ALL TIGHT DESCRIPTIONS OF 3-STARS
IN 3-POLYTOPES WITH GIRTH 5

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Abstract

Lebesgue (1940) proved that every 3-polytope $P_5$ of girth 5 has a path of three vertices of degree 3. Madaras (2004) refined this by showing that every $P_5$ has a 3-vertex with two 3-neighbors and the third neighbor of degree at most 4. This description of 3-stars in $P_5$s is tight in the sense that no its parameter can be strengthened due to the dodecahedron combined with the existence of a $P_5$ in which every 3-vertex has a 4-neighbor.

We give another tight description of 3-stars in $P_5$s: there is a vertex of degree at most 4 having three 3-neighbors. Furthermore, we show that there are only these two tight descriptions of 3-stars in $P_5$s.

Also, we give a tight description of stars with at least three rays in $P_5$s and pose a problem of describing all such descriptions. Finally, we prove a structural theorem about $P_5$s that might be useful in further research.

Keywords: 3-polytope, planar graph, structure properties, $k$-star.

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1. Introduction

Let $\delta$ be the minimum vertex degree, and $g$ be the girth (the length of a shortest cycle) in a given 3-polytope. We recall that the graphs of 3-polytopes are precisely the 3-connected planar graphs due to Steinitz’s famous theorem [22].

The degree of a vertex $v$ or a face $f$, which is the number of edges incident with $v$ or $f$ in a 3-polytope, is denoted by $d(v)$ or $d(f)$, respectively. A $k$-vertex is a vertex $v$ with $d(v) = k$. By $k^+$ or $k^-$ we denote any integer not smaller or not greater than $k$, respectively. Hence, a $k^+$-vertex $v$ satisfies $d(v) \geq k$, etc.

Let $P_5$ be the set of (finite) 3-polytopes with $g = 5$, and $P^*_5$ be the 3-polytopes with $\delta = 5$. We note that $P_5$ and $P^*_5$ are in 1–1 correspondence due to the vertex–face duality, so structural results on $P_5$ are easily translated to the language of $P^*_5$ and vice versa.

The early interest of researchers to the structure of $P^*_5$ was motivated by the Four Color Problem. Already in 1904, Wernicke [23] proved that every graph in $P^*_5$ contains a 5-vertex adjacent to a $6^-$-vertex, and Franklin [12] in 1922 strengthened this to the existence of at least two $6^-$-neighbors. Franklin’s result is precise, as shown by putting a vertex inside each face of the dodecahedron and joining it with the five boundary vertices.

In 1940, Lebesgue [17] gave, in particular, an approximate description of the neighborhoods of 5-vertices in $P^*_5$ and proved that every 3-polytope in $P_5$ has a 5-face incident with four 3-vertices and the fifth 5-vertex, which face includes a path of three 3-vertices. In 2004, Madaras [19] refined the last mentioned result by Lebesgue as follows.

**Theorem 1** (Madaras [19]). Every 3-polytope with girth 5 has a 3-vertex adjacent to two 3-vertices and another vertex of degree at most 4, which is tight.

In dual terms, Theorem 1 reads equivalently as follows.

**Theorem 2** (Madaras [19]). Every 3-polytope with minimum degree 5 has a 3-face adjacent to two 3-faces and another face of degree at most 4, which is tight.

Nowadays, a lot of structural results on $P_5$ and $P^*_5$ can be found in the literature; for example, see [1–9, 13–16, 18–21].

We need a few definitions. A $k$-star $S_k(v; v_1, \ldots, v_k)$ in a 3-polytope consists of the central vertex $v$ and its neighbor vertices $v_1, \ldots, v_k$, in no particular order. A $k^+$-star has at least $k$ rays. In this note, we deal with $3^+$-stars in $P_5$.

