ON THE $H$-FORCE NUMBER OF HAMILTONIAN GRAPHS 
AND CYCLE EXTENDABILITY

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Abstract

The $H$-force number $h(G)$ of a hamiltonian graph $G$ is the smallest cardinality of a set $A \subseteq V(G)$ such that each cycle containing all vertices of $A$ is hamiltonian. In this paper a lower and an upper bound of $h(G)$ is given. Such graphs, for which $h(G)$ assumes the lower bound are characterized by a cycle extendability property. The $H$-force number of hamiltonian graphs which are exactly 2-connected can be calculated by a decomposition formula.

Keywords: cycle, hamiltonian graph, $H$-force number, cycle extendability.

2010 Mathematics Subject Classification: 05C45.

1. Introduction

Throughout this paper, only finite graphs without loops or multiple edges are considered. The number of vertices of a graph $G$, i.e., its order will be denoted by $n$. We use the standard graph terminology according to [3].

Let $G$ be a hamiltonian graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A nonempty vertex set $X \subseteq V(G)$ is called a hamiltonian cycle enforcing set (for short, $H$-force set) of $G$ if every $X$-cycle of $G$ (i.e., a cycle of $G$ containing all vertices of $X$) is a hamiltonian one. Let $h(G)$ denote the smallest cardinality of an $H$-force set of $G$ and call it the $H$-force number of $G$. The concepts of $H$-force set and $H$-force number were first given by Fabrici et al. (see [4]) and studied there for several special families of hamiltonian graphs. Timková (see [9]) determined the $H$-force number of generalized dodecahedral graphs. Note also, that the concepts of $H$-force set and $H$-force number were extended to hamiltonian digraphs and hypertournaments in [10] and [7], respectively.
The authors in [4] observed that the $H$-force number $h(G)$ of a hamiltonian graph $G$ satisfies

- $h(G) = 1$ if and only if $G$ is a cycle,
- $h(G) = n$ if and only if $G$ is 1-hamiltonian (that is, if $G$ is hamiltonian and $G - v$ is hamiltonian for every $v \in V$).

For a hamiltonian graph $G$, we define sets $S = S(G) = \{x \in V \mid G - x$ is hamiltonian$\}$ and $T = T(G) = \{x \in V \mid G - x$ is 2-connected$\}$. Then, we have $S \subseteq T$. Let $s(G) = |S(G)|$ and $t(G) = |T(G)|$.

**Proposition 1.** Let $G$ be a hamiltonian graph and $P$ be a path of $G$ containing no branch vertex of $G$, i.e., no vertex of degree at least 3 in $G$. Then, every smallest $H$-force set $F \subseteq V(G)$ contains at most one vertex of $P$.

Let $H$ be the family of hamiltonian graphs that do not contain adjacent vertices of degree 2. Also, let $G'$ be the graph formed from a hamiltonian graph $G$ by replacing each maximal path not containing a branch vertex by a single vertex. Then, $G'$ is hamiltonian and has no adjacent vertices of degree 2, so $G' \in H$. Because $h(G') = h(G)$, it is sufficient to restrict our study to the family $H$.

The main results of this paper are Theorems 2, 7, 8 and 11. Theorem 2 shows that $s(G)$ and $t(G)$ form bounds for the $H$-force number $h(G)$. After this theorem, we discuss some consequences. Theorem 7 contains a decomposition formula for the $H$-force number of hamiltonian graphs which are exactly 2-connected. In Theorem 8 hamiltonian graphs $G$ for which $S(G)$ is an $H$-force set are characterized by a cycle extendability property. Eventually, a sum formula for hamiltonian graphs $G$ with $s(G) < h(G)$ is proved in Theorem 11.

2. Results and Proofs

**Theorem 2.** Let $G \in H$. Then

$$s(G) \leq h(G) \leq t(G).$$

The proof of this theorem requires the following exchange property.

