UNION OF DISTANCE MAGIC GRAPHS

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Abstract

A distance magic labeling of a graph $G = (V, E)$ with $|V| = n$ is a bijection $\ell$ from $V$ to the set $\{1, \ldots, n\}$ such that the weight $w(x) = \sum_{y \in N_G(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu$, called the magic constant. In this paper, we study unions of distance magic graphs as well as some properties of such graphs.

Keywords: distance magic labeling, magic constant, sigma labeling, graph labeling, union of graphs, lexicographic product, direct product, Kronecker product, Kotzig array.

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1. Definitions

All graphs $G = (V, E)$ are finite undirected simple graphs. For standard graph theoretic notation and definitions we refer to Diestel [10]. For a graph $G$, we use $V(G)$ for the vertex set and $E(G)$ for the edge set of $G$. The open neighborhood $N(x)$ (or more precisely $N_G(x)$, when needed) of a vertex $x$ is the set of all vertices adjacent to $x$, and the degree $d(x)$ of $x$ is $|N(x)|$, i.e., the size of the neighborhood of $x$. By $N[x]$ (or $N_G[x]$) we denote the closed neighborhood $N(x) \cup \{x\}$ of $x$. By $C_n$ we denote a cycle on $n$ vertices.

Different kinds of labelings have been an important part of graph theory for years. See a dynamic survey [14] which covers the field. The subject of our investigation is the distance magic labeling. A distance magic labeling of a graph $G$ of order $n$ is a bijection $\ell : V \to \{1, 2, \ldots, n\}$ such that there exists a positive

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integer $\mu$ such that the weight $w(v) = \sum_{u \in N(v)} \ell(u) = \mu$ for all $v \in V$, where $N(v)$ is the open neighborhood of $v$. The constant $\mu$ is called the magic constant of the labeling $\ell$. Any graph which admits a distance magic labeling is called a distance magic graph. Closed distance magic graphs are a variation of distance magic graphs, where the sums are taken over the closed neighborhoods $N_G[x]$ instead of the open ones $N_G(x)$, see [3, 4].

The concept of distance magic labeling has been motivated by the equalized incomplete tournaments (see [11, 12]). Finding an $r$-regular distance magic labeling is equivalent to finding equalized incomplete tournament $EIT(n, r)$ [12]. In an equalized incomplete tournament $EIT(n, r)$ of $n$ teams with $r$ rounds, every team plays exactly $r$ other teams and the total strength of the opponents that team $i$ plays is $k$. Thus, it is easy to notice that finding an $EIT(n, r)$ is the same as finding a distance magic labeling of some $r$-regular graph on $n$ vertices.

From the point of view of this application it is interesting to find disconnected $r$-regular distance magic graphs (tournaments which could be played simultaneously in different locations). Therefore in the paper we show examples of distance magic graphs $G$ such that the union of $t$ disjoint copies of $G$, denoted $tG$, is distance magic as well.

We recall four graph products (see [16]). All four, the Cartesian product $G \Box H$, lexicographic product $G \circ H$, direct product $G \times H$ and the strong product $G \boxtimes H$ are graphs with the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $(g', h')$ are adjacent in:

- $G \Box H$ if $g = g'$ and $h$ is adjacent to $h'$ in $H$, or $h = h'$ and $g$ is adjacent to $g'$ in $G$,
- $G \times H$ if $g$ is adjacent to $g'$ in $G$ and $h$ is adjacent to $h'$ in $H$,
- $G \boxtimes H$ if $g = g'$ and $h$ is adjacent to $h'$ in $H$, or $h = h'$ and $g$ is adjacent to $g'$ in $G$, or $g$ is adjacent to $g'$ in $G$ and $h$ is adjacent to $h'$ in $H$,
- $G \circ H$ if either $g$ is adjacent to $g'$ in $G$ or $g = g'$ and $h$ is adjacent to $h'$ in $H$.