We say that $S_k(v; v_1, \ldots, v_k)$ is an $(a; b_1, \ldots, b_k)$-star, or a star of type $(a; b_1, \ldots, b_k)$, where $b_1 \geq \cdots \geq b_k$, if $d(v) = a$ and $d(v_i) = b_i$ whenever $1 \leq i \leq k$.

A set $D = \{T_1, \ldots, T_n\}$ of star-types is a description for $P_5$ if every graph in $P_5$ has a star of one of the types from $D$. A description $D$ is tight if all descriptions $D - T_i$ with $1 \leq i \leq n$ are invalid, which means that for every $i$ there is a graph in $P_5$ that has no stars of types from $D - T_i$, but it has a star of type $T_i$.\]
Madaras [19] constructed a polytope in $P_5$ in which every 3-vertex has at least one 4\,+\,-neighbor (see Figure 1). In what follows, this construction is called $M_{04}$. The tightness of the description of 3-stars given in Theorem 1 is implied by the dodecahedron (which has no (3; 4, 3)-stars) together with $M_{04}$.

One of the purposes of our paper is to augment Theorem 1 by giving another tight description of 3-stars in $P_5$.

**Theorem 3.** Every 3-polytope with girth 5 has a vertex of degree at most 4 having three 3-neighbors, which is a tight description of 3-stars in $P_5$.

Here, the tightness follows from the facts that the dodecahedron has no stars of type (4; 3, 3), while $M_{04}$ avoids the (3; 3, 3)-star.

Our next result is that there are only two tight descriptions of 3-stars in $P_5$.

**Theorem 4.** In $P_5$, there are precisely two tight descriptions of 3-stars:

(a) $D_{04} = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$, given by Theorem 1 (Madaras [19]), and

(b) $D_{15} = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$, given by Theorem 3.

A 3-vertex is weak if it has two 3-neighbors and a 4-neighbor. For further attempts to find tight descriptions of 3\,+\,-stars in $P_5$, the following structural result seems useful.

**Theorem 5.** Every 3-polytope of girth 5 has one of the following configurations (see Figure 2):

(a) a 3-vertex with three 3-neighbors;

(b) a 4-vertex with four 3-neighbors, at least one of which is weak;

(c) a 4-vertex with a 4-neighbor and three 3-neighbors, at least two of which are weak.

It is easy to see that Theorem 5 implies Theorems 1 and 3, as well as the next fact.

**Corollary 6.** The following tight descriptions of 3\,+\,-stars in $P_5$ hold:

(i) $D_{04} = \{(3; 3, 3, 3), (3; 4, 3, 3)\}$ (Madaras [19]);

(ii) $D_{15} = \{(3; 3, 3, 3), (4; 3, 3, 3)\}$;

(iii) $D_1 = \{(3; 3, 3, 3), (4; 3, 3, 3), (4; 4, 3, 3, 3)\}$.

The tightness of $D_{04}$ and $D_{15}$ follows from the dodecahedron combined with $M_{04}$. In Figure 3, we see a half of a graph $H_1$ whose every vertex has a 4-neighbor. (Note that in $M_{04}$ only each 3-vertex has a 4-neighbor.) So the type (4; 4, 3, 3, 3) cannot be dropped from $D_1$, while the first and second types cannot be dropped due to the dodecahedron and $M_{04}$, respectively.
It looks like the following tempting problem is hard.

**Problem 7.** Find all tight descriptions of $3^+$-stars in $P_5$.

In fact, we were not able to solve even the following two much more modest problems.

**Problem 8.** Is it true that $D_2 = \{(3; 3, 3, 3), (3; 4, 4, 4), (4; 3, 3, 3)\}$ is a tight description of $3^+$-stars in $P_5$?

We note that if $D_2$ is a description, then it is tight due to the dodecahedron, $H_1$, and $M_{04}$.

**Problem 9.** Is it true that \{(3; 3, 3, 3), (3; 4, 4, 3), (3; 4, 4, 4), (4; 3, 3, 3, 3), (4; 4, 4, 3, 3), (4; 4, 4, 4, 3)\} is a description of $3^+$-stars in $P_5$?

