i \) is a positive integer. Then for each \( i \), there exists \( m_iZ \in F \) such that \( p_i^{s_i} \mid m_i \), so \( m_iZ \subseteq p_i^{s_i}Z \in F \) since \( F \) is a filter. Thus \( mZ = \wedge_{i=1}^n p_i^{s_i}Z \in F \).

(iv) This follows from (i), (ii), (iii), and Theorem 2.8.

Remark 2.10 shows that prime filters which are maximal are abundant.

**Proposition 2.12.** If \( G \) is a 2-absorbing subfilter of a filter \( F \) of \( L \), then \( (G :_L F) \) is a 2-absorbing filter of \( L \).

**Proof.** Let \( a,b,c \in L, a \vee b \vee c \in (G :_L F), a \vee c \notin (G :_L F), \) and \( b \vee c \notin (G :_L F) \). We must to show that \( a \vee b \in (G :_L F) \). There exist \( x_1,x_2 \in L \) such that \( a \vee c \vee x_1, b \vee c \vee x_2 \notin G \) but \( (a \vee b) \vee [(c \vee x_1) \wedge (c \vee x_2)] = (a \vee b \vee c) \vee (x_1 \wedge x_2) \in G \) since \( G \) is a filter. Now \( G \) is a 2-absorbing filter gives \( a \vee [(c \vee x_1) \wedge (c \vee x_2)] = (a \vee c \vee x_1) \wedge (a \vee c \vee x_2) \in G \) or \( b \vee [(c \vee x_1) \wedge (c \vee x_2)] = (b \vee c \vee x_1) \wedge (b \vee c \vee x_2) \in G \) or \( a \vee b \in G \). If \( a \vee b \in G \), we are done. If \( a \vee [(c \vee x_1) \wedge (c \vee x_2)] \notin G \), then by Lemma 1.1 (i), \( a \vee c \vee x_1 \notin G \) which is a contradiction. Similarly, \( b \vee [(c \vee x_1) \wedge (c \vee x_2)] \notin G \). This completes the proof.
Proposition 2.13. If $G$ is a 2-absorbing subfilter of a filter $F$ of $L$, then $(G :_L F)$ is a prime filter if and only if $(G :_L x)$ is a prime filter for all $x \in F \setminus G$.

Proof. Let $a, b \in L$, $x \in F \setminus G$, and $a \lor b \in (G :_L x)$. Then $a \lor b \lor x \in G$ gives $a \lor x \in G$ or $b \lor x \in G$ or $a \lor b \in G$. If $a \lor x \in G$ or $b \lor x \in G$ we are done. If $a \lor b \in G$, then $(a \lor b) \lor F \subseteq G$ since $G$ is a filter; so $a \lor b \in (G :_L F)$. By assumption, $a \in (G :_L F)$ or $b \in (G :_L F)$; hence $a \in (G :_L x)$ or $b \in (G :_L x)$. Thus $(G :_L x)$ is a prime filter of $L$. Conversely, suppose that $a \lor b \in (G :_L F)$ for some $a, b \in L$ with $a, b \notin (G :_L F)$. It follows that $a \lor x \notin G$ and $b \lor y \notin G$ for some $x, y \in F \setminus G$ (so $x \land y \notin G$ by Lemma 1.1 (i)). As $a \lor b \lor (x \land y) = (a \lor b \lor x) \land (a \lor b \lor y) \in G$, we have $a \lor b \in (G :_L (x \land y))$: hence $a \lor (x \land y) = (a \lor x) \land (a \lor y) \in G$ or $b \lor (x \land y) = (b \lor x) \land (b \lor y) \in G$ since $(G :_L (x \land y))$ is a prime filter which is a contradiction. Thus $a \in (G :_L F)$ or $b \in (G :_L F)$ which implies that $(G :_L F)$ is a prime filter of $L$. 

Let $G$ be a proper subfilter of a filter $F$ of $L$. We say that $p \in \text{Spec}(L)$ is an associated prime filter of $F$ with respect to $G$ if there is an element $x \in F \setminus G$ such that $(G :_L x) = p$. The set of associated prime filters of $F$ with respect to $G$ is denoted $\text{Ass}_L(G :_L F)$.