**Lemma 3.** Let $G \in H$ and let $F \subseteq V$ be a smallest $H$-force set of $G$. Then, for every vertex $v \in F \setminus T$ there exists a vertex $u \in T$ such that $(F \setminus \{v\}) \cup \{u\}$ is an $H$-force set of $G$.

**Proof.** Suppose there exists a vertex $v \in V \setminus T$. Then $G$ is exactly 2-connected. Let $C$ be any fixed hamiltonian cycle of $G$ and $w$ be a cut-vertex of $G - v$. Then, $C$ consists of two $v$-$w$-paths $P_1$ and $P_2$ both of which have at least one inner vertex but no inner vertex in common. Since $G$ is not a cycle, $C$ has a chord.
But, there is no chord connecting an inner vertex of \( P_1 \) with an inner vertex of \( P_2 \). Let \( F \subseteq V \) be a smallest \( H \)-force set of \( G \) (i.e., \( |F| = h(G) \)) and suppose \( v \in F \).

**Case 1.** The cut-vertex \( w \) of \( G - v \) can be chosen so that each \( P_i \), for \( i = 1, 2 \), has a chord of \( C \), say \( x_i y_i \). Then, the subpath \((x_i, y_i)\) of \( P_i \) contains an inner vertex \( z_i \) such that \( z_i \notin F \). Otherwise, the \( x_i-y_i \)-path on \( C \) which passes \( v \) forms together with \( x_i y_i \) a non-hamiltonian \( F \)-cycle. By the choice of \( F \), \( F \setminus \{v\} \) is not an \( H \)-force set of \( G \), i.e., \( G \) contains a non-hamiltonian \((F \setminus \{v\})\)-cycle \( C' \) not passing \( v \). Since \( z_1 \) and \( z_2 \) belong to different components of \( G - \{v, w\} \) and since \( w \) is a cut-vertex of \( G - v \), every \( z_1-z_2 \)-path of \( G - v \) is passing \( w \) which contradicts the fact that \( C' \) is a cycle.

**Case 2.** By any choice of the cut-vertex \( w \) of \( G - v \) only one of \( P_1 \) and \( P_2 \) has a chord. Suppose for a fixed \( w \) that \( P_1 \) has no chord. Then \( P_1 \) has only one inner vertex \( u \) where \( d_G(u) = 2 \). Since every hamiltonian cycle of \( G \) passes the edge \( uv \), \( F' := (F \setminus \{v\}) \cup \{u\} \) is also an \( H \)-force set of \( G \). Moreover, we have \( u \in T \) because otherwise there exists a cut-vertex \( z \) of \( G - u \) which is also a cut-vertex of \( G - v \). Hence, \( C \) consists of two \( v-z \)-paths (with no common inner vertices) such that both of them have at least one chord, a contradiction. That proves the assertion.

**Proof of Theorem 2.** Let \( F \subseteq V \) be any smallest \( H \)-force set of \( G \). Suppose that \( S \) contains a vertex \( x \) such that \( x \notin F \). A hamiltonian cycle \( C \) of \( G - x \) is, obviously, a non-hamiltonian \( F \)-cycle of \( G \). That is a contradiction and proves \( S \subseteq F \) and, consequently, \( s(G) \leq h(G) \).

Let \( F \subseteq V \) be a smallest \( H \)-force set of \( G \). If \( F \subseteq T \) then \( h(G) \leq t(G) \) trivially holds. Otherwise, there exists an \( x \in F \setminus T \). By Lemma 3 there is a \( y \in T \) such that \((F \setminus \{x\}) \cup \{y\} \) is an \( H \)-force set of \( G \), too. The repeated use of the above exchange property finally yields a smallest \( H \)-force set \( F' \subseteq T \) and proves the upper bound.

From the proof of Theorem 2, we have \( S \subseteq F \) and we can choose \( F \) such that \( F \subseteq T \).