The graph $G \circ H$ is also called the composition and denoted by $G[H]$ (see [17]). The product $G \times H$ is also known as Kronecker product, tensor product, categorical product and graph conjunction. The direct product is commutative, associative, and it has several applications, for instance it may be used as a model for concurrency in multiprocessor systems [19]. Some other applications can be found in [18].

Some product related graphs, which are distance magic or closed distance magic can be found in [1–5, 9, 21, 22].

**Theorem 1.1** [21]. Let $r \geq 1$, $n \geq 3$, $G$ be an $r$-regular graph and $C_n$ be the cycle of length $n$. Then the graph $G \circ C_n$ admits a distance magic labeling if and only if $n = 4$. 

Theorem 1.2 [2]. Let $G$ be an arbitrary regular graph. Then $G \times C_4$ is distance magic.

Theorem 1.3 [22]. The Cartesian product $C_n \square C_m$ is distance magic if and only if $n \equiv m \equiv 2 \pmod{4}$ and $n = m$.

Theorem 1.4 [2]. A graph $C_m \times C_n$ is distance magic if and only if $n = 4$ or $m = 4$, or $m \equiv n \equiv 0 \pmod{4}$.

Theorem 1.5 [3]. A graph $C_m \boxtimes C_n$ is distance magic if and only if at least one of the following conditions holds:

1. $m \equiv 3 \pmod{6}$ and $n \equiv 3 \pmod{6}$.
2. $\{m, n\} = \{3, x\}$ and $x$ is an odd number.

Let $K(n; r)$ denote the complete $r$-partite graph $K(n, n, \ldots, n)$.

Theorem 1.6 [8]. The Cartesian product $K(n; r) \square C_4$ is distance magic if and only if $n > 2$, $r > 1$ and $n$ is even.

The $d$-dimensional hypercube is denoted $Q_d$ where the vertices are binary $d$-tuples and two vertices are adjacent if and only if the $d$-tuples differ precisely in one position.

Theorem 1.7 [15]. A hypercube $Q_d$ has a distance magic labeling if and only if $d \equiv 2 \pmod{4}$.

The circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges $(x_i, x_{i+s_j})$ for $i = 0, \ldots, n-1$, $j = 1, \ldots, k$ where $i + s_j$ is taken modulo $n$.

Theorem 1.8 [7]. Let $p \geq 2$ and $n = p^2 - 1$ when $p$ is odd and $n = 2(p^2 - 1)$ when $p$ is even. Then $C_n(1, p)$ is a distance magic graph.

Theorem 1.9 [6]. If $p > 1$ is odd, then $C_{2p(p+1)}(1, 2, \ldots, p)$ is a distance magic graph.

By $tG$ we denote $t$ disjoint copies of a graph $G$. Here are some examples of disconnected distance magic graphs.

Theorem 1.10 [13, 20]. Let $nr$ be odd, $t$ be even, $r > 1$ and $t \geq 2$. Then $tK(n; r)$ is distance magic if and only if $r \equiv 3 \pmod{4}$.

Theorem 1.11 [20]. Let $m \geq 1$, $n \geq 2$ and $p \geq 3$. Then $mC_p \circ K_n$ has a distance magic labeling if and only if $n$ is even or $mnp$ is odd or $n$ is odd and $p \equiv 0 \pmod{4}$.
Theorem 1.12 [9]. Let $m$ and $n$ be two positive even integers such that $m \leq n$. The graph $G = tK_{m,n}$ is distance magic if and only if the following conditions hold:

- $m + n \equiv 0 \pmod{4}$, and
- $1 = 2(2tn + 1)^2 - (2tm + 2tn + 1)^2$ or $m \geq (\sqrt{2} - 1)n + \frac{\sqrt{2} - 1}{2r}$.

Theorem 1.13 [3]. Given $n \geq 2$ and $t \geq 1$, the union $tK_n$ is closed distance magic if and only if $n(t + 1) \equiv 0 \pmod{2}$.