Compare the next Theorem with Theorem 2.6 in [10].

Theorem 2.14. Let $G$ be a 2-absorbing subfilter of a filter $F$ of $L$. If $(G :_L F)$ is a prime filter of $L$, then $\text{Ass}_L(G :_L F)$ is a totally ordered set.

Proof. Let $p, q \in \text{Ass}_L(G :_L F)$. Then there are elements $x, y \in F \setminus G$ such that $(G :_L x) = p$ and $(G :_L y) = q$. Suppose that $q \not\subseteq p$. We have to show that $(G :_L x) \subseteq (G :_L y)$. Let $z \in (G :_L x)$ (so $z \lor x \in G$). There exists $w \in (G :_L y)$ such that $w \not\in (G :_L x)$: so $w \lor y \in G$ and $w \lor x \notin G$. Clearly, $x \land y \not\in G$. If $z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G$, then $z \lor y \in G$ by Lemma 1.1 (i) and so $z \in (G :_L y)$. Now assume that $z \lor (x \land y) \notin G$, so $(z \lor w) \lor (x \land y) = (z \lor w \lor x) \land (z \lor w \lor y) \in G$ since $G$ is a filter; hence $z \lor w \in (G :_L (x \land y))$. By Proposition 2.13 and Lemma 1.1 (i), $(G :_L (x \land y))$ is a prime filter gives $z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G$ and $w \lor (x \land y) = (w \lor x) \land (w \lor y) \not\in G$. Thus $z \lor y \in G$ and so $z \in (G :_L y)$.

Compare the next Theorem with Theorem 2.7 in [10].

Theorem 2.15. Let $G$ be a 2-absorbing subfilter of a filter $F$ of $L$ such that $(G :_L F) = p \cap q$ for some prime filters $p, q$ of $L$.

(i) If $x \in F \setminus G$ and $p \subseteq (G :_L x)$, then $(G :_L x)$ is a prime filter of $L$.

(ii) If $x, y \in F \setminus G$ and $p \subseteq (G :_L x) \cap (G :_L y)$, then either $(G :_L x) \subseteq (G :_L y)$ or $(G :_L y) \subseteq (G :_L x)$. Therefore $\text{Ass}_L(G :_L F)$ is the union of two totally ordered sets.
Proof. (i) Let \( a, b \in L \) and \( a \lor b \in (G :_L x) \). Then \( a \lor b \lor x \in G \) gives \( a \lor x \in G \) or \( b \lor x \in G \) or \( a \lor b \in G \). If \( a \lor x \in G \) or \( b \lor x \in G \) we are done. If \( a \lor b \in G \), then \( (a \lor b) \lor y \leq G \) since \( G \) is a filter; so \( a \lor b \in (G :_L F) \leq p \). thus either \( a \in p \subseteq (G :_L x) \) or \( b \in p \subseteq (G :_L x) \).

(ii) Suppose that \((G :_L y) \not\subseteq (G :_L x)\). We have to show that \((G :_L x) \subseteq (G :_L y)\). Let \( z \in (G :_L x) \) (so \( z \lor x \in G \)). There exists \( w \in (G :_L y) \) such that \( w \not\in (G :_L x) \); so \( w \lor y \in G \) and \( w \lor x \not\in G \). Clearly, \( x \land y \not\in G \). If \( z \lor (x \land y) = (z \lor x) \land (z \lor y) \in G \), then \( z \lor y \in G \) by Lemma 1.1 (i) and so \( z \in (G :_L y) \). Now assume that \( z \lor (x \land y) \not\in G \), so \( (z \lor w) \lor (x \land y) = (z \lor w \lor x) \land (z \lor w \lor y) \in G \) since \( G \) is a filter; hence \( z \lor w \in G \) since \( w \lor (x \land y) = (w \lor x) \land (w \lor y) \not\in G \) and \( z \lor (x \land y) \not\in G \). Thus \( z \lor w \in (G :_L F) \subseteq p \). If \( w \in p \subseteq (G :_L x) \), then \( w \lor x \in G \) that is a contradiction; hence \( z \in p \subseteq (G :_L y) \).