**Corollary 4.** Let \( G \in \mathcal{H} \). Then,

(i) \( s(G) = n \) if and only if \( h(G) = n \).

(ii) If \( s(G) = n - 1 \), then \( h(G) = n - 1 \).

**Proof.** Statement (i) is an immediate consequence of the lower bound in Theorem 2.

If \( s(G) = n - 1 \), then the lower bound of Theorem 2 implies \( h(G) \geq n - 1 \), and by (i) we have \( h(G) \neq n \) which proves (ii).
The graph $G$ of order 20 shown in Figure 1 is hamiltonian (the bold painted edges form a hamiltonian cycle) with $S = V \setminus \{x, y\}$ and with $V \setminus \{x\}$ as a smallest $H$-force set confirms that the converse of statement (ii) does not hold.

Theorem 2 has the following two consequences. A planar graph is called outerplanar if it can be embedded in the plane in such a way that every vertex is incident with the unbounded face.

**Theorem 5.** Let $G \in \mathcal{H}$ be outerplanar. Then $h(G)$ corresponds to the number of vertices of degree 2 whose two neighbours are adjacent.

**Proof.** Let $G \in \mathcal{H}$ be outerplanar and let $x \in V$. If $d_G(x) \geq 3$ then $x \notin T$ and also $x \notin S$. Assume otherwise $d_G(x) = 2$ and let $y, z \in V$ denote the neighbours of $x$. If $yz \notin E$ then $x \notin T$ and also $x \notin S$. If $yz \in E$ then $G - x$ is hamiltonian which yields $x \in S$ and, consequently, $x \in T$. Hence, $S = T$ and the statement can be deduced from Theorem 2.

In [4], the $H$-force number of an outerplanar hamiltonian graph $G$ different from a cycle was proved to be equal to the number of leafs of the weak dual of $G$. The weak dual of an outerplanar graph $G$ is a tree and is obtained from the dual of $G$ by removing the vertex corresponding to the unbounded face.

**Theorem 6.** For $G \in \mathcal{H}$, $h(G) = 2$ if and only if $t(G) = 2$.

**Proof.** Suppose first $h(G) = 2$. Then by Lemma 3 there exists a smallest $H$-force set $F = \{x, y\}$ of $G$ such that $F \subseteq T$. Assume that there exists a vertex
\( v \in T \setminus F \) which means that \( G - v \) is 2-connected. Then, \( G - v \) and, consequently, \( G \) has two different \( x\)-\( y \)-paths with no common inner vertices. Hence, \( G \) has an \( F \)-cycle not passing \( v \), a contradiction. That proves \( F = T \) and \( t(G) = 2 \).

Suppose now \( t(G) = 2 \). Since \( G \) is not a cycle we have \( h(G) \geq 2 \). And, by Theorem 2 we have \( h(G) \leq 2 \) which completes the proof.

In [4], hamiltonian graphs with \( H \)-force number 2 have been characterized already by a condition on crossed chords of a hamiltonian cycle. In [4] they also noted that every hamiltonian graph with \( h(G) = 2 \) is planar.

Now, we give a decomposition formula with respect to the \( H \)-force number of a hamiltonian graph which is exactly 2-connected. To that end, let \( G \in \mathcal{H} \) be a graph with vertices \( u, v \in V(G) \) such that \( G - \{u, v\} \) is disconnected, i.e., \( u, v \notin T \).

Any given hamiltonian cycle \( C \) of \( G \) can be divided into two \( u \)-\( v \)-paths \( P_1 \) and \( P_2 \) which have no inner vertices in common. For \( i = 1, 2 \), let \( G_i \) denote the graph which results from \( G[V(P_i)] \) (the subgraph of \( G \) induced by \( V(P_i) \)) by introducing an additional vertex \( w_i \) \((w_1 \neq w_2) \) and edges \( uv, uw_i, vw_i \). Obviously, \( G_i \) is also a member of \( \mathcal{H} \).