We say that an $r$-regular graph $G$ has a $p$-partition if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ (that is, $V(G) = V_1 \cup V_2 \cup \cdots \cup V_p$ where $V_i \cap V_j = \emptyset$ for $i \neq j$) such that for every $x \in V(G)$

$|N(x) \cap V_1| = |N(x) \cap V_2| = \cdots = |N(x) \cap V_p|$.

Analogously we say that an $r$-regular graph $G$ has a closed $p$-partition if there exists a partition of the set $V(G)$ into $V_1, V_2, \ldots, V_p$ such that for every $x \in V(G)$

$|N[x] \cap V_1| = |N[x] \cap V_2| = \cdots = |N[x] \cap V_p|$.

We show that if a distance magic graph $H$ has a 2-partition, then $tH$ is distance magic for every positive integer $t$. Moreover, for an $r$-regular graph $G$ the products $G \circ H$ and $G \times H$ are distance magic as well, and thus we generalize Theorems 1.1 and 1.2.

2. Distance Magic Graphs

Lemma 2.1. Let $G$ be an $r$-regular graph of order $n$ with a 2-partition (closed 2-partition). If $G$ is a distance magic (closed distance magic) graph, then $tG$ is a distance magic (closed distance magic) graph for any positive integer $t$.

**Proof.** Let $\ell$ be a distance magic (closed distance magic) labeling of $G$ with the magic constant $\mu$. In each copy $G^1, G^2, \ldots, G^t$ of $G$ we apply the partition defined above such that $V^1_j \cup V^2_j$ is the partition of the $j$-th copy $G^j$ of $G$. Define

$$\ell'(x) = \begin{cases} 
\ell(x) + (j - 1)n, & \text{if } x \in V^1_j, \\
\ell(x) + (t - j)n, & \text{if } x \in V^2_j.
\end{cases}$$

Obviously, $\ell'$ is a distance magic (closed distance magic) labeling of the graph $tG$ with the magic constant $\mu' = \mu + (t - 1)nr/2$ (closed magic constant $\mu' = \mu + (t - 1)n(r + 1)/2$). □
We will now use Kotzig arrays as a tool. A Kotzig array was defined in [23] to be a \( j \times k \) matrix, each row being a permutation of \( \{0, 1, \ldots, k - 1\} \) and each column having a constant sum.

**Lemma 2.2** [23]. A Kotzig array of size \( j \times k \) exists whenever \( j > 1 \) and \( j(k - 1) \) is even.

The following lemma shows that even if an \( r \)-regular distance magic graph \( G \) has no 2-partition, the union \( tG \) can be distance magic.

**Lemma 2.3.** Let \( p \geq 2 \) and \( G \) be an \( r \)-regular graph of order \( n \) having a \( p \)-partition (closed \( p \)-partition). If \( G \) is a distance magic (closed distance magic) graph, then for \( t \geq 0 \) where \( p(t - 1) \) is even the graph \( tG \) is also distance magic (closed distance magic).

**Proof.** Let \( \ell \) be a distance magic (closed distance magic) labeling of \( G \) with the magic constant \( \mu \). In each copy \( G^1, G^2, \ldots, G^t \) of \( G \) we apply the partition defined above such that \( V_1^j \cup V_2^j \cup \cdots \cup V_k^j \) is the partition of \( j \)-th copy \( G^j \) of \( G \).

Let \( A = (a_{i,j}) \) be a Kotzig array of size \( p \times t \). Define

\[
\ell'(x) = \ell(x) + na_{n_i,j}, \quad x \in V_i^j.
\]

Obviously, \( \ell' \) is the distance magic (closed distance magic) labeling of the graph \( tG \) with a magic constant \( \mu' = \mu + (t - 1)nr/2 \) (closed magic constant \( \mu' = \mu + (t - 1)n(r + 1)/2 \)).

We will now present some examples of graphs that have the desired 2-partition.