Theorem 2.16. If \( G \) is a 2-absorbing subfilter of a filter \( F \) of \( L \), then \((G :_L F)\) is a prime filter if and only if \((G :_L H)\) is a prime filter of \( L \) for all subfilters \( H \) of \( F \) containing \( G \).

Proof. By Proposition 2.13 and Theorem 2.14, the set \( \{ (G :_L x) : x \in H \setminus G \} \) is a totally ordered set of prime filters of \( L \); so \((G :_L H) = \cap_{x \in H}(G :_L x)\) is a prime filter of \( L \). Conversely, suppose that \( x \land y \in (G :_L F) \) with \( x, y \not\in (G :_L F) \). Then there exist \( a, b \in F \setminus G \) (so \( a \land b \not\in G \)) such that \( x \lor a, y \lor b \not\in G \), so \( x \land y \in (G :_L (a \land b)) \). Now \((G :_L (a \land b))\) is a prime filter gives \( x \lor (a \land b) = (x \lor a) \land (x \lor b) \in G \) or \( y \lor (a \land b) = (y \lor a) \land (y \lor b) \in G \) which is a contradiction. Thus \((G :_L F)\) is prime.

3. 2-Absorbing Avoidance Theorem

Let \( F, F_1, F_2, \ldots, F_n \) be filters of \( L \). We call a covering \( F \subseteq \cup_{i=1}^{n} F_i \) efficient if no \( F_i \) is superfluous. Analogously, we say that \( F = \cup_{i=1}^{n} F_i \) is an efficient union if none of the \( F_i \) may be excluded. Any cover or union consisting of filters of \( L \) can be reduced to an efficient one, called an efficient reduction, by deleting any unnecessary terms.

Theorem 3.1. If \( G \) is a 2-absorbing subfilter of a filter \( F \) of \( L \) and \( x \in F \setminus G \), then either \((G :_L x)\) is a prime filter of \( L \) or there exists an element \( a \in L \) such that \((G :_L a \lor x)\) is a prime filter of \( L \).

Proof. By Proposition 2.12 and Theorem 2.8 (iii), \((G :_L F)\) is a prime filter of \( L \) or \((G :_L F)\) is an intersection of two prime filters of \( L \). We split the proof into two cases:

Case 1. \((G :_L F) = p\), where \( p \) is a prime filter of \( L \). We show that \((G :_L x)\) is a prime filter of \( L \). Clearly, \( p \subseteq G :_L x \). Suppose that \( a, b \in L \) and
Let $a \lor b \in G :_L x$. Then $a \lor b \lor x \in G$; hence $a \lor x \in G$ or $b \lor x \in G$ or $a \lor b \in G$. If either $a \lor x \in G$ or $b \lor x \in G$, we are done. So we may assume that $a \lor b \in G$. As $G$ is a filter, $(a \lor b) \lor F \subseteq G$; thus $a \lor b \in p$ and so $a \in p$ or $b \in p$. Therefore, $a \in G :_L x$ or $b \in G :_L x$ and the assertion follows.

**Case 2.** $(G :_L F) = p \cap q$, where $p$ and $q$ are distinct prime filters of $L$. If $p \subseteq (G :_L x)$, then the result follows by an argument like that in the Case 1. So we may assume that $p \not\subseteq (G :_L x)$. There is an element $a \in p$ such that $a \lor x \notin G$. By Theorem 2.8 (ii), $p \lor q \subseteq (G :_L x)$; so $q \subseteq G :_L a \lor x$ and the result follows by a similar proof to that of Case 1. ■

Compare the next lemma with Lemma 1 in [7].

**Lemma 3.2.** Let $F_i$ and $F_j$ $(i = 1, 2, \ldots, n)$ be filters such that $F \subseteq \cup_{i=1}^{n-1} F_i$ is an efficient covering of filters of $L$, where $n \geq 3$. Then The intersection of any $n - 1$ of the filters $F \cap F_i$ coincides with $H = \cap_{i=1}^{n-1} (F \cap F_i)$.