**Theorem 7.** Let \( G \in \mathcal{H} \) with \( u, v \in V(G) \) such that \( G - \{u, v\} \) is disconnected, and let \( G_1, G_2 \) be graphs derived from \( G \) as described above. Then,

\[
h(G) = h(G_1) + h(G_2) - 2.
\]

**Proof.** On the one hand, from \( u, v \notin T(G_i) \) and Lemma 3 it follows that \( G_i \) has a smallest \( H \)-force set \( F_i \subseteq V(G_i) \) such that \( u, v \notin F_i \). \( F_i \) contains \( w_i \) because \( G_i - w_i \) is hamiltonian. Let \( F := (F_1 \setminus \{w_1\}) \cup (F_2 \setminus \{w_2\}) \) and let \( C_F \) denote an \( F \)-cycle of \( G \). \( F_i \setminus \{w_i\} \) is not empty for \( i = 1, 2 \) which implies that neither \( G_1 \) nor \( G_2 \) contains \( C_F \) as a cycle. Suppose that \( C_F \) is not a hamiltonian cycle of \( G \). Then, without loss of generality, there exists a vertex \( x \in V(G) \setminus V(G_2) \) which is not contained in \( F \). Let \( P_{F_1} \) denote the \( u \)-\( v \)-path of \( C_F \) which is contained in \( G_1 \). Then, the cycle obtained by connecting \( P_{F_1} \) with the \( u \)-\( v \)-path \((u, w_1, v)\) is an \( F_1 \)-cycle of \( G_1 \) which is not hamiltonian, a contradiction. Consequently, \( F \) is an \( H \)-force set of \( G \) and

\[
h(G) \leq |F| = |F_1 \setminus \{w_1\}| + |F_2 \setminus \{w_2\}| = (|F_1| - 1) + (|F_2| - 1)
\]

\[= h(G_1) + h(G_2) - 2.\]

On the other hand, Lemma 3 implies that \( G \) has an \( H \)-force set \( F \subseteq V(G) \) where \( |F| = h(G) \) and \( u, v \notin F \). Clearly, \( F_i := (F \cap V(G_i)) \cup \{w_i\} \) is a subset of \( V(G_i) \). If \( C_i \) denotes an \( F_i \)-cycle of \( G_i \), then \( C_i \) contains \( w_i \) and also the vertices \( u \) and \( v \). Hence, \( C_i - w_i \) is a \( u \)-\( v \)-path of \( G_i \) and also of \( G \). By connecting the \( u \)-\( v \)-paths \( C_1 - w_1 \) and \( C_2 - w_2 \) we obtain an \( F \)-cycle \( \tilde{C} \) in \( G \). If \( C_i \) for \( i = 1 \) or \( 2 \) would not be hamiltonian in \( G_i \), then \( \tilde{C} \) could not be hamiltonian in \( G \).
This contradicts the fact that $F$ is an $H$-force set of $G$ and implies that $F_i$ is an $H$-force set of $G_i$. Hence,

$$h(G) = |F| = (|F_1| - 1) + (|F_2| - 1) \geq (h(G_1) - 1) + (h(G_2) - 1) = h(G_1) + h(G_2) - 2$$

which proves the statement of Theorem 7.

If, for example, $G_t$ denotes the hamiltonian graph which consists of a “chain” of $t \geq 1$ cube graphs (see Figure 2) then by induction and using Theorem 7 we obtain for the $H$-force-number $h(G_t) = 2t + 2$.

Next, we will give a characterization of hamiltonian graphs $G$ such that $S(G)$ is an $H$-force set of $G$ and, consequently, $h(G) = s(G)$. To this end, let us consider the concept of cycle extendable graphs (which was first investigated by Hendry in [5]) and weaken it in a suitable sense.