**Observation 1.** If

1. \( G = C_n \square C_n \) for \( n = m \) and \( n \equiv m \equiv 2 \pmod{4} \),
2. \( G = C_n \times C_m \) for \( n = 4 \) or \( m = 4 \), or \( m \equiv n \equiv 0 \pmod{4} \),
3. \( G = K(n; r) \square C_4 \) for \( n > 2 \), \( r > 1 \) and \( n \) even,
4. \( G = Q_d \) for \( d \equiv 2 \pmod{4} \),
5. \( G = C_{p^2-1}(1, p) \) for \( p \) odd,
6. \( G = C_{2(p^2-1)}(1, p) \) for \( p \) even,
7. \( G = C_{2p(p+1)}(1, 2, \ldots, p) \) for \( p \) odd,

then \( G \) has a 2-partition.

**Proof.** 1. Let \( V(C_m \square C_n) = \{v_{i,j} : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\} \), where \( N(v_{i,j}) = \{v_{i-1,j}, v_{i+1,j}, v_{i,j-1}, v_{i,j+1}\} \) and the addition in the first suffix is taken modulo \( m \) and in the second suffix modulo \( n \). Let \( V_1 = \{v_{i,j} : i = 0, 1, \ldots, m - 1, j = 0, 2, \ldots, n - 2\}, V_2 = \{v_{i,j} : i = 0, 1, \ldots, m - 1, j = 1, 3, \ldots, n - 1\} \). Notice that for any \( v \in G \) we obtain \( |N(v) \cap V_1| = |N(v) \cap V_2| = 2 \).
2. Let \( V(C_m \times C_n) = \{v_{i,j} : 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \), where \( N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j}, v_{i,j+1}, v_{i+1,j+1}\} \) and the addition in the first suffix is taken modulo \( m \) and in the second suffix modulo \( n \). Let \( V_1 = \{v_{i,j} : i \equiv 0, 1 \pmod{4}, j = 0, 1, \ldots, n-1\} \), \( V_2 = \{v_{i,j} : i \equiv 2, 3 \pmod{4}, j = 0, 1, \ldots, n-1\} \). Notice that for any \( v \in G \) we obtain \(|N(v) \cap V_1| = |N(v) \cap V_2| = 2\).

3. Let \( V(K(n;r)) = \{v_i : i = 1, \ldots, n, j = 1, \ldots, r\} \), \( C_4 = xuywx \), and \( H = K(n;r) \square C_4 \). Let \( V_1 = \{(v_i, x), (v_i, u), (v_{i/2+i}, y), (v_{i/2+i}, w)\} \), where \( i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r\), \( V_2 = \{(v_{i/2+i}, x), (v_{i/2+i}, u), (v_i, y), (v_i, w)\} \), where \( i = 1, 2, \ldots, n/2, j = 1, 2, \ldots, r\). Obviously for any \( v \in G \) we obtain \(|N(v) \cap V_1| = |N(v) \cap V_2| = n(r-1)/2 + 1\).

4. Let us define the set of vertices of \( Q_n \) as the set of binary strings of length \( n \), that is, \( V = \{a_1, a_2, \ldots, a_n\}; a_i \in \{0, 1\} \). Two vertices are adjacent if and only if the corresponding strings differ in exactly one position. Then \( V_1 = \{(a_1, \ldots, a_n)\}, \) where \( a_1 + \cdots + a_{n/2} \) is even, \( V_2 = \{(a_1, \ldots, a_n)\}, \) where \( a_1 + \cdots + a_{n/2} \) is odd. Notice that each vertex has \( n/2 \) neighbours in \( V_1 \) and \( n/2 \) in \( V_2 \).

5. Let \( V(G) = \{x_0, x_1, \ldots, x_{p^2-2}\} \), where \( N(x_i) = \{x_i-p, x_{i-1}, x_{i+1}, x_{i+p}\} \) and the addition in the suffix is taken modulo \( n \). Let \( V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 1, \ldots, p-1\} \), \( V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, p\} \). Notice that for any \( v \in G \) we obtain \(|N(v) \cap V_1| = |N(v) \cap V_2| = 2\).