**Proof.** It suffices to show that the intersection of any $n - 1$ of the filters $F \cap F_i$ is contained in $H$. Since $F \subseteq \cup_{i=1}^{n-1} F_i$ is an efficient covering, we have $F = \cup_{i=1}^{n} (F \cap F_i)$ is an efficient union consisting of subfilters of $F$, so $F$ is not contained in the union of any $n-1$ of the filters $F \cap F_i$; hence there exists an element $c_n \in F_n$ which is not in $\cup_{i=1}^{n-1} (F \cap F_i)$. If $x \in \cap_{i=1}^{n-1} (F \cap F_i)$, then the element $x \land c_n$ in $F$ can not be in $F_i$ for $1 \leq i \leq n-1$; thus $x \land c_n \in F_n$. By Lemma 1.1 (i), $x \in F_n$ and so $x \in H$, as needed. ■

**Proposition 3.3.** Let $F_i$ $(i = 1, 2, \ldots, n)$ be filters such that $F \subseteq \cup_{i=1}^{n-1} F_i$ is an efficient covering of filters of $L$, where $n \geq 3$. If $F_i \not\subseteq (F_j :_L x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then no $F_i$ for $1 \leq i \leq n$ is a 2-absorbing filter of $L$.

**Proof.** Assume to the contrary, $F_k$ is a 2-absorbing filter of $L$ for some $k = 1, \ldots, n$. By Lemma 3.2, $\cap_{i \neq j} (F_i \cap F_j) \subseteq F \cap F_k$. Clearly, $F \not\subseteq F_k$, so there is an element $b \in F$ with $b \notin F_k$. Now Theorem 3.1 gives either $(F_k :_L b)$ is a prime filter or there exists $a \in L$ such that $(F_k :_L (a \lor b))$ is a prime filter of $L$. Suppose first that $(F_k :_L b)$ is a prime filter. By assumption, there is an element $a_i \in F_i \setminus (F_k :_L b)$ for all $i \neq k$; so $(\lor_{i \neq j} a_i) \lor b \in \cap_{i \neq j} (F \cap F_i) \setminus (F \cap F_k)$ since $(\lor_{i \neq j} a_i) \lor b \in F \cap F_k$ implies that $(\lor_{i \neq j} a_i) \in (F_k :_L b)$ and so there is $a_i \in (F_k :_L b)$ for some $i \neq k$ that is a contradiction. If $(F_k :_L (a \lor b))$ is a prime filter of $L$ for some $a \in L$, then there exists $c_i \in F_i \setminus (F_k :_L (a \lor b))$ for all $i \neq k$. Therefore $(\lor_{i \neq j} c_i) \lor (a \lor b) \in \cap_{i \neq j} (F \cap F_i) \setminus (F \cap F_k)$ which is a contradiction. Thus $F_k$ is not a 2-absorbing filter, as required. ■

The following theorem is a lattice counterpart of Theorem 3.2 in [10] describing the structure of 2-absorbing submodules.
Theorem 3.4 (2-Absorbing Avoidance Theorem). Let $F, F_1, F_2, \ldots, F_n$ ($n \geq 2$) be filters of $L$ such that at most two of $F_1, F_2, \ldots, F_n$ are not 2-absorbing. If $F \subseteq \bigcup_{i=1}^{n} F_i$ and $F_i \not\subseteq (F_j : L, x)$ for all $x \in L \setminus F_j$ whenever $i \neq j$, then $F \subseteq F_i$ for some $i$ with $1 \leq i \leq n$.

Proof. By Remark 2.3 (i), we may assume that $n \geq 3$. Let $F \not\subseteq F_i$ for all $i$ with $1 \leq i \leq n$. Then $F \subseteq \bigcup_{i=1}^{n} F_i$ is an efficient covering of filters of $L$. Then by Proposition 3.3, no $F_i$ is 2-absorbing that contradicts the assumption. Therefore $F \subseteq F_i$ for some $i$ with $1 \leq i \leq n$.

References


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