A cycle $C$ of a graph $G$ is called extendable if $G$ contains a $V(C)$-cycle $C'$ which has exactly one vertex more than $C$. A graph $G$ is called cycle extendable if $G$ contains a cycle and if every non-hamiltonian cycle is extendable. Cycle extendable graphs are obviously hamiltonian ones.

In [5], Hendry raised the problem whether every hamiltonian chordal graph is cycle extendable or not. Jiang proved in [6] that every planar hamiltonian chordal graph is also cycle extendable. Moreover, a hamiltonian graph which is an interval graph or a split graph has been proved to be cycle extendable, see [1] and also [2].

Now, we call a non-hamiltonian cycle $C$ of a graph $G$ weakly extendable if $G$ contains a $V(C)$-cycle of length $n - 1$. And, a graph $G$ is called weakly cycle extendable if $G$ is hamiltonian and if every non-hamiltonian cycle is weakly extendable. Trivially, every cycle extendable graph is weakly cycle extendable. Every outerplanar graph which belongs to $\mathcal{H}$ is also weakly cycle extendable.

**Theorem 8.** Let $G \in \mathcal{H}$. Then, the following conditions are equivalent.

(i) $S(G)$ is an $H$-force set, i.e., $h(G) = s(G)$.

(ii) $G$ is weakly cycle extendable.
Proof. Suppose that $S = S(G)$ is an $H$-force set and that $G$ contains a cycle $C$ which is not weakly extendable. Then, $G - x$ is not hamiltonian for each $x \in V(G) \setminus V(C)$ which implies $x \notin S$. Hence, $C$ is an $S$-cycle which contradicts our claim that $S$ is an $H$-force set. Thus, $G$ is weakly cycle extendable.

Now, let $G$ be weakly cycle extendable and suppose that $S$ is not an $H$-force set. If $S$ is empty then $G - x$ is not hamiltonian for each $x \in V(G)$. Since $G$ is not a cycle, there exists a cycle $C$ in $G$ of length at most $n - 2$, and $C$ is not weakly extendable, a contradiction. So, suppose that $S$ is not empty and let $C$ be a non-hamiltonian $S$-cycle of $G$. Then, $C$ is weakly extendable, i.e., $G$ has a $V(C)$-cycle $C'$ of length $n - 1$. Suppose $C'$ does not contain a vertex $x \in V(G)$. Then $G - x$ is hamiltonian and, consequently, $x \in S$. That together with $x \in V(G) \setminus V(C') \subseteq V(G) \setminus V(C) \subseteq V(G) \setminus S$ yields a contradiction which proves that $S$ is an $H$-force set.

Hence, every weakly cycle extendable graph $G \in \mathcal{H}$ has a uniquely determined smallest $H$-force set. In Figure 3, a not weakly cycle extendable graph with a unique smallest $H$-force set (the two black vertices) is presented.

![Figure 3](image.png)

Theorem 9. Let $G \in \mathcal{H}$.

(i) If $s(G) \geq n - 1$, then $G$ is weakly cycle extendable.

(ii) If $s(G) \leq 1$, then $G$ is not weakly cycle extendable.

Proof. (i) If $s(G) = n$ then $G$ is 1-hamiltonian which implies that every non-hamiltonian cycle of $G$ is weakly extendable. If $s(G) = n - 1$ then every $S$-cycle is hamiltonian. For every other non-hamiltonian cycle $C$ of $G$, there is an $x \in S$ which is not contained in $C$. Since $G - x$ is hamiltonian, $C$ is a cycle of $G - x$ and, consequently, weakly extendable in $G$.