6. Let \( V(G) = \{x_0, x_1, \ldots, x_{2p^2-3}\} \), where \( N(x_i) = \{x_i-p, x_{i-1}, x_{i+1}, x_{i+p}\} \) and the addition in the suffix is taken modulo \( n \). Let \( V_1 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 0, 1, \ldots, 2p\} \), \( V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, 2p-1\} \). Notice that for any \( v \in G \) we obtain \(|N(v) \cap V_1| = |N(v) \cap V_2| = 2\).

7. Let \( V(G) = \{x_0, x_1, \ldots, x_{2p(p+1)-1}\} \), where \( N(x_i) = \{x_i-p, x_{i-1}, x_{i+1}, x_{i+p}\} \) and the addition in the suffix is taken modulo \( n \). Let \( V_1 = \{x_{i+j(p+1)} : i = 0, 1, \ldots, p, j = 0, 2, \ldots, 2p-2\} \), \( V_2 = \{x_{i+j(p-1)} : i = 0, 1, \ldots, p-1, j = 1, 3, \ldots, 2p-1\} \). Notice that for any \( v \in G \) we obtain \(|N(v) \cap V_1| = |N(v) \cap V_2| = p\).

Below we show some interesting properties of distance magic unions of graphs.

**Theorem 2.4.** If \( G \) is an \( r \)-regular graph of order \( t \) and \( H \) is \( p \)-regular such that \( tH \) is distance magic, then the product \( G \circ H \) is distance magic.

**Proof.** Let \( \ell \) be a distance magic labeling of the graph \( tH = H_1 \cup H_2 \cup \cdots \cup H_t \) with a magic constant \( \mu \). For any \( u \in V(H) \) let \( u_j \) be the corresponding vertex belonging to \( V(H_j) \), \( j = 1, 2, \ldots, t \). Let \( V(G) = \{1, 2, \ldots, t\} \). Notice that for any \( i = 1, 2, \ldots, t \) we have \( \sum_{v \in V(H_i)} \ell(v) = \frac{|H|\mu}{p} \).

Define the labeling \( \ell' \) of \( G \circ H \) as \( \ell'(j, u) = \ell(u_j) \) for \( u \in V(H) \), \( u_j \in V(H_j) \), \( j = 1, 2, \ldots, t \). Obviously, \( \ell' \) is a bijection. Moreover, for any \( (g, h) \in V(G \circ H) \)
we obtain
\[
\sum_{(j,u) \in N_{G \circ H}((g,h))} \ell'(j,u) = \sum_{j \in N_G(g)} \sum_{u \in V(H)} \ell'(j,u) + \sum_{u \in N_H(h)} \ell'(g,u)
\]
\[
= r \sum_{u \in V(H)} \ell(u_j) + \sum_{u \in N_H(h)} \ell(u_g) = (2r + 1)\mu,
\]
for any \((g,h) \in V(G \circ H)\).

**Theorem 2.5.** If \(G\) is an \(r\)-regular graph of order \(t\) and \(H\) is \(p\)-regular such that \(tH\) is closed distance magic, then the product \(G \circ H\) is closed distance magic.

Using the same technique we can prove an analogous theorem for closed distance magic labeling.

**Theorem 2.5.** If \(G\) is an \(r\)-regular graph of order \(t\) and \(H\) is \(p\)-regular such that \(tH\) is closed distance magic, then the product \(G \circ H\) is closed distance magic.

Notice that the assumption that \(H\) is a regular graph is not necessary, as shown in the observation below.

**Observation 2.** Let \(G\) be an \(r\)-regular graph of order \(t\). If \(m\) and \(n\) are two positive even integers such \(m + n \equiv 0 \pmod{4}\) and either \(2(2tn + 1)^2 - (2tm + 2tn + 1)^2 = 1\) or \(m \geq (\sqrt{2} - 1)n + \sqrt{2} - 1\), then the product \(G \circ K_{m,n}\) is distance magic.