In Section 2, we illustrate the constructions $M_{04}$ and $H_1$ and configurations in Theorem 5. Section 3 contains proofs of Theorems 5 and 4. Note that Theorem 5 implies Theorem 1 and is proved shorter than Theorem 1 in Madaras [19].

2. **Constructions** $M_{04}$ and $H_1$ and Configurations in Theorem 5

![Figure 1. (Madaras [19]) $M_{04}$: every 3-vertex has a 4-neighbor.](image)
Figure 2. Configurations in Theorem 5.

Figure 3. (A half of) $H_1$: every vertex has a 4-neighbor.
3. Proofs

3.1. Proving Theorem 5

Suppose that $P$ is a counterexample to Theorem 5. Euler’s formula $|V| - |E| + |F| = 2$ for $P$ may be written as

$$\sum_{v \in V} \left( \frac{3d(v)}{2} - 5 \right) \leq \sum_{v \in V} \left( \frac{3d(v)}{2} - 5 \right) + \sum_{f \in F} \left( d(f) - 5 \right) = -10,$$

where $V$, $E$, and $F$ are the sets of vertices, edges and faces of $P$, respectively.

Let us assign a charge $\mu(v) = \frac{3d(v)}{2} - 5$ to every vertex $v$ in $V$, so that the charge of vertices, depending on theirs degree, is $-\frac{1}{2}$, $1$, $\frac{5}{2}$, and so on. Using the properties of $P$ as a counterexample, we define a local redistribution of $\mu$’s, preserving their sum, such that the new charge $\mu'(v)$ is non-negative for all $v \in V$. This will contradict the fact that the sum of the new charges is at most $-10$, according to (1). Our rules of discharging are:

**R1.** Every 3-vertex $v$ receives from every adjacent 4-vertex either $\frac{1}{2}$ if $v$ is weak or $\frac{1}{4}$ otherwise.

**R2.** Every 4-vertex receives $\frac{1}{2}$ from every adjacent 5-vertex.

To complete the proof of Theorem 5, we first observe that every 3-vertex $v$ receives from its 4-neighbors either $\frac{1}{2}$ if $v$ is weak, or at least $2 \times \frac{1}{4}$ otherwise, so $\mu'(v) \geq -\frac{1}{2} + \frac{1}{2} = 0$.

Now if $d(v) = 4$, then $\mu(v) = 1$, and we are easily done unless $v$ has at least three 3-neighbors but no 5+neighbors. If so, then $v$ either gives $4 \times \frac{1}{4}$ to its four 3-neighbors, or at most $\frac{1}{2} + 2 \times \frac{1}{4}$ to its three 3-neighbors, which yields $\mu'(v) \geq 1 - 1 = 0$.

Finally, for $d(v) \geq 5$ we have $\mu'(v) \geq \frac{3d(v)}{2} - 5 - d(v) \times \frac{1}{2} = d(v) - 5 \geq 0$, as desired.

3.2. Proving Theorem 4

Suppose that $D$ is a tight description of 3-stars for $P_5$. Since the dodecahedron has 3-stars only of the type $(3;3,3,3)$, it follows that $(3;3,3,3) \in D$. Now we look at the graph $M_{04}$: it has 3-stars only of the types $(3;4,3,3)$ and $(4;3,3,3)$. As $M_{04}$ obeys $D$, at least one of these types should appear in $D$.

**Case 1.** $(3;4,3,3) \in D$. Since $D' = \{(3;3,3,3), (3;4,3,3)\}$ is a tight description by Theorem 1, we have $D = \{(3;3,3,3), (3;4,3,3)\}$ due to the minimality of $D$.

**Case 2.** $(4;3,3,3) \in D$. Since $D' = \{(3;3,3,3), (4;3,3,3)\}$ is a tight description by Theorem 3, we have $D = \{(3;3,3,3), (4;3,3,3)\}$, as desired.
References


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