(ii) If $s(G) = 0$ then $G$ has no cycle of length $n - 1$, i.e., every non-hamiltonian cycle is not weakly extendable. If $s(G) = 1$ then, obviously, $G$ has at least five vertices. Let be $S = \{x\}$ and let $C$ be a hamiltonian cycle of $G - x$. Moreover, let $y$ and $z$ be two neighbors of $x$. Then, $C$ passes $y$ and $z$ and consists of two $y$-$z$-paths $P_1$ and $P_2$ with no common inner vertex. At least one of these paths has more than one inner vertex. Otherwise, because of $n \geq 5$, each of $P_1$ and
$P_2$ would have exactly one inner vertex which implies $s(G) > 1$, a contradiction. Suppose, now, that $P_1$ has at least two inner vertices. Then, $V(P_2) \cup \{x\}$ is the vertex set of a cycle $C'$ of length at most $n - 2$. $C'$ cannot be weakly extendable in $G$ because otherwise there would exist a $V(C')$-cycle of length $n - 1$ in $G$ which is different from $C$. That contradicts the claim $S(G) = \{x\}$. \hfill \blacksquare

For every integer $n \geq 9$ and all $k$ with $2 \leq k \leq n - 2$ we were able to construct a weakly cycle extendable graph of order $n$ with $H$-force number $k$.

Now, let $\mathcal{F} = \mathcal{F}(G)$ for a given graph $G \in \mathcal{H}$ denote the family of all $H$-force sets of $G$. As is easily seen, $\mathcal{F} = \{X \subseteq V \mid X \notin \mathcal{F}\}$ is an independence system on $V$ which means that $\mathcal{F}$ satisfies the following two properties.

1. $\emptyset \in \mathcal{F}$.
2. $X \in \mathcal{F}, Y \subseteq X$ implies $Y \in \mathcal{F}$.

In general, the independence system $(V, \mathcal{F})$ is not also a matroid which means that the property

3. If $X, Y \in \mathcal{F}$ and $|X| = |Y| + 1$, then there exists an $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{F}$.

is not satisfied for every graph $G \in \mathcal{H}$ (see, also [8]). Consider the hamiltonian graph $G$ with vertex set $V = \{1, 2, \ldots, 7\}$ which consists of the cycle $(1, 2, \ldots, 7)$ and the chords $14$ and $36$. For $G$ we have $\{1, 2, 3, 4\} \in \mathcal{F}$ and $\{1, 2, 3, 6, 7\} \in \mathcal{F}$ but, property (M3) is not satisfied for these two sets.

**Theorem 10.** If $G$ is a weakly cycle extendable graph, then $(V, \mathcal{F})$ is a matroid.

**Proof.** Let $X, Y \in \mathcal{F}$ be two sets where $|X| = |Y| + 1$. As $G$ is weakly cycle extendable, $G$ contains a $Y$-cycle $C$ of length $n - 1$. Let $v \in V$ be the only vertex which does not belong to $C$. Hence, $X \setminus \{v\}$ is a subset of $V(C)$. If there is a vertex $x \in X \setminus \{v\}$ with $x \notin Y$, then we have $Y \cup \{x\} \in \mathcal{F}$ and, consequently, $Y \setminus \{x\} \in \mathcal{F}$. Otherwise, we have $Y = X \setminus \{v\}$. That yields $Y \cup \{v\} = X \in \mathcal{F}$ and proves the property (M3). \hfill \blacksquare

The maximal independent sets of the matroid $(V, \mathcal{F})$, which are the members of $\mathcal{F}$ of maximal cardinality, are just the vertex sets of the cycles of length $n - 1$ of $G$.

If $\mathcal{C} = \mathcal{C}(G)$ denotes the set of all cycles in $G$ which are not weakly extendable, then let $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ denote a partition of $\mathcal{C}$, i.e., $\mathcal{C}$ is the union of $m \geq 1$ nonempty and disjoint subsets $\mathcal{C}_i$ of $\mathcal{C}(G)$. We call a partition $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ vertex-unsaturated (for short, unsaturated) if $V(\mathcal{C}_i)$ where

$$V(\mathcal{C}_i) := \bigcup_{C \in \mathcal{C}_i} V(C)$$

is different from $V(G)$ for $i = 1, 2, \ldots, m$. Now, let $p(G)$ denote the smallest integer $m$ for which there exists an unsaturated partition $(\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m)$ of $\mathcal{C}(G)$. 