**Proof.** The graph \(tK_{m,n}\) is distance magic by Theorem 1.12. Let \(\ell\) be a distance magic labeling of the graph \(tK_{m,n} = K_{m,n}^1 \cup K_{m,n}^2 \cup \cdots \cup K_{m,n}^t\) with the magic constant \(\mu\). For any \(u \in V(K_{m,n})\) let \(u_j\) be the corresponding vertex belonging to \(V(K_{m,n}^j), j = 1, 2, \ldots, t\). Let \(V(G) = \{1, 2, \ldots, t\}\). We have \(\sum_{v \in V(K_{m,n})} \ell(v) = 2\mu\) for any \(i = 1, 2, \ldots, t\). Define the labeling \(\ell'\) of \(G \circ H\) as \(\ell'(j,u) = \ell(u_j)\) for \(u \in V(K_{m,n}), u_j \in V(K_{m,n}^j), j = 1, 2, \ldots, t\). As in the proof of Theorem 2.4 we have
\[
\sum_{(j,u) \in N_{G \circ K_{m,n}^j}((g,h))} \ell'(j,u) = \sum_{j \in N_G(g)} \sum_{u \in V(H)} \ell'(j,u) + \sum_{u \in N_H(h)} \ell'(g,u)
\]
\[
= r \sum_{u \in V(H)} \ell(u_j) + \sum_{u \in N_H(h)} \ell(u_g) = (2r + 1)\mu,
\]
for any \((g,h) \in V(G \circ H)\).

**Theorem 2.6.** If \(G\) is an \(r\)-regular graph of order \(t\) and \(H\) is such that \(tH\) is distance magic, then the product \(G \times H\) is distance magic.

**Proof.** Let \(\ell\) be a distance magic labeling of the graph \(tH = H_1 \cup H_2 \cup \cdots \cup H_t\) with the magic constant \(\mu\). For any \(u \in V(H)\) let \(u_j\) be the corresponding vertex
belonging to \( V(H_j), \ j = 1, 2, \ldots, t \). Let \( V(G) = \{1, 2, \ldots, t\} \). Set the labeling \( \ell' \) of \( G \times H \) as \( \ell'(j, u) = \ell(u_j) \) for \( u \in V(H), \ u_j \in V(H_j), \ j = 1, 2, \ldots, t \). Therefore

\[
\begin{align*}
\sum_{(j, u) \in N_G(\times) \times N_H(h)} \ell'(j, u) &= \sum_{j \in N_G(g)} \sum_{u_j \in N_{H_j}(h_j)} \ell(u_j) = \sum_{j \in N_G(g)} \mu = r \mu,
\end{align*}
\]

for any \((g, h) \in V(G \times H)\).

Now we present a theorem, which is a corollary of Lemma 2.1 and Theorems 2.4 and 2.6.

**Theorem 2.7.** If \( G \) is an \( r \)-regular graph and \( H \) is a \( p \)-regular distance magic graph with a 2-partition, then the products \( G \circ H \) and \( G \times H \) are both distance magic.

Notice that even if \( G \) and \( H \) are both regular distance magic graphs with 2-partitions, then the product \( G \square H \) is not necessarily distance magic (for instance \( G = H = C_4 \)).

Below are presented some families of disconnected distance magic graphs.

**Theorem 2.8.** If

1. \( H = C_n \square C_m \) for \( n = m \) and \( m \equiv n \equiv 2 \pmod{4} \),
2. \( H = C_n \times C_m \) for \( n = 4 \) or \( m = 4 \), or \( m \equiv n \equiv 0 \pmod{4} \),
3. \( H = K(n; r) \square C_4 \) for \( n > 2, \ r > 1 \) and \( n \) even,
4. \( H = Q_d \) for \( d \equiv 2 \pmod{4} \),
5. \( H = C_{p^2-1}(1, p) \) for \( p \) odd,
6. \( H = C_{2(p-1)}(1, p) \) for \( p \) even,
7. \( H = C_{2p(p+1)}(1, 2, \ldots, p) \) for \( p \) odd,

then \( tH \) is distance magic. Moreover, if \( G \) is an \( r \)-regular graph, then the products \( G \circ H \) and \( G \times H \) are distance magic as well.