Theorem 11. Let $G \in \mathcal{H}$ be a graph that is not weakly cycle extendable. Then,

$$h(G) = s(G) + p(G).$$

Proof. First, let $(C_1, C_2, \ldots, C_m)$ be an unsaturated partition of $C(G)$ such that $m = p(G)$. For $i = 1, 2, \ldots, m$ let $v_i \in V(G) \setminus V(C_i)$ be any fixed vertex. We prove that $X := S(G) \cup \{v_1, \ldots, v_m\}$ is an H-force set which implies $h(G) \leq s(G) + p(G)$. For this purpose, let $C$ be any non-hamiltonian cycle of $G$.

If there exists a $V(C)$-cycle $C'$ of length $n - 1$ in $G$, then $S(G)$ contains a vertex $v$ such that $\{v\} = V(G) \setminus V(C')$. Hence, $v \notin V(C)$ and, consequently, $X \notin V(C)$. If there is no $V(C)$-cycle of length $n - 1$ in $G$, then $G$ contains a $V(C)$-cycle $C'' \subseteq C(G)$. In this case there exists a partition set $C_i, 1 \leq i \leq m$, such that $C'' \subseteq C_i$. Then

$$v_i \in V(G) \setminus V(C_i) \subseteq V(G) \setminus V(C'') \subseteq V(G) \setminus V(C)$$

implies $X \notin V(C)$. Thus, every $X$-cycle is hamiltonian and $X$ is an $H$-force set.

Assume now that there exists an $H$-force set $X$ of $G$ with less than $s(G) + p(G)$ vertices. Since, by Theorem 8, $S(G)$ is not an $H$-force set, there exists a nonempty subset $Y \subseteq V(G) \setminus S(G)$ such that $X = S(G) \cup Y$. Because of the assumption we have $|Y| < p(G)$. Note that every cycle $C \subseteq C(G)$ is an $S(G)$-cycle because otherwise there would exist an $x \in S(G) \setminus V(C)$ such that $V(G) \setminus \{x\}$ is the vertex set of a cycle $C'$ of length $n - 1$ in $G$ with $V(C) \subseteq V(C')$, a contradiction with respect to $C \in C(G)$. Since, moreover, every $X$-cycle is hamiltonian, we have that for every $C \in C(G)$ there exists a vertex $y \in Y$ such that $y \notin V(C)$.

For every $y \in Y$, let us define $D_y = \{C \in C(G) \mid y \notin V(C)\}$. Then, we have

$$C(G) = \bigcup_{y \in Y} D_y$$

and, because of $C(G) \neq \emptyset$, there exists a vertex $y_1 \in Y$ such that $D_{y_1} \neq \emptyset$.

Now, we are able to construct an unsaturated partition of $C(G)$. To this end, let $C_1 := D_{y_1}$ and $Y_1 := Y \setminus \{y_1\}$. We may assume that the partition sets $C_1, \ldots, C_k$ with $k \geq 1$ are already constructed. If $Y_k$ contains a vertex $y_{k+1}$ such that the set

$$D_{y_{k+1}} \setminus \bigcup_{i=1}^{k} C_i$$

is not empty, then let

$$C_{k+1} := D_{y_{k+1}} \setminus \bigcup_{i=1}^{k} C_i.$$

This procedure terminates after at most $|Y| - 1$ steps and yields an unsaturated partition $(C_1, \ldots, C_m)$ with $m < p(G)$ which contradicts the definition of $p(G).$
As an immediate consequence of Theorem 11 we have

**Corollary 12.** Let $G \in \mathcal{H}$ be a not weakly cycle extendable graph. Then, the following conditions are equivalent.

1. $h(G) = s(G) + 1$,
2. $(C(G))$ is unsaturated.

**References**


Received 27 July 2015
Revised 23 February 2016
Accepted 23 February 2016