**Proof.** We obtain that \( tH \) is distance magic by Lemma 2.1, Observation 1 and Theorems 1.3, 1.4, 1.6, 1.7, 1.8 and 1.9, respectively. By Theorem 2.7 we obtain now that \( G \circ H \) and \( G \times H \) are distance magic.

We conclude this section with an observation that can be obtained easily by applying Theorems 1.10, 1.11, 2.4 and 2.6.

**Observation 3.** If \( G \) is an \( r \)-regular graph of order \( t \) and

1. \( H = K(n; p) \) for \( n \) odd, \( t \geq 2 \) even and \( p \equiv 3 \pmod{4} \),
2. $H = C_p \circ \overline{K}_n$ for $t \geq 1$, $n \geq 3$ and $p \geq 3$, $tnp$ odd or $n$ odd and $p \equiv 0 \pmod{4}$, then the products $G \circ H$ and $G \times H$ are distance magic.

3. Closed Distance Magic Graphs

We start with the following observations about closed distance magic graphs:

**Observation 4** [4]. Let $u$ and $v$ be vertices of a closed distance magic graph. Then $|N(u) \cup N(v)| = 0$ or $|N(u) \cup N(v)| > 2$.

**Observation 5** [3]. If $G$ is an $r$-regular graph on $n$ vertices having a closed distance magic labeling with a magic constant $\mu'$, then $\mu' = \frac{(r+1)(n+1)}{2}$.

We will present now two examples of graphs that have a closed 3-partition.

**Observation 6.** If
1. $G = C_3$, or
2. $G = C_n \boxtimes C_m$ for $n = 3$ and $m$ odd, or $m \equiv n \equiv 3 \pmod{6}$,
then $G$ has the closed 3-partition.

**Proof.** 1. Let $V(C_3) = \{v_0, v_1, v_2\}$. Let $V_i = \{v_i\}$ for $i = 0, 1, 2$.
2. Let $V(C_m \boxtimes C_n) = \{v_{i,j} : 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, where $N(v_{i,j}) = \{v_{i-1,j-1}, v_{i-1,j+1}, v_{i,j-1}, v_{i,j+1}, v_{i+1,j-1}, v_{i+1,j+1}, v_{i,j+1}, v_{i,j+1}\}$ and the addition in the first suffix is taken modulo $m$ and in the second suffix modulo $n$. Let $V_p = \{v_{i,j} : i + j \equiv p \pmod{3}\}$. Notice that for any $v \in G$ we obtain $|N[v] \cap V_1| = |N[v] \cap V_2| = |N[v] \cap V_3| = \frac{mn}{3}$.

**Theorem 3.1.** If
1. $G = C_3$, or
2. $G = C_n \boxtimes C_m$ for $n = 3$ and $m$ odd, or $m \equiv n \equiv 3 \pmod{6}$,
then $tG$ is closed distance magic if and only if $t$ is odd.

**Proof.** Notice that if $G = C_3$ then it is closed distance magic. Note that $G = C_n \times C_m$ for $n = 3$ and $m$ odd, or $m, n \equiv 3 \pmod{6}$, is closed distance magic by Theorem 1.13. Since $G$ has a closed 3-partition, then the graph $tG$ is closed distance magic by Lemma 2.3 for odd $t$. Observe that $G$ is an $r$-regular graph with $r$ even. Suppose now that $t$ is even. Then $|V(tG)|$ is even as well and $\frac{(r+1)(|V(tG)|+1)}{2}$ is not an integer. Therefore the graph $G$ is not closed distance magic by Observation 5.
By Lemma 4 it is now obvious that $tC_n$ is closed distance magic if and only if $t$ is odd and $n = 3$. Moreover, by Theorem 2.4 we obtain immediately the following observation.

**Observation 7.** When $G$ is an $r$-regular graph with $r$ odd and
1. $H = C_3$, or
2. $H = C_n \boxtimes C_m$ for $n = 3$ and $m$ odd, or $m \equiv n \equiv 3 \pmod{6}$,

then the product $G \circ H$ is closed distance magic.

**References